Chapters 2.3 - Degree Sequences; 2.4 - Graphs and Matrices

Degree sequence of a graph is a sequence of it's vertex degrees.

A finite sequence of nonnegative integers is **graphical** if is is a degree sequence of some graph.

Theorem 2.10 Havel-Hakimi A non-increasing sequence $s : d_1, d_2, \ldots, d_n (n \geq 2)$ of non-negative integers, where $d_1 \geq 1$, is graphical if and only if the sequence

$$s_1 : d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n.$$

is graphical.

Proof:

1: *Example 2.11* Decide whether the sequence s : .... is graphical.

How to store graph?

Let $G = (V, E)$ be a graph, where $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$.

Adjacency matrix of G is $n \times n$ matrix $A = [a_{ij}]$ where

$$a_{i,j} = \begin{cases} 1 & v_i v_j \in E \\ 0 & \text{otherwise} \end{cases}$$

Incidence matrix of G is $n \times m$ matrix $B = [b_{ij}]$ where

$$b_{i,j} = \begin{cases} 1 & v_i \in e_j \\ 0 & \text{otherwise} \end{cases}$$

Theorem 2.13 $A^k_{i,j}$ counts the number of $v_i - v_j$ walks of length k.

2: Find an example of two different graphs with the same degree sequence.

3: Is $5,5,3,3,2,2,2,2,2$ graphical? Justify your answer.

4: *2.39* Let A be the adjacency matrix for P_4. Determine A^4 without computing A or performing matrix multiplication.

5: Let G be a bipartite graph. Show that A^k will have zero entries for each value of k.

6: An edge e is a **bridge** if $G - e$ has more components than G. Show that if G has no odd vertices, then G has no bridges.

7: Prove that every graph with at least two vertices has at least two vertices of equal degree.

8: Show that $G \,\square\, H$ is connected if and only if both G and H are connected. ($G \,\square\, H$ denotes the Cartesian product of G and H.)

9: *Open problem* In a connected graph, pick any three paths of maximum length. Is there always a vertex that lies on all of them?

Reading for next time: Chapters 3.1, 3.2