Red-Blue algorithm and Chapter 4.3 Counting Spanning Trees of K_n

Let G be a graph. A **cut** is $\{uv : u \in X, v \notin X, uv \in E(G)\}$ for some $X \subseteq V(G)$.

1: Let $X \subseteq V(G)$ create a cut C. Let $u \in X$ and $v \notin X$. Show that every $u - v$ path contains at least one edge of C.

Red-Blue meta algorithm for MST. Let G be a graph and w be a weight assignment to $E(G)$. Assume that all weights are distinct. Start with all edges being uncolored. Apply the following rules as long as possible.

- if $e \in E$ is in a cycle C where e is the heaviest edge, color e red
- if there is a cut where $e \in E$ is the lightest edge, color e blue.

Blue edges for a minimum spanning tree.

2: Show that red edge cannot be in MST.

3: Show that blue edge must be in MST.

4: Show that blue edges form a tree

5: Show that every edge gets colored.

6: Show that no edge satisfies both red and blue criteria. (i.e. every edge has one color).

7: What is the number of spanning trees of C_5?

8: Count the number of spanning trees of K_4.

Theorem 4.15 (Cayley 1889) The number of distinct trees of order n with a specified vertex set is n^{n-2}.

Proof by double counting (there are many others).

$\mathcal{F}_{n,k}$ be the set of all rooted forests with k components on n (distinguishable) vertices. Assume edges directed away from the roots.

9: What is $|\mathcal{F}_{n,1}|$?

10: What is $|\mathcal{F}_{n,n}|$?

Forest F *contains* forest F' if F' can be obtained from F by deleting edges.

Call a sequence $F_1, \ldots, F_k$ of forests *refining sequence* if $F_i \in \mathcal{F}_{n,i}$ and F_i contains F_{i+1} for all i.

For a fixed forest F_k in $\mathcal{F}_{n,k}$ denote by

- $N(F_k)$ the number of rooted trees containing F_k.
- $N^\star(F_k)$ the number of refining sequences ending in F_k.

11: Suppose $F_1 \in F_{n,1}$ contains F_k. How many refining sequences are there that start at F_1 and end at F_k?

12: Express $N^\star(F_k)$ using $N(F_k)$ and the previous question.

Now we try to count $N^\star(F_k)$ the other way by adding edges to F_k and building a tree.

13: How many ways can you build F_{k-1} that contains F_k?

14: Express $N^\star(F_k)$ by iterating the previous question.

15: Combine results of $N^\star(F_k)$ to count $N(F_k)$.

16: Use $N(F_k)$ to count the number of rooted trees on n vertices (pick the correct k).

17: Count the number of spanning trees of K_7.