Riemann sums

If we want to approximate an area we can slice it into little strips each of which can be approximated by a rectangle; we then add up the individual rectangles. To get a better approximation we can make the slices "smaller". This is the underlying idea of *Riemann sums*. Given a function $f(x)$ and an interval $[a, b]$ we start by partitioning $[a, b]$ up into a partition into pieces by first choosing points

$$x_0 = a < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

These are the bases of the rectangle and the "width" of the ith rectangle is $\Delta x_i = x_i - x_{i-1}$. To find the heights we choose points c_i so that $x_{i-1} \leqslant c_i \leqslant x_i$ we then have that the "height" is $f(c_i)$. So then we have that the Riemann sum (which is an approximation of the area under the curve $y = f(x)$ in the interval $[a, b]$) is

$$\sum_{k=1}^{n} f(c_k)\Delta x_k.$$

Here "Σ" (sigma) is used to indicate doing a sum. (Note that sigma and sum both start with "s", almost as if we had planned it that way!)

In general we have that

$$\sum_{k=1}^{n} a_k$$

is a convenient way to compress $a_1 + a_2 + \cdots + a_n$, the "k" here is a dummy variable that starts at 1 and goes to n (giving n terms). Because this is a sum then this behaves as we expect sums to behave, i.e., we can break sums apart, pull out constants, etc.

To indicate the slices getting smaller we let $\|P\| = \max\{\Delta x_k\}$. We are interested in functions where the limit as $\|P\| \to 0$ exists, we call such functions integrable (all continuous functions are integrable) and denote the limit by

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k = \int_a^b f(x)\,dx.$$

The $\int$ sign is a stretched out "S" and indicates the idea that we are summing up little pieces. The "x" is a dummy variable and can be replaced by any other variable, the result will be the same.

$$\int_a^b f(x)\,dx = \int_a^b f(u)\,du = \int_a^b f(y)\,dy = \cdots.$$

While our starting point is thinking of finding area, it is important to remember that the result of the integration can be positive or negative, so more appropriately it is signed area.

Note that we have used the $\int$ sign to indicate antiderivative (also called indefinite integrals), we will soon discover that there is a nice connection with $\int_a^b$ (also called definite integrals).

Properties of integrals

Properties of integration follow from the definition of Riemann sums (as well as some geometric intuition).

$$\int_a^b f(x)\,dx = \begin{bmatrix}\text{area above}\\ x\text{-axis}\end{bmatrix} - \begin{bmatrix}\text{area below}\\ x\text{-axis}\end{bmatrix}.$$

We can find the values of some integrals by finding the area is composed of combinations of triangles, rectangles and circles (right now this is the only way we can handle integrals of the form $\sqrt{r^2 - x^2}$).

If our upper and lower bound match then there is no "area" and so the integral is 0, i.e.,

$$\int_a^a f(x)\,dx = 0.$$

Changing the order of integration changes the sign, i.e.,

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

Integration is "linear" in the sense that we can pull constants out as well as break it up over addition, i.e.,

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx \quad \text{and}$$

$$\int_a^b \big(f(x) + g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

This for example allows us to break the problem of integrating several things added together into individual parts (this is especially convenient when we need to do one technique for integrating one part and a different technique for integrating another part).

We can break the interval we are integrating into pieces (this is convenient, for example, when we have piecewise functions), i.e.,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

This is true for any relationship of a, b, c. We can also reverse this and combine several integrals together to make a single integral.

If $f(x) \leqslant g(x)$ on $[a, b]$ then

$$\int_a^b f(x)\,dx \leqslant \int_a^b g(x)\,dx.$$

In particular, if $m \leqslant f(x) \leqslant M$ on $[a, b]$ then

$$m(b-a) \leqslant \int_a^b f(x)\,dx \leqslant M(b-a).$$

Using some basic sum identities we have the following:

$$\int_a^b 1\,dx = b - a$$
$$\int_a^b x\,dx = \frac{1}{2}(b^2 - a^2)$$
$$\int_a^b x^2\,dx = \frac{1}{3}(b^3 - a^3)$$

For a function $f(x)$ we define the average value of the function on the interval $a \leqslant x \leqslant b$ by

$$\text{average} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Quiz 11 problem bank

1. Use a Riemann sum to approximate the area under the curve $y = \dfrac{4x}{1+x^2}$ from $x = 1$ to $x = 4$ by using a partition with three equal-width parts, and choosing the *left* hand points on each part of the partition.

2. Use a Riemann sum to approximate the area under the curve of the function $f(t) = \dfrac{10}{1+t^4}$ for $-\frac{5}{2} \leqslant t \leqslant \frac{5}{2}$ by using a partition with five equal parts and choosing the *center* points on each part of the partition.

3. Let $h(x)$ be a function such that

$$\int_0^2 h(x)\,dx = 1,\ \int_0^3 h(x)\,dx = 2,\ \int_0^4 h(x)\,dx = 6,$$
$$\int_1^5 h(x)\,dx = 5,\ \text{and}\ \int_2^5 h(x)\,dx = 7.$$

   Find $\displaystyle\int_1^3 h(x)\,dx$.

4. Given that $\displaystyle\int_{-7}^{8} \big(4f(x) + g(x)\big)\,dx = 10$ and $\displaystyle\int_{-7}^{8} \big(2f(x) + 3g(x)\big)\,dx = 0$, determine $\displaystyle\int_{-7}^{8} \big(5f(x) - 3g(x)\big)\,dx$.

5. Find $\displaystyle\int_{-5}^{5} \sin(te^{-t^2})\,dt$.

6. Find $\displaystyle\int_{-1}^{3} \big(2|x| + 3\big)\,dx$.

7. Find $\displaystyle\int_0^1 \big(1 + \sqrt{1-x^2}\big)^2\,dx$.

8. Find $\displaystyle\int_1^{\sqrt{2}} \sqrt{2-y^2}\,dy$.

9. Determine a so that the average value of the function of $f(x) = 2x + 5$ for the interval $a \leqslant x \leqslant a + 2$ is 11.

10. Looking over attendance records the calculus instructor has discovered as the term progressed that fewer students came to lecture (thus breaking their super-sized heart). In particular, t weeks into the semester there were $150 - 20t + t^2$ students per lecture. Find the average number of students per lecture through the twelfth week (i.e., from $t = 0$ to $t = 12$).