Chapters 2.1 - The Degree of a Vertex

Degree of a vertex v is the number of edges incident with v (loop counts $2\times$), denoted by $deg(v)$ or $d(v)$.

In digraph we count **in-degree** $d^-(v)$ and **out-degree** $d^+(v)$.

Neighborhood of a vertex v is the set of vertices adjacent to v, denoted by $N(v)$.

Note $deg(v) = |N(v)|$ for *simple* graphs.

Vertex v is **isolated** if $d(v) = 0$.

Vertex v is **leaf** if $d(v) = 1$.

The **minimum degree** of G is $\delta(G) = \min_{v \in V(G)} d(v)$.

The **maximum degree** of G is $\Delta(G) = \max_{v \in V(G)} d(v)$.

Theorem 2.1 If a graph G has m edges when

$$\sum_{v \in V(G)} deg(v) = 2m$$

A vertex of even degree is called an **even vertex**, while a vertex of odd degree is an **odd vertex**.

Corollary 2.3 Every graph has an even number of odd vertices.

Theorem 2.4 Let G be a graph of order n. If

$$deg(u) + deg(v) \geq n - 1$$

for every two nonadjacent vertices u and v of G, then G is connected and $diam(G) \leq 2$.

Graph G is r**-regular** if $r = \delta(G) = \Delta(G)$.

3-regular graphs are called **cubic**. Petersen's graph.

Theorem 2.6 Let r and n be integers with $0 \leq r \leq n - 1$. There exists an r-regular graph of order n if and only if at least one of r and n is even. (Harary graphs)

Theorem 2.7 For every graph G and every integer $r \geq \Delta(G)$, there exists an r-regular graph H containing G as and induced subgraph.

1: Is Theorem 2.4 tight? Find a disconnected graph where $deg(u) + deg(v) = n - 2$.

2: Show that if G of order n has $\delta(G) \geq (n-1)/2$, then G is connected.

3: Is it possible that among a group of seven people that each person has exactly three friends in the group? Explain.

4: *2.1* Give an example of the following or explain why no such example exists:

(a) a graph of order 7 whose vertices have degrees 1, 1, 1, 2, 2, 3, 3.
(b) a graph of order 7 whose vertices have degrees 1, 2, 2, 2, 3, 3, 7.
(c) a graph of order 4 whose vertices have degrees 1, 3, 3, 3.?

5: *2.3* The degree of each vertex of a certain graph of order 12 and size 31 is either 4 or 6. How many vertices of degree 4 are there?

6: *2.25*

(a) Let v be a vertex of a graph G. Show that if $G - v$ is 3-regular, then G has odd order.
(b) Let G be an r-regular graph, where r is odd. Show that G does not contain any component of odd order.

7: Show that if a graph G on n vertices is isomorphic with $\overline{G}$ then either n or $n-1$ is divisible by 4.

8: If G is a k-regular graph then is $\overline{G}$ also a regular graph? If so what is the degree of a vertex?