Chapter 5.3 Vertex and Edge Cuts

A graph G is 2-connected if $G - v$ is connected for any $v \in V(G)$ and $|V(G)| \geq 3$.

Theorem (Ear-decomposition) Every 2-connected graph G can be constructed from a cycle by successive addition of paths that have both endpoints in already constructed graph.

The paths in the construction are called **ears**, one edge is also an ear.

1: Construct a 3D cube and Petersen from cycle by adding ears.

Recall if G is 2-connected than for every two vertices u, v there exists a cycle in G containing both u and v.

2: Prove the ear decomposition theorem.

Vertex-cut U in a graph G is $U \subset V(G)$ such that $G - U$ is disconnected.

Vertex-connectivity (or connectivity) $\kappa(G)$ is the minimum cardinality of a vertex-cut or $n-1$ if $G = K_n$.

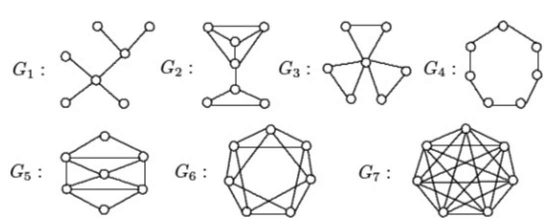
Graph G is k**-connected** if $\kappa(G) \geq k$.

Edge-cut X in a graph G is $X \subset E(G)$ such that $G - X$ is disconnected.

Edge-connectivity $\lambda(G)$ is the minimum cardinality of an edge-cut.

Graph G is k**-edge-connected** if $\kappa(G) \geq k$.

3: Find $\kappa(G)$ and $\lambda(G)$ for the following graphs.



4: Show that $\lambda(K_n) = n - 1$.

Recall that $\delta(G)$ is the minimum of the degrees of vertices in a graph G.

Theorem 5.11 For every graph G

$$\kappa(G) \leq \lambda(G) \leq \delta(G).$$

5: Show that for every graph $\lambda(G) \leq \delta(G)$.

6: Show that for every graph $\kappa(G) \leq \lambda(G)$.

7: Find a graph G such that $\kappa(G) < \lambda(G) < \delta(G)$.

8: Find $\kappa(G)$ and $\lambda(G)$ for the Petersen's graph.

Recall that cubic graph is a 3-regular graph, that is every vertex has degree 3.

Theorem 5.12 If G is a cubic graph, then $\kappa(G) = \lambda(G)$.

9: Show that if G is a cubic graph with $\kappa(G) = 1$ then $\lambda(G) = 1$.

10: Show that if G is a cubic graph with $\kappa(G) = 2$ then $\lambda(G) = 2$.

11: Prove Theorem 5.12. (use previous questions)

12: Show that if G is a graph of order n and size $m \geq n-1$, then $\kappa(G) \leq \left\lfloor \frac{2m}{n} \right\rfloor$.
Use Theorem 5.11 and compute average degree of G.

kth power G^k of a graph G is a graph where $V(G^k) = V(G)$ and $uv \in V(G^k)$ if $d_G(u,v) \leq k$.

G^2 and G^3 are called square and cube of G.

13: Draw P_5^2 and C_9^2.

We show that graph power can be computed by taking power of slightly modified adjacency matrix. It somehow justifies to call it a power.

14: Let A be the adjacency matrix of G. Let A' be obtained from A by adding 1 to every diagonal entry. Show that $uv \in E(G^k)$ iff $A^k_{u,v} \neq 0$.

15: Show that if G is connected then G^2 is 2-connected.

Reading for next time - Chapter 5.4.