Notation for derivatives

We can think of the derivative $f'(x)$ as a function, where evaluating the function at any point gives us the slope of the tangent line to the curve $y = f(x)$. In general we have

$$\frac{d}{dx}\big(f(x)\big) = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

(The notation "d/dx" comes from Leibniz and should be read as "the derivative with respect to x".)

We can denote the derivative at a particular point a by $f'(a)$, or using the Leibniz notation we have

$$\frac{d}{dx}\big(f(x)\big)\bigg|_{x=a}$$

(the "|" indicates that we are evaluating, we will see this notation later when we get to integration).

If the derivative exists, we note that for small h that $f(x+h) - f(x) \approx f'(x)h$. One consequence of this is that $f(x+h)-f(x)$ must be small and so we have that if the derivative exists at a point then the function *must* be continuous at that point. Conversely, if the function is not continuous then we automatically can conclude that there is no derivative.

Rules for derivatives

Actually using the limit definition to calculate the derivative is tedious and can get quickly complicated. So we will build up a collection of rules to allow us to calculate the derivatives of functions. Here are a few basic ones to get us started:

1. $\frac{d}{dx}\big(1\big) = 0.$
2. $\frac{d}{dx}\big(x\big) = 1.$
3. $\frac{d}{dx}\big(x^a\big) = ax^{a-1}.$
4. $\frac{d}{dx}\big(e^x\big) = e^x.$
5. $\frac{d}{dx}\big(f(x)+g(x)\big) = f'(x)+g'(x).$
6. $\frac{d}{dx}\big(kf(x)\big) = kf'(x).$
7. $\frac{d}{dx}\big(f(x)g(x)\big) = f'(x)g(x)+f(x)g'(x).$
8. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x)-f(x)g'(x)}{\big(g(x)\big)^2}.$

The first three rules tell us how to take the derivative of x^a, the basic idea is to bring the power of x down in front and then subtract 1 from the power (note if you ever encounter a $\sqrt{x}$ or similar term it is best to rewrite is as $x^{1/2}$ before applying the rule). The fourth rule is for the differentiation of the exponential function. The remaining rules give us ways to take the derivative of combinations of functions. For instance the fourth rule says if we have two functions being added (or subtracted) then to take the derivative we take the derivative of each piece and add (or subtract) the result. The fifth rule says if we have a constant times a function the derivative is the same constant times the derivative of the function. The sixth rule tells us how to take the derivative of a product (hence it is known as the *product rule* and is one of the **most important** results we will learn in this class), similarly the seventh rule tells us how to take the derivative of a quotient (hence it is known as the *quotient rule* it is not one of the most important results we will learn in this class, but still good to know.

There is one other way to combine functions and that is by composition. This leads to a rule for derivatives known as the chain rule, but we have not covered that and it will appear on a future quiz.

Once we have the derivatives of the basic functions and the rules for how to take derivatives of various ways of combining functions we will be able to take derivatives of all the functions that we encounter. That is why it is important for us to learn these rules!

Using the derivative

The reason that we are going through all of this is that we can use the derivative of a function to give us information about the function. For instance, by knowing the slopes of the tangent lines we can know whether the function is increasing or decreasing (i.e., where the function is going "up" the slopes of the tangent lines, or the derivative, is positive and where the function is going "down" the slopes of the tangent lines, or the derivative, is negative). This helps us to graph a curve and let us identify important features on the curve (we will discuss these ideas in more detail later).

Derivatives also show up in economic analysis. For example in economics we are often concerned with the cost of manufacturing goods. An important measurement that comes up in analysis is the *marginal cost* which corresponds to the cost of producing the next item. So for example we have

$$\underbrace{C(q+1)-C(q)}_{=\text{ marginal cost}} = \frac{C(q+1)-C(q)}{(q+1)-q} \approx C'(q),$$

i.e., the marginal cost can be interpreted as a slope of a line between $\big(q, C(q)\big)$ and $\big(q+1, C(q+1)\big)$ which is approximately the slope of the tangent line. In practice it is more convenient to use the derivative as the definition of the marginal cost in doing the analysis and answering questions about that is happening. Note that a similar thing happens with the *marginal revenue*.

Derivatives also arise **frequently** in physics. Before we discuss an example let us introduce higher order derivatives. Since $f'(x)$ is a function of x we can take its derivative, we get a function denoted $f''(x)$ which is the second derivative of x. Since $f'(x)$ tells us something about how the function is changing (via slopes of tangent lines), then $f''(x)$ also tells us something about how the function looks by seeing how the slopes of the tangent lines are changing. (This last idea is known as concavity and we will come back to this later.)

Of course, we can take the derivative of the second derivative and get what is known as the third derivative; and yet another derivative would give us the fourth derivative and so on. In general, the nth derivative of $f(x)$ is denoted by $f^{(n)}(x)$ or in Leibniz notation $d^n\big(f(x)\big)/dx^n$.

As an example of this, suppose that we have a function $s(t)$ which measures distance in some units. Then the first derivative, $s'(t)$. is how position is changing and is called the *velocity* (velocity is different from speed in that velocity has a sign (alternatively, an orientation), i.e., speed is $|s'(t)|$), while the second derivative, $s''(t)$ is how the velocity is changing and is called the *acceleration*.

Quiz 5 problem bank

1. Find $\dfrac{d}{dt}\left(\dfrac{t^3-2t^2+2t-4}{t^2+2}\right)$.

2. Determine the value x for which $f(x)=x^3-9x^2+5x-e^7$ has the *lowest* instantaneous rate of change; also give the instantaneous rate of change at that point.

3. Two curves passing through the same point intersect *perpendicularly* if their tangent lines are perpendicular. Choose a and b so that $y=\frac{1}{e}e^x$ and $y=ax^2+bx$ intersect perpendicularly at the point $(1,1)$.

4. Express $\dfrac{d^2}{dx^2}\big(f(x)g(x)\big)$ in terms of $f(x)$, $f'(x)$, $f''(x)$, $g(x)$, $g'(x)$ and $g''(x)$.

5. Express $\dfrac{d}{dx}\big(f(x)g(x)h(x)\big)$ in terms of $f(x)$, $g(x)$, $h(x)$, $f'(x)$, $g'(x)$ and $h'(x)$.

6. Given that $f(1)=2$, $f'(1)=-1$, $g(1)=3$ and $g'(1)=2$ find $h(1)$ and $h'(1)$ where $h(x)=x^2f(x)-3\sqrt{x}g(x)$.

7. Given that $y=3x-7$ is tangent to $y=f(x)$ at $x=2$, find the tangent line to $y=xf(x)$ at $x=2$.

8. Given that $y=51x-117$ is tangent to $y=x^2g(x)$ at $x=3$, find the tangent line to $y=g(x)$ at $x=3$.

9. Newton's first law states that an object in motion tends to stay in motion. Suppose a particle is being moved along the curve $y=8+4e^x-7x$ for $x<0$. At time $x=0$ the outside force stops acting on the particle and it continues in its current motion. Determine the (positive) location where the particle will hit the x-axis.

10. Given that the distance of a particle is given by $s(t)=t^4-24t^2+\pi^{17}t+e^3+6$, determine *all* values t so that the acceleration of the particle is zero.