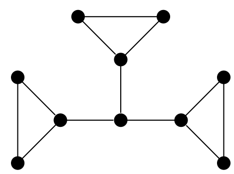
Chapter 8.2 - Factorization I

1-factor of a graph G is a spanning 1-regular subgraph (same as perfect matching)

k-factor of a graph G is a spanning k-regular subgraph

1: Decide if the following graph has a 1-factor



A (connected) component of a graph is **odd** or **even** according to its number of vertices. Denote the number of odd components of a graph G by $k_o(G)$.

2: Let G be a graph and $S \subseteq V(G)$. Show that if G has a 1-factor then for every $S \subseteq V(G)$ holds $k_o(G - S) \leq |S|$.

Theorem 8.10 (Tutte) A graph G contains a 1-factor iff $k_o(G - S) \leq |S|$ for every $S \subseteq V(G)$.

Proof. We only need to show $\Leftarrow$. We proceed by induction on the order n of graph G. Base case could be $n = 1, n = 2$. Now induction step.

3: Is it true that n is even? Is there a proper subset $S \subseteq V(G)$ such that $k_o(G - S) = |S|$? (that is, $S \neq \emptyset$)

Let S be the maximum size set such that $k_o(G - S) = |S| = k$. Let $G_1, \ldots, G_l$ be components of $G - S$.

4: Show that all components $G_1, \ldots, G_l$ are odd. That gives $k = l$.

Now our plan is to match one vertex from every G_i with S. We create an auxiliary bipartite graph H, where one part of the bipartition are vertices from S and in the other part there are k vertices where each vertex i corresponds to one G_i. Then H contains all edges is, where $s \in S$ has a neighbor in G_i. We want to find a perfect matching in H.

5: Show that H satisfies Hall's condition for the side corresponding to graphs G_is.

Let M be a perfect matching in H. The perfect matching M corresponds to a matching M_S, where every vertex $s \in S$ is matched to some vertex v_i in G_i.

Now in order to finish the perfect matching in G, we need to match vertices in $G_i - v_i$ to each other. We try to use induction on $G_i - v_i$.

Let $W \subseteq G_i - v_i$. Since $G_i - v_i$ has even order, both W and $k_o(G_i - v_i - W)$ have the same parity.

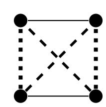
6: Show that $k_o(G_i - v_i - W) \le |W|$ to verify the induction hypothesis. (Hint: try combine W with S).

7: Let G be a bridgeless 3-regular graph. Let $S \subseteq V(G)$ and let H be an odd component of $G - S$. Is it possible that there are at most two edges uv, where $u \in S$ and $v \in V(H)$?

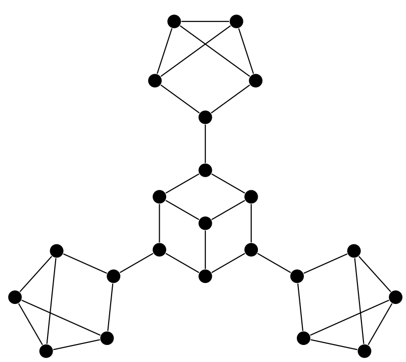
Theorem 8.11(Petersen's Theorem) Every 3-regular bridgeless graph contains a 1-factor.

8: Prove Theorem 8.11 by verifying Tutte's condition from Theorem 8.10.

A graph G is **1-factorable** if its edges can be decomposed into 1-factors.
Decomposition of K_4 into three 1-factors:



9: Determine which of the cubic graphs has a 1-factor and which is 1-factorable.



10: Find a cubic bridgeless graph G and two edges e and f that do not share any vertex such that there is no perfect matching M in G where $e \notin M$ and $f \notin M$.

11: Is Peterson's graph 1-factorable?

12: Let G be a bridgeless cubic graph. Let e be any edge of G. Show that there exists a perfect matching M in G such that $e \notin G$.

13: Let G be a bridgeless cubic graph. Let e be any edge of G. Show that there exists a perfect matching M in G such that $e \in G$.