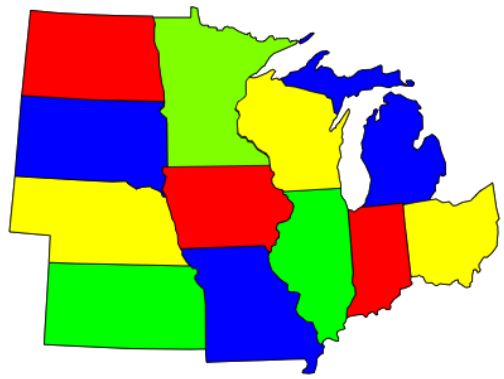
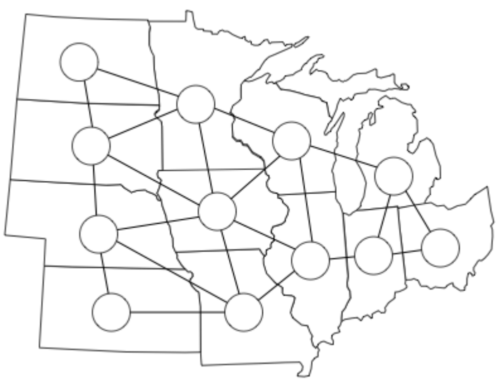
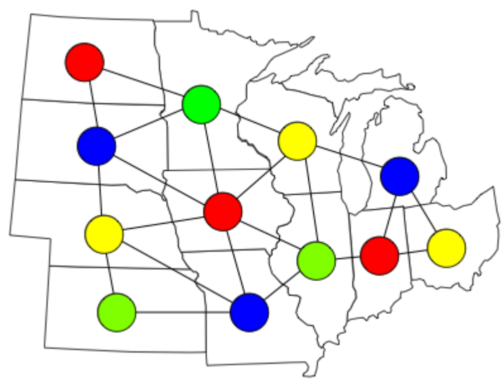
Chapter 10.2 - Graphs Coloring

Problem: Color regions of the plane such that regions sharing border get different colors. Show that 4 colors is enough (if regions connected) for any set of regions. (Restating: Is it true that every planar graph is 4-colorable? Answer is yes.)



The problem can be turned into a graph problem by having a vertex for every region.





Let G be a graph and C be a set of colors. **Coloring** is a mapping $c : V(G) \to C$ such that $c(u) \neq c(v)$ for all $uv \in E(G)$. Sometimes called **proper coloring**.

A graph G is k**-colorable** is there exists a (proper) coloring of G using k colors.

Chromatic number of G, denoted by $\chi(G)$ is the minimum k such that G is k-colorable.

1: Decide what is the chromatic number of C_k. (try $3 \leq k \leq 7$)

Let c be a (proper) coloring of G. If V_{red} is the set of vertices colored red then V_{red} is an independent set.

Coloring G by k colors is a decomposition of $V(G)$ into in k independent sets. $V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$.

2: Show that $\chi(G) = 2$ iff G is bipartite (and has at least one edge).

3: What is $\chi(K_n)$?

A **clique** in a graph G is a subgraph that is isomorphic to a complete graph.
The **clique number**, $\omega(G)$, is the order of the largest clique in G.

Recall $\Delta(G)$ is the maximum degree of a vertex in G.

4: Show that for every graph G holds $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$.

Theorem 10.8 Brook's theorem. Let G be a connected graph, that is not a complete graph or an odd cycle. Then $\chi(G) \leq \Delta(G)$.

Notice that Brook's theorem tells us that $\chi(G) \leq \Delta(G) + 1$ holds with equality iff G is a complete graph or an odd cycle.

Let's prove Brook's theorem. Let G be a connected graph that is not complete or an odd cycle. Also assume that we have proved the theorem for all smaller graphs (we are proving by induction on the number of vertices). Let $\Delta(G) = \Delta$.

5: Solve the case that $\Delta = 2$

So we assume $\Delta \geq 3$.

6: Show that G is 2-connected. (use induction)

7: Prove the case where G has a vertex v of degree less than Δ (greedy coloring)

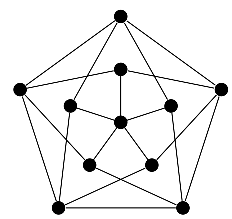
Now we assume G is Δ-regular. We still want to use greedy coloring, but guarantee that the last vertex has 2 neighbors with the same color.

8: Assume that there is a vertex v such that $G - v$ is 2-connected. Prove Brook's Theorem.

9: Assume that there is a vertex v such that $G - v$ is not 2-connected. Prove Brook's Theorem. Notice that $G - v$ is still connected. Consider block decomposition of $G - v$ and see where are neighbors of v.

Question: Do you need a large clique for a large chromatic number?

10: What is the chromatic number of the Grötzsch's graph? Notice it is triangle-free.



Theorem (Erdős) For every k, l there exists a graph of girth l and chromatic number k.

The proof is probabilistic and we skip it.

Mycielski construction is a construction to create a triangle free graph of arbitrary chromatic number.

Start with a graph G, duplicate every vertex and connect new vertex to the duplicates.

11: Apply the Mycielski operation on C_5.

12: Show that the Mycielski construction is increasing the chromatic number.

13: Determine $\chi(G)$ where G is the Petersen graph.

14: Show that $\chi(G)\alpha(G) \geq |V|$ (recall $\alpha(G)$ is the independent number). IMPORTANT!

15: Show that $\chi(G+H) = \chi(G)+\chi(H)$ (recall $G+H$ is the join of G and H, between G and H is a complete bipartite graph).

16: Show that $\chi(G\Box H) \geq \max\{\chi(G), \chi(H)\}$ (recall $G\Box H$ is the Cartesian product of G and H).

17: Show that every planar graph is 6-colorable.

18: Show that every outer-planar graph is 3-colorable.

A graph G is k-**critical** if $\chi(G) = k$ but $\chi(H) < k$ for every proper subgraph H of G. One can think that G is minimal graph with $\chi(G) = k$.

19: Show that the set of all 3-critical graphs is equal to the set of all odd cycles.

20: Prove that Brooks' Theorem for $\Delta \geq 3$ is equivalent to the following statement: If $G = (V, E)$ is k-critical for $k \geq 4$ and G is not complete, then $|E| \geq (|V|(k-1)+1)/2$.

21: *Open* Reed's conjecture: $\chi(G) \leq \lceil(\omega(G) + \Delta(G) + 1)/2\rceil$