Derivatives of inverse functions

Graphically a function is related to its inverse by flipping across the line $y = x$ (i.e., interchanging the roles of x and y. In particular a tangent line to $f(x)$ at $\big(a, f(a)\big)$ will when flipped across the line $y = x$ will become a tangent line to $f^{-1}(x)$ at $\big(f(a), a\big)$. We can make this precise by using implicit differentiation. Namely if $y = f^{-1}(x)$ then we have $x = f(y)$, taking the derivative of both sides have $1 = f'(y)\frac{dy}{dx}$, or rearranging:

$$\boxed{\frac{d}{dx}\big(f^{-1}(x)\big) = \frac{1}{f'(y)} = \frac{1}{f'\big(f^{-1}(x)\big)}}$$

So when there is a function which has a derivative, we can now find the derivative of the inverse function. As a simple example, take the exponential function $f(x) = e^x$ which has as its inverse $f^{-1}(x) = \ln x$. Since the derivative of e^x is again e^x we have

$$\frac{d}{dx}\big(\ln x\big) = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

More generally:

$$\boxed{\frac{d}{dx}\big(\ln|x|\big) = \frac{1}{x}}$$

Combining these results with the chain rule and basic algebra we also have the following:

- $\frac{d}{dx}\big(a^x\big) = a^x \ln a$
- $\frac{d}{dx}\big(\log_a x\big) = \frac{1}{x \ln a}$

Logarithms are very useful, they satisfy several nice properties, i.e., $\ln(ab) = \ln a + \ln b$ and $\ln(a^b) = b \ln a$. This allows us to take a complicated expression composed by multiplying multiple functions and/or raising functions to a function and then taking a log to simplify expressions. This is known as *logarithmic differentiation*. As a simple example consider $y = x^x$. First we take the log of both sides giving $\ln y = x \ln x$ and now we take derivatives of both sides, i.e., $\frac{y'}{y} = \ln x + x\frac{1}{x} = \ln x + 1$ or $y' = y(\ln x + 1) = x^x(\ln x + 1)$.

Inverse trig functions

Among the inverse functions that will prove useful are the inverse trig functions (sometimes denoted with "arc"). The key is to use various trig identities to rewrite our expressions. As an example (a personal favorite!) if $y = \arctan x$ then $x = \tan y$ or taking the implicit derivative we have $1 = (\sec^2 y)y'$ or $y' = 1/\sec^2 y = 1/(\tan^2 y + 1) = 1/(x^2 + 1)$. Similar analysis gives the other inverse trig functions and so we have the following.

1. $\frac{d}{dx}\big(\arctan x\big) = \frac{1}{1 + x^2}$
2. $\frac{d}{dx}\big(\arcsin x\big) = \frac{1}{\sqrt{1 - x^2}}$
3. $\frac{d}{dx}\big(\operatorname{arcsec} x\big) = \frac{1}{x\sqrt{x^2 - 1}}$

Related rates

Derivatives measure rate of change. In many problems it often occurs that the change of one variable will be connected with the change of another variable; in particular if we know how one of the variables is changing we should be able to say something about how the other variable is changing since their rates of change are related. Related rate problems are usually easy to identify since they will give a rate and ask for another rate.

There is a slight twist compared to what we did previously in that we usually think of how something changes with time and so instead of taking a derivative with respect to one of our variables we will almost always take it with respect to time t (or some similarly appropriate variable), so it is important to have the chain rule and to apply it correctly.

In general there are a few simple steps to solve a problem about related rates.

1. Find a relationship (i.e., an equation) between the variables that are changing. It often helps to draw a picture if one is possible. There are only a few handful of variations that we will encounter. Most involve either the Pythagorean Theorem, similar triangles or areas and volumes of basic shapes. (You will be expected to know the area of rectangles and circles and triangles, and the volumes of boxes and cylinders.)
2. Take the derivative (of both sides) with respect to t to get a relationship for the rates.
3. Plug in all the values that you know to find the value that you are looking for.

The hardest part about these problems is almost always finding the relationship. Another thing to watch out for is they may only give the value for one variable, in which case you might need to solve for the value of the other variable using the relationship. Finally, an important tool in solving this type of problem is being able to strip out all the unnecessary information and translating a word problem into something like a calculus problem ("math-a-nese").

A classic example is a 10 foot ladder that is sliding down a wall and you notice that when the bottom of the ladder is 6 feet from the wall that it is sliding away from the wall at a rate of 1 foot per second. How fast is it sliding down the wall? To answer this we can draw a simple picture and see that the ladder the floor and the wall make a right triangle with a fixed hypotenuse of length 10. If we let x denote the distance of the bottom of the ladder to the wall and y the height of the ladder then we have $x^2 + y^2 = 100$ (note that when $x = 6$ it is easy to see from this relationship that $y = 8$). The fact that the ladder is moving away from the wall tells us that $dx/dt = 1$ and that we are after dy/dt. Taking the derivative of both sides we have $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$. Plugging in what we know we have $2{\cdot}6{\cdot}1 + 2{\cdot}8{\cdot}\frac{dy}{dt} = 0$ so that $\frac{dy}{dt} = -\frac{3}{4}$ feet per second.

Note that units will always do what you think they should do. And so we do not need to keep track of them in our calculations if we know what the end units should be, they will work themselves out correctly!

Quiz 7 problem bank

1. Find $\dfrac{d}{dx}\big(\ln\big(\ln\big(\ln|x|\big)\big)\big)$.

2. Find $\dfrac{d}{dx}\left(\dfrac{(1+x^2)^{\sin x}}{e^{5x}} + 3x\right)$.

3. Given that $f(x) = x^3 + e^{x-1}$, and that $g(x) = f^{-1}(x)$, find $g'(2)$.

4. Find $\dfrac{d}{d\theta}\big(\ln(\sec\theta + \tan\theta)\big)$, simplify the answer as much as possible.

5. Find $\dfrac{d}{dx}\big(\arctan x - \frac{1}{2}\arctan\big(\frac{1}{2}\big(x - \frac{1}{x}\big)\big)\big)$ for $x \neq 0$, simplify the answer as much as possible.

6. From a small island in the middle of a large lake you set out in a canoe and head east, and a short time later your friend sets out in a canoe and heads north. After a few hours your friend calls you on your walkie-talkie to see how far you have gone. You respond "I am fine, but haven't kept track of distance but right now I am going at a speed of one mile per hour." Your friend comments back, "That's pretty good, I have gone a few miles and right now am only doing one-third of a mile per hour." Glancing at your walkie-talkie you see that it is indicating that you and your friend are currently five miles apart and are moving apart from each other at a speed of one mile per hour. Find the distances that you and your friend have traveled in the canoe.

7. While studying for the quiz some students decide to take a break and bake brownies. They stir the brownie batter in a bowl which has a hemispherical shape with a radius of 5 inches, they then pour it into the only pan they have which is a large circular cake pan (i.e., the shape of a cylinder) which is 12 inches in diameter and 2 inches deep. One student observes that the depth of the batter in the pan raised at a constant rate of 1/6 of an inch per second. Find the rate at which the depth of the batter is falling in the mixing bowl when the depth of the batter in the mixing bowl is 3 inches. (Hint: the volume of batter of depth h in a hemispherical bowl with radius r is $\pi(rh^2 - \frac{1}{3}h^3)$.)

8. Starfleet intelligence has recently learned of a new threat. A new Borg vessel has been discovered that can change its shape, they are calling it the B-1000 (short for Borg-1000). The B-1000 has some limitations, first the only shape it can have is a three dimensional box and second the volume is always fixed, i.e., the box cannot deviate from a fixed volume. From an earlier observation you saw that the B-1000 had dimensions 20 meters by 15 meters by 10 meters. Currently though you can only see two sides. You notice that the length is currently 12 meters and is increasing at a rate of 1 meter per minute; the width is currently 25 meters and is decreasing at a rate of 2 meters per minute. What is the current depth of the B-1000 and how fast is it changing?

9. While visiting family you attend a pumpkin drop, an event where a very large pumpkin is lifted by a crane to a height and then dropped resulting in a beautiful moment of pumpkin explosion. You pull out your camera to record the pumpkin from the moment it is released and follow it down until it hits the ground. You happen to notice that you are standing 32 feet away from where the pumpkin will land, and you also notice that the pumpkin is being lifted to a height of 64 feet before being dropped. Knowing your basic physics you determine the pumpkin's height at a given time t will be $64 - 16t^2$ where t is the number of seconds after the pumpkin is dropped. If θ measures the angle of elevation your camera makes with the ground, what is $\frac{d\theta}{dt}$ at the moment the pumpkin will hit the ground?

10. A particle moves along the curve implicitly defined by $xy^4 - yx^4 = x - y^2$. When the particle passes through the point $(1, 1)$ its x coordinate is changing 1/4 units per second. How fast is the y coordinate changing?