Linear approximation

Sometimes we do not need the exact value of a function (or it might be very hard to compute) but an estimate will do. In that case we can use the tangent lines at a point that we understand what is going on to get an estimate at a point *nearby* that we want to know what is going on. (Note a key word to look for in a problem is "estimate", if we see it then we know we are doing some form of linear approximation.)

Recalling one of the definitions of the derivative we have that at a point a that

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{\Delta f}{\Delta x},$$

where Δf is a small (but measurable!) change in the function and similarly Δx is a small change in the input. In particular we have that if x is close to a, i.e., Δx is small, then

$$f'(a) \approx \frac{\Delta f}{\Delta x} \quad \text{or} \quad \boxed{\Delta f \approx f'(a)\Delta x.}$$

One example of where this might come in is if we want to estimate the result of a small change in x will have on the change in f or vice-versa. It also is useful for getting an estimate on how much error in our input will effect the error in our output.

If we replace Δf by $f(x)-f(a)$ and Δx by $x-a$ and rearrange then we have

$$f(x) \approx \underbrace{f(a)+f'(a)(x-a)}_{=L(x)}.$$

The right hand side, the $L(x)$ also known as the local linearization, is the same as the tangent line at $x = a$ and this states that the value of the function for x near a is approximately the same as the value of the tangent line. We can use this if we know the value of the function *somewhere near* x and want an estimate for the function *at* x. For instance suppose we want an estimate for $\sqrt{65}$, we know $\sqrt{64} = 8$ and 65 is near 64 and so we will approximate the function $f(x) = \sqrt{x}$ around $a = 64$ using the tangent line and use that to estimate $\sqrt{65}$. So for example we have $f'(x) = 1/(2\sqrt{x})$ and so for x near 64 we have

$$\sqrt{x} \approx f(64) + f'(64)(x-64) = 8 + \frac{1}{16}(x-64).$$

So we can conclude that $\sqrt{65} \approx 8 + \frac{1}{16} = 8.0625$ (the actual value is $8.06225\ldots$).

Sometimes these relationships are expressed using *differential* notation, i.e.,

$$\boxed{dy = f'(x)\,dx}$$

where dy and dx are the differentials in y and x.

Global extrema

Extrema are maximums (the largest possible value) and minimums (the smallest possible value) of the function. There are two types of extremums. One is global extremum which says that you are either the largest or smallest possible value for all x we are considering. The other is local extremum which says you are either the largest or smallest possible value for all x nearby (i.e., if we zoom in close enough and ignore everything else it is the largest or smallest possible value). Note that a global extremum is also a local extremum but a local extremum might not be a global extremum.

Extreme Value Theorem: If a function is continuous on a closed interval (an interval which includes the endpoints) then the function has a global maximum and a global minimum on the interval.

Knowing that there is an extreme value is useful, but we still have to find where it is. One important observation is that we can use the derivative to help us locate where the extremums *might* occur. This is because the derivative tells us locally what our function is doing and we can use that information to rule out points. For example we have the following.

$$\begin{aligned} f' > 0 &\quad\longleftrightarrow\quad f \text{ is increasing} \\ f' < 0 &\quad\longleftrightarrow\quad f \text{ is decreasing} \end{aligned}$$

But if we are at a maximum or a minimum then we are neither increasing or decreasing (if we were we would move slightly and be even more maximer or minimer but those aren't even words so of course this is not possible). So we can conclude that if we are at an extremum that we must be at a point where one of three things happens: (i) $f'(c) = 0$; (ii) $f'(c)$ is undefined; (iii) we are at a boundary (and so unable to move in one direction). We call these points *critical points*.

While extremum will happen at critical points, just because we are at a critical point does not mean we are at a minimum or a maximum. For example the functions $y = x^3$ and $y = \sqrt[3]{x}$ both have critical points at 0, but 0 is not an extremum for either function.

So our method to find Global extrema is as follows:

1. Make a list of x values where the global extrema can occur.
    (a) Endpoints of the interval.
    (b) The derivative is 0.
    (c) The derivative is undefined.
2. Plug each of the above values into the function. Largest answer is the global max; smallest answer is the global min.
3. Profit!

Note that while we only have one global maximum it can occur at multiple points. For instance $y = \sin x$ on the interval $0 \leqslant x \leqslant 4\pi$ has a global maximum of 1 and it occurs at two points, $\frac{1}{2}\pi$ and $\frac{5}{2}\pi$.

Rolle's Theorem and Mean Value Theorem

Rolle's Theorem states that if a function is continuous on an interval $a \leqslant x \leqslant b$, differentiable for $a < x < b$ and $f(a) = f(b)$ then there is some point c in between a and b (i.e., $a < c < b$) so that $f'(c) = 0$.

The Mean Value Theorem takes Rolle's Theorem and turns it slightly. Namely, if a function is continuous on an interval $a \leqslant x \leqslant b$ and differentiable for $a < x < b$ then there is some point c between a and b (i.e., $a < c < b$) so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(The right hand side is the slope of the secant line through $\big(a, f(a)\big)$ and $\big(b, f(b)\big)$ and also the average rate of change between a and b while the left hand side is the slope of the tangent line at c or the instantaneous rate of change. So the theorem says that if our function is "smooth" then at some point between a and b our instantaneous rate of change exactly matched the average rate of change.)

An important corollary to the Mean Value Theorem is that if a function has a derivative of 0 on an interval then the function is constant on that interval. (Previously we had seen that the derivative of a constant was 0, this says the opposite is also true.) In particular if two functions have the same derivative then they must differ by a constant.

Quiz 8 problem bank

1. For $y = 5x^3 - 4e^{3x-6}$, if $x = 2 \pm 0.05$, use linearization to estimate the corresponding range for y.

2. You have recently been hired as the chief architect for one of the pyramids being constructed by Pharaoh Sneferu in Egypt. After some consultation the pharaoh has agreed to a pyramid design that is 500 cubits wide and 300 cubits high (a cubit is the system of measurement used in ancient Egypt). The volume of a pyramid is $\frac{1}{3}b^2h$ where b is the length of one side of the base and h is the height; so that the pyramid will require $25,000,000$ cubic cubits of stone. After getting in touch with your stone contractor you discover that there are only $23,000,000$ cubic cubits of stone available. The pharaoh gives the go ahead to build a (slightly) smaller pyramid, but with the same proportions as before. Estimate how many cubits smaller the base of the pyramid will end up being.

3. Use linearization to give an estimate for $\sqrt[3]{1018}$.

4. Use the following information to get an estimate for $g(f(2.1))$.

x	0	1	2	3
$f(x)$	-1	3	0	2
$f'(x)$	1	-2	3	0
$g(x)$	3	0	1	2
$g'(x)$	1	3	2	2

5. You and a classmate are preparing to give a presentation in your astronomy course. You have decided that the best way to show how a star gets sucked into a black hole is through a modern interpretive dance where you will be playing the part of a large blue class O star and your partner will be the black hole. You will represent these two astronomical features using paper mache, and you are responsible for making your star. Initially you were planning to blow up a spherical balloon to a diameter of 16 inches before covering it in paper mache, but in your excitement you ended up blowing the balloon to a diameter of 17 inches. Using linear approximation get an estimate of how much more surface area the 17 inch balloon has as compared to the 16 inch balloon (i.e., an estimate of how much more paper you will need to make your model). (Hint: the surface area of a sphere of radius r is $4\pi r^2$.)

6. List all critical points of $g(x) = x^{2/3}\big(5x^2 - 8x - 40\big) + 137e^{\pi}$.

7. List all critical points of $f(t) = t\sqrt{4e^t} - (t+2)^2$.

8. Find the global maximum and global minimum for $h(x) = x^2 - 2\arctan(x^2)$ for $0 \leqslant x \leqslant \sqrt[4]{3}$. (Hint: $h(\sqrt[4]{3}) = \sqrt{3} - \frac{2}{3}\pi \approx -0.3623$.)

9. Consider the following function which is continuous and differentiable for all x (you do not need to prove this!):
$$f(x) = \begin{cases} x^3 + 2x^2 - 4x + 2 & \text{if } x \leqslant 1; \\ 3x - 2 & \text{if } x > 1. \end{cases}$$
For the interval $-4 \leqslant x \leqslant 2$ find *all* values of c that satisfies the Mean Value Theorem. (Hint: $f(-4) = -14$.)

10. Find the *unique* value c that satisfies the Mean Value Theorem for the function $f(x) = \arctan(\sin x)$ on the interval $0 \leqslant x \leqslant \pi/2$.