Chapters 4.1 - Bridges; 4.2 - Trees

Edge $e = uv$ in a connected graph G is a **bridge** if $G - e$ is disconnected.

Edge e in a graph G is a **bridge** if the number of connected components in $G - e$ is more than in G.

Theorem 4.1 An edge e of a graph G is a bridge if and only if e lies on no cycle of G.

A graph is **acyclic** if it has no cycles.

A graph G is **tree** if G is acyclic and connected.

1: List all non-isomorphic trees on 4 vertices

End-vertex or **leaf** is a vertex if degree one.

Tree is a **star** if it has exactly one vertex that are not a leaf.

Tree is a **double-star** if it has exactly two vertices that are not leaves.

Tree is G a **caterpillar** if G has at least 3 vertices and removing all leaves from G gives a path, the path is called **spine** of the caterpillar.

Sometimes a tree G has a vertex called **root**, then G is **rooted tree**.

An acyclic graph is called a **forrest**.

Theorem 4.2 A graph G is a tree if and only if every two vertices of G are connected by a unique path.

Theorem 4.3 Every nontrivial tree has at least two end-vertices.
Hint: Take longest path.

Theorem 4.4 Every tree of order n has size $n - 1$. (recall order $= |V|$ and size $= |E|$)

2: *4.7* (a) Draw all forests of order 5. (b) Draw all trees of order 6.

3: Show that if T is a tree and $\Delta(T) = k$ then T has at least k leaves.

4: *4.9* Show that a graph of order n and size $n-1$ need not be a tree

5: Show that sequence of natural numbers $d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$ is a degree sequence of some tree iff $\sum_i d_i = 2n - 2$.

6: Prove that every n vertex graph with m edges has at least $m - n + 1$ cycles (different cycles, but not necessarily disjoint cycles).

7: Prove that a graph G is a tree if and only if G contains no cycle, but $G+uv$ does for each pair of non-adjacent vertices u, v in G.

8: Let G be a connected graph that has neither C_3 nor P_4 as an induced subgraph. Prove that G is a complete bipartite graph.

9: *Open problem* In a 3-regular graph, is there always a cycle whose length is a power of 2? Is it true for the Petersen's graph?

Reading for next time: Chapters 4.2, 4.3