Chapter 8.2 - Factorization II

Recall k-**factor** of a graph G is a spanning k-regular subgraph

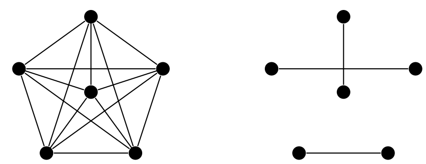
A graph G is k-**factorable** if its edges can be decomposed into k-factors.

Theorem 8.15 Every r-regular bipartite graph, $r \geq 1$, is 1-factorable.

1: Prove Theorem 8.15 by induction on r.

Theorem 8.14 K_{2k} is 1-factorable.

2: Find 1-factorization of K_6. Use the following 1-factor F_1 to find the 1-factorization.



K_6 F_1

3: Generalize the 1-factorization of K_6 to factorization of any K_{2k} for $k \geq 3$.

Theorem 8.16 A graph G is 2-factorable if and only if G is r-regular for some positive even integer r.

4: Show that if G is 2-factorable, then G is r-regular, where r is some even integer.

Now we plan to prove the other direction of Theorem 8.16. Let G be a $2k$-regular graph on n vertices for some integer k. We want to show G has a 2-factor and then use induction like in 8.15.

Notice that G is Eulerian so it has an Eulerian trail T. Remember, edges do not repeat but vertices may repeat.

Let G have vertices $v_1, \ldots, v_n$. Create a bipartite graph H, whose vertex set $U = \{u_1, \ldots, u_n\}$ and $W = \{w_1, \ldots, w_n\}$ and u_i, w_j is an edge if $v_i v_j$ appear in this order next to each other on the trail T.

5: Suppose $G = K_5$. Find some Eulerian trail T in G and construct the corresponding graph H.

6: Let F_H be a 1-factor in H. What is correspondence of edges in F_H and edges in G and why? (Use K_5.)

This way we managed to find one 2-factor F in G. Since $G - F$ is $(2k-2)$-regular, we can use induction. □

Spanning subgraph of G is called a **factor**. A graph is **factorable** into factors $F_1, F_2, \ldots, F_k$ if edges of the factors form a partition of edges of G. If all factors are isomorphic to F, then G is F**-factorable.**

7: Let $3K_3$ be a disjoint union of three triangles. Show that K_9 is $3K_3$-factorable.

8: Show that K_{2k} can be factorized into $k-1$ 2-factors and one 1-factor. (Hint: use that K_{2k} is 1-factorable.)

9: Let G be a connected graph on at least 4 vertices such that every edge of G belongs to a 1-factor in G. Show that G is 2-connected

10: Prove that if the bridges of a 3-regular graph G lie on a single path, then G has a 1-factor.

11: Is there a 2-factorization of K_7 in which no 2-factor is a Hamiltonian cycle?

12: Show that K_{2k} can be factorized into $k-1$ Hamilton cycles and one 1-factor. (Hint: construction like where K_{2k} is 1-factorable.)

13: Find a multiset of cycles in Petersen's graph such that every edge is in exactly two of the cycles.

14: *Open* Let G be a bridgeless cubic graph. Is there a multiset of cycles in G such that every edge is in exactly two of the cycles. (cycle double cover conjecture)