Coloring Planar Graphs

1: Let G be a planar graph. Show that $\delta(G) \leq 5$. Recall that $\delta(G)$ is the minimum degree of G.

2: Show that every planar graph is 6-colorable.

Theorem Every planar graph is 5-colorable.

We do the proof by induction on the number of vertices. We use Kempe chains.

Let c be a coloring of G. A **Kempe chain** in colors 1 and 2 is a maximal connected subgraph of G where all vertices are colored 1 and 2.

3: Let $K \subseteq V(G)$ be a Kempe chain in G for a coloring c. Let c' be obtained from c by swapping colors 1 and 2 on K but nowhere else.

$$c'(v) = \begin{cases} 1 & \text{if } v \in K \text{ and } c(v) = 2 \\ 2 & \text{if } v \in K \text{ and } c(v) = 1 \\ c(v) & \text{otherwise} \end{cases}$$

Show that c' is a proper coloring.

Proof of the 5-color theorem. Let G be a plane graph on n vertices. Assume that all planar graphs on at most $n-1$ vertices are 5-colorable.

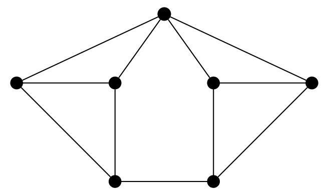
4: Show that G is 5-colorable if it contains a vertex v of degree at most four.

Let v be a vertex of degree 5 in G. By induction, there is a 5-coloring c of $G - v$. If the neighbors of v use at most 4 colors in c, the coloring c can be extended to v. Hence assume that the neighbors of v are colored by $1, 2, 3, 4, 5$ (in clockwise order in the drawing on G).

5: Use Kempe chain in colors 1 and 3 and another one in colors 2 and 4 to show that there exists a coloring c' of $G - v$ such that the neighbors of G have at most 4 colors.

A graph G is k**-critical** if $\chi(G) = k$ but $\chi(H) < k$ for every proper subgraph H of G. One can think that G is minimal graph with $\chi(G) = k$.

6: Show that the following graph (called Moser's spindle) is 4-critical.



The set of k-critical graphs is completely known only for $k = 3$. Notice that knowing this set as algorithmic implications.

7: Show that the set of all 3-critical graphs is equal to the set of all odd cycles.

Theorem Kostochka-Yancey: If G is a 4-critical graph, then

$$|E(G)| \geq \frac{5|V(G)| - 2}{3}.$$

Theorem Grötzsch: Every triangle-free planar graph is 3-colorable.

Let G be a minimal counterexample. That is, triangle-free *plane* graph that is not 3-colorable. By the minimality, it is 4-critical and every smaller triangle-free planar graph is 3-colorable.

8: Suppose G contains a 4-face. Try to identify opposite vertices of the 4-face and argue that G is 3-colorable.

9: Suppose G has no 4-faces. Use Euler's formula to give an upper bound on the number of edges in G.

10: Suppose G has no 4-faces. Use the bound from the previous question together with Kostochka-Yancey and get a contradiction.

11: Let G be a graph with $\chi(G) = k$. Let c be a k-coloring of G by colors $C = \{1, \ldots, k\}$ Show that for every color $a \in C$, there is a vertex v in $V(G)$ such that the neighbors of v are colored by all $C \setminus a$ colors.

12: Show that Moser's spindle can be draw in the plane with crossings such that every edge is a straight line and all edges have the same length.

13: *Bonus* Let there be 3 teams playing each other n times. In a match, winning team gets 3 points, looser 0 points. If there is a tie, both teams get 1 point. Is it possible that there is a sequence of matches, where the team that has most points has least number of wins and the team that has least number of points has most number of wins?

14: *Open* Let G be a graph whose vertices are points in the plane $\mathbb{R}^2$ and two vertices are adjacent iff their Euclidean distance is 1. The graph G has infinitely many vertices and it is not planar. What is $\chi(G)$?