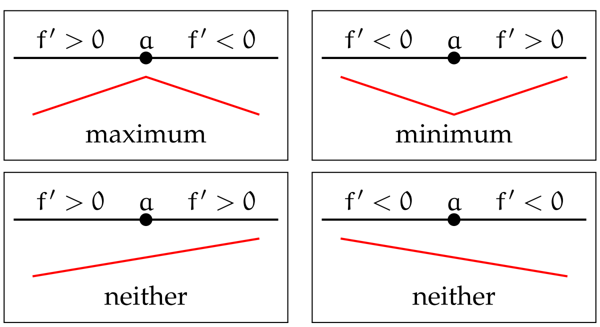
Local extremum and the first derivative test

When we are looking for local extremum then we look at the critical points (and endpoints if we have them, but we won't always). So we know where to look so now the important thing is how to identify whether a critical point is a local maximum, a local minimum or neither. One way to help distinguish between the possibilities is to use information about how the function is behaving near the critical point by using the first derivative.

$f' > 0$	$\longleftrightarrow$	function is increasing
$f' < 0$	$\longleftrightarrow$	function is decreasing

We are now ready to discuss "the first derivative test". We start by finding all the critical points, undefined points, and boundary points. We represent all of these points on the real number line. Between any pair of points the first derivative will always have the same sign (either positive or negative); to determine the sign just pick any point in the interval and evaluate. There are four possibilities for how the sign of the derivative behaves around a critical point and these are shown below.



In addition to helping to classify critical points this also helps us to answer on what intervals the function is increasing and on what intervals the function is decreasing.

Concavity and the second derivative test

Concavity tells us how the function is "bending", so that concave up is that the function is trying to bend up while concave down is that the function is trying to bend down.

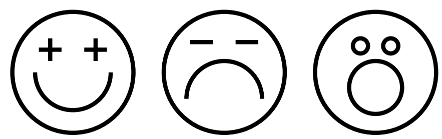
Put more concretely the graph is *concave up* when the derivatives are increasing (i.e., f' is increasing) which corresponds to when $f'' > 0$ (i.e., a function is increasing when the derivative is > 0). Similarly the graph is *concave down* then the derivatives are decreasing (i.e., f' is decreasing) which corresponds to when $f'' < 0$ (i.e., a function is decreasing when the derivative is < 0).

A point where concavity changes is called an inflection point. To find inflection points we take the second derivative; see where it is 0 or undefined and then mark off intervals and test each interval for the concavity. Any point where the concavity changes is an inflection point. (This process is similar to what we did to use the first derivative test to find and test local extremum, but this is not surprising since an inflection point also corresponds to the maximums and minimums of the first derivative.)

Since concavity tells us how the curve is "shaped" we can also use it to tell us whether a critical point is a maximum or a minimum. However there are some limitations. First, we can only use the second derivative test on a point where the first and second derivatives exist, and second the test might be inconclusive (unlike the first derivative test which works for both kinds of critical points and is conclusive). We have the following rule: If $f'(c) = 0$ and $f''(c)$ exists then

$f''(c) > 0$	$\longrightarrow$	$\big(c, f(c)\big)$ is a local minimum;
$f''(c) < 0$	$\longrightarrow$	$\big(c, f(c)\big)$ is a local maximum;
$f''(c) = 0$	$\longrightarrow$	the test is inconclusive.

A convenient way to remember this is with the following adorable picture.



Sketching a curve

The idea about sketching a curve is to find the interesting points (i.e., critical points, inflection points, intercepts and asymptotes). Mark the points and then connect them with appropriately shaped curves based on the signs of the first and second derivatives (a souped-up version of connect the dots).

Optimization

In optimization problems we are trying to minimize or maximize some value. (It is easy to spot optimization problems because they will be the ones that ask you to find the "largest" or "smallest" or really any kind of "-est" word.)

How to solve optimization problems:

1. Find appropriate labels. In particular there are essentially two things that need labels: (a) the value that we are trying to optimize; (b) the value(s) that we can vary in our optimization problem. It is often useful to draw a picture, if possible.

2. Find a function for what we are trying to optimize in terms of what we can vary (this is always the hardest part!). We need to get the equation down to a single variable, this is done using constraints (i.e., relationships that the variables must satisfy).

3. Use techniques for finding local max/min to find optimal values.

These problems are no different than other problems where we are looking for the maximum and minimum. The only difficulty is that we are usually not given a function and we need to set it up (and sometimes it is buried deep inside of a word problem).

For instance we might need to find the maximum area that we can enclose with 400 feet of fencing if the area is a rectangle and one side does not need fencing, since it is on a river. Then what we are trying to optimize is the area so we label it A. The area is the length times the width of the rectangle let us label these as x and y (where x will be the top and bottom and y will be a single side, the other side being the river). Now we know that $A = xy$, but we need to get down to a single variable. To do this we look back and notice that we haven't used the information about the 400 feet of fencing, which is a constraint. This tells us that $2x+y = 400$ so we can rearrange this to $y = 400-2x$ so that $A = x(400-2x) = 400x-2x^2$. We now look for critical points and to do this we use the first derivative $A' = 400-4x$, setting it to 0 and solving we get $x = 100$. We can then conclude that $y = 200$ so that the maximum area is $100\cdot 200 = 40,000$ square feet.

When doing optimization problems it is good to stop and see if your answer is reasonable. For instance if our answer to the previous problem was $x = -200$ then we would look for a mistake. Also make sure that you answer the question that is asked. In this last example we are asked to find the maximum area, so our answer should be an area. If we had been asked for the dimensions then our answer should be 100×200, i.e., the dimensions, and so on.

Quiz 9 problem bank

1. On what intervals is $h(x) = x^2 - 2\arctan(x^2)$ increasing and on what intervals is it decreasing?

2. Find the *two* critical points of $y = x^{2/3}e^{-2x/3}$ and use the first derivative test to determine if they are local min's or max's.

3. Find the *two* inflection points for $y = \theta^2 + \sin(2\theta)$ for $0 \leqslant \theta \leqslant \pi$, also identify the intervals where the function is concave up and where the function is concave down.

4. Verify the function $f(x) = \cos(x^3 - 2x)$ has a critical point at $x = 0$. Use the second derivative test to determine if it is a maximum or a minimum.

5. Find and classify the location of the critical points of $f(x) = 2x^5 - 5x^4 - 10x^3 + 13$.

6. Find all inflection points for $y = xe^{-6x^2}$, and determine the intervals where the function is concave up and where the function is concave down.

7. For $y = \dfrac{1}{1+x^2}$, find the value $a > 0$, where the y-intercept of the tangent line at $x = a$ is maximal.

8. What is the area of the largest rectangle that you can make where the bottom edge is on the x-axis and the top two corners lie on the parabola $y = 12 - x^2$?

9. For $a \geqslant 0$ find the point on the curve $y = \sqrt{x}$ closest to the point $(a, 0)$.

10. Past analysis of previous parties has led to the development of a chip index where the higher the chip index the better the party. In particular, the chip index is N^2P where N is the number of nacho chip bags that you have and P is the number of potato chip bags that you have. Given that you have \$18 for your chip fund and a bag of nacho chips cost \$3 and a bag of potato chips cost \$1, how many bags of each chip should you buy to maximize the chip index and thus have a most awesome party.