Tangent lines

Now that we have limits we see that the slope of the tangent line at $x = a$ can be found by taking the limit of slopes of secant lines between a and a point that is moving closer to a. We call the slope of the tangent line at $x = a$ and denote it by $f'(a)$. So we have

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{b\to a} \frac{f(b) - f(a)}{b - a}.$$

(The second definition is an alternative definition, but is based on the same idea where we take the limit of slopes of secant lines.) We note that if $f(x)$ and x have units then the units of $f'(a)$ are (the units of f)/(the units of x); as an example if f measures cost in dollars and x measures number of items then the units of the derivative would be (dollars)/(item) (or "dollars per item").

Once we have the slope of the tangent line it is easy to find the tangent line (since we also have the point $\big(a, f(a)\big)$. Namely, since in general a line has the form

$$y - y_0 = m(x - x_0)$$

then we can substitute $(x_0, y_0) = \big(a, f(a)\big)$ and $m = f'(a)$ to get

$$y - f(a) = f'(a)(x - a) \quad \text{or} \quad y = f(a) + f'(a)(x - a).$$

Of course we can also recover information if we have the tangent line. For instance if we know the tangent line and at what value of $x = a$ it is tangent at we can recover $f(a)$ and $f'(a)$ (by looking at the y coordinate of the tangent line at $x = a$ and the slope respectively). But it is also important to note that this is *all* the information that we know about $f(x)$, i.e., we cannot say anything about f for x away from a.

Problem solving advice

Solving problems (and in Calculus we have many problems) goes through four stages. It is useful to consciously think about these stages, and eventually to have them become automatic for us.

1. *Understand the problem.* Make sure to identify what we are given, what we are trying to find, and what connects them. Understand all of the words and symbols. Are there similar problems we have tried in the past?

2. *Make a plan.* Outline the steps that need to be carried out. Understand what pieces of the problem there are and what order things need to be solved in. Don't be intimidated by problems that look big and complex, problems break down into smaller and simple pieces.

3. *Carry out the plan.* The important thing is to follow through the steps carefully, small mistakes snowball into big mistakes. As you carry out the plans be responsive to modifying the plan, perhaps a better method will reveal itself.

4. *Look back.* Is your answer reasonable? Can you check it by some other method? What were the indicators that will help you identify the proper method to find a solution in the future?

The best problem solving advice: *Try something. If it doesn't work, try something else. But never give up!*

Test taking advice

When taking tests, don't get too hung up on a problem. Go until you are stuck, think for a minute then go to another problem. When you return you will find that the problem has gotten easier and now you can make more progress.

Make sure to give yourself time to review your work for errors (as much as possible). If tests allow partial credit do your best to answer as much as you can. In general, make sure to grab the low hanging fruit, do the easy problems first! If something seems impossible or computations are becoming ridiculous, stop and think for a minute, usually this is a sign a mistake has been made or that there's a better way.

Quiz 4 problem bank

1. Given that $y = 3x + 4$ is tangent to $y = f(x)$ at $x = 2$, determine $f(2)$ and $f'(2)$.

2. Given that $y = 5x - 3$ is tangent to $y = g(x)$, determine a tangent line to $y = 2g(x)$.

3. Given that $y = -2x + 4$ is tangent to $y = h(x)$, determine a tangent line to $y = h(x + 2)$.

4. Given that $y = 4x - 7$ is tangent to $y = f(x)$ and that $f(x)$ is invertible, determine a tangent line to $y = f^{-1}(x)$.

5. Determine c so that for $f(x) = x^2$ the average rate of change between a and b equals the instantaneous rate of change at c.

6. Find the instantaneous rate of change for $y = (\sin x - \cos x)^2 + \sin(2x)$ at $x = \frac{2}{3}\pi$.

7. Find the value a so that the tangent line to $y = \dfrac{1}{\sqrt{x}}$ at $x = a$ is parallel to the line $y = -4x - 137$.

8. Find all lines of the form $y = k$ which are tangent to the curve $y = x^2 - 4x + 3$.

9. Find all lines of the form $y = kx$ which are tangent to the curve $y = x^2 - 3x + 4$.

10. Find a and b so that $y = ax^2 + bx + 5$ is tangent to the line $y = 2x - 3$ at $(2, 1)$.