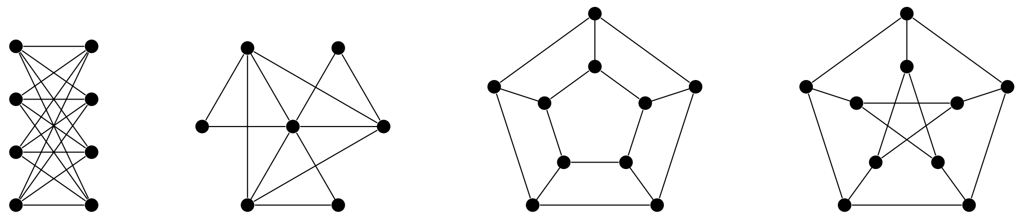
# Chapter 6.2 - Hamiltonian Graphs

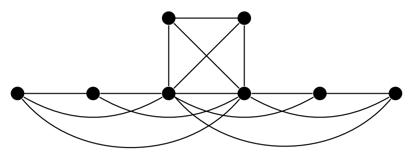
A graph $G$ on $n$ vertices is **Hamiltonian** if it contains a cycle of length $n$. The cycle is called **Hamiltonian cycle**. (Imagine you want to visit every vertex of a graph once. You don't care about edges.) A path containing all vertices is called **Hamiltonian path**.

**1:** Decide for the following graphs if they are Hamiltonian, have Hamiltonian path or nothing.



**2:** Is there a Hamiltonian graph that is not 2-connected? (i.e, connectivity is 1)

**3:** Is the following graph Hamiltonian? Why?



Recall that $k(G)$ denotes the number of connected components of a graph $G$.

**Theorem 6.5** If $G$ is a Hamiltonian graph, then for every nonempty proper set $S$ of vertices of $G$,

$$k(G - S) \leq |S|.$$

**4:** Prove Theorem 6.5

**Theorem 6.6** Let $G$ be a graph of order $n \geq 3$. If

$$\deg u + \deg v \geq n$$

for each pair $u, v$ of nonadjacent vertices of $G$, then $G$ is Hamiltonian.

*Proof* Fix $n$. Let $G$ be a counterexample maximizing the number of edges (why can we take it?). Notice $G \neq K_n$ so $G$ has $u, v$ nonadjacent vertices. By maximality of the number of edges, $G + uv$ contains a Hamilton cycle $C$ and $C$ contains edge $uv$. Then $C - uv$ is a Hamilton path $P$ with endpoints $u$ and $v$.

**5:** How to finish the proof? How to use neighbors of $u$ and $v$ and $P$ to find a different Hamilton cycle? Find construction or contradiction with $\deg u + \deg v \geq n$.

**6:** Show that Theorem 6.6 is sharp by finding a graph $G$ where for every pair of non-adjacent vertices $u$ and $v$ satisfy $deg(u) + deg(v) \geq n - 1$ but $G$ is not Hamiltonian.

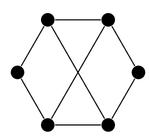
**Theorem 6.8** Let $u$ and $v$ be nonadjacent vertices in a graph $G$ of order $n$ such that $\deg u + \deg v \geq n$. Then $G + uv$ is Hamiltonian if and only if $G$ is Hamiltonian.

**7:** Prove Theorem 6.8. (see proof of Theorem 6.6)

The **closure** $C(G)$ of a graph $G$ of order $n$ is obtained from $G$ by adding edges between pairs of vertices $u$ and $v$ where $\deg u + \deg v \geq n$.

Theorem 6.8 implies that $G$ is Hamiltonian iff $C(G)$ is Hamiltonian.

**8:** Find closure of the following graph



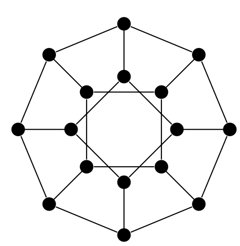
**Theorem 6.11** Let $G$ be a graph of order $n \geq 3$. If for every integer $j$ with $1 \leq j < \frac{n}{2}$, the number of vertices of $G$ with degree at most $j$ is less than $j$, then $G$ is Hamiltonian.

Proof: We show that the closure of $G$ is a complete graph. Suppose for contradiction that it is not. Let $u$ and $w$ be two non-adjacent vertices of the closure $C(G)$ such that $\deg_{C(G)} u + \deg_{C(G)} w$ is as large as possible. Let $k = \deg_{C(G)} u \leq \deg_{C(G)} w$.

**9:** Finish the proof. Show $k < \frac{n}{2}$. Consider $W$ the set of vertices non-adjacent to $w$ and look at $|W|$.

**10:** *6.13* Give an example of a graph $G$ that is
(a) Eulerian but not Hamiltonian.
(b) Hamiltonian but not Eulerian.
(c) Hamiltonian and has an Eulerian trail but is not Eulerian.
(d) neither Eulerian nor Hamiltonian, but has an Eulerian trail.

**11:** Is the following graph Hamiltonian?



**12:** *6.21* Let $G$ be a graph of order $n \geq 3$ such that $\deg u + \deg v \geq n - 1$ for every two nonadjacent vertices $u$ and $v$. Prove that $G$ must contain a Hamiltonian path.

**13:** For $n \geq 2$, prove by induction on $n$ that the maximum number of edges in a simple non-Hamiltonian $n$-vertex graph is $\binom{n-1}{2} + 1$.

**14:** Let $G$ be a graph that is not a forest and the shortest cycle in $G$ has length at least 5. Prove that $\overline{G}$ is Hamiltonian.