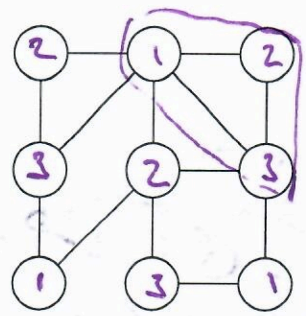
Lecture 1 - Proper Coloring

Let G be a graph and C be a set of colors. A *(proper) coloring* is a mapping $\varphi : V(G) \to C$ such that for every $uv \in E(G)$ holds $\varphi(u) \neq \varphi(v)$.

$C = \{1,2\}$

- If $|C| = k$, then G is *k-colorable.*
- Smallest k such that G is k-colorable is called the *chromatic number*, denoted by $\chi(G)$.
- If $\chi(G) = k$, then we say G is a *k-chromatic.*
- In a coloring φ, vertices colored with the same color are *color class.*

1: Determine the chromatic number of the following graph.

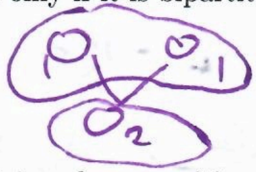


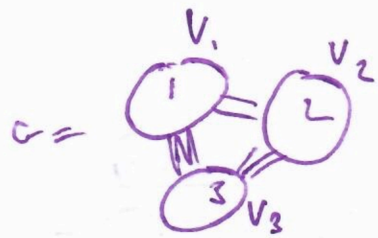
3

$\chi(G) = 3$

2: Show that

- Graph is 1-colorable if and only if it is edgeless.
- Graph is 2-colorable if and only if it is bipartite.



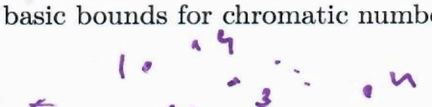


Note: A k-coloring of $G = (V, E)$ is a decomposition V as $V_1 \cup \cdots \cup V_k$ into independent sets.

Deciding 3-colorability is an NP-complete problem. That means no efficient algorithm. Thus giving upper and lower bound of chromatic number for various classes of graphs is valuable.

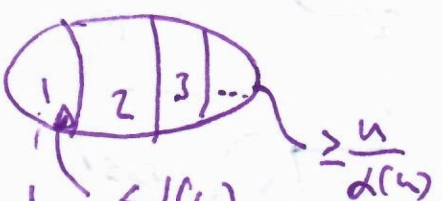
3: Show these basic bounds for chromatic number.

(a) $\chi(G) \leq n$.



(b) If H is a subgraph of G, then $\chi(H) \leq \chi(G)$. In particular, $\omega(G) \leq \chi(G)$.

(c) $\chi(G) \geq n/\alpha(G)$.



$\geq \frac{n}{\alpha(G)}$

$\leq \alpha(G)$

G

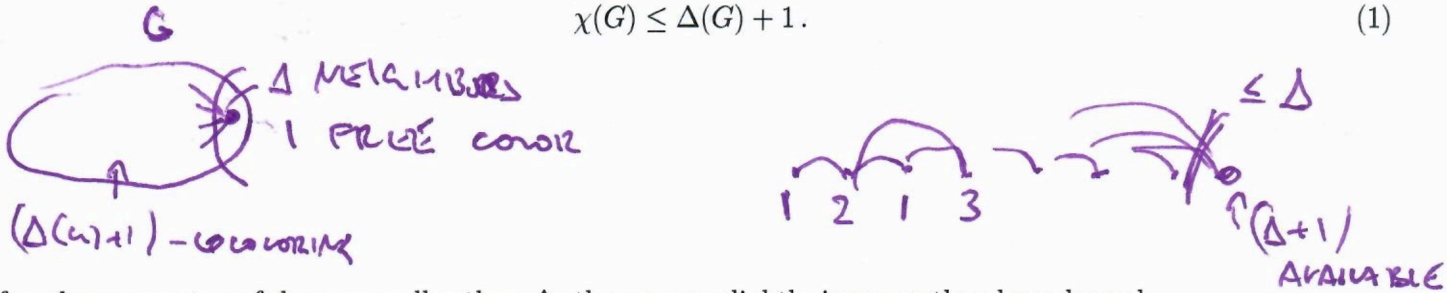
$\chi(G)$

$\omega = 4$

1 Bounds by Δ

4: Show that for any graph G,

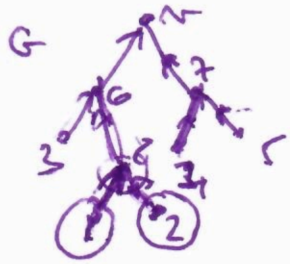
$$\chi(G) \leq \Delta(G) + 1. \tag{1}$$



If we have a vertex of degree smaller than Δ, the we can slightly improve the above bound.

5: Show that if graph G (Δ CONNECTED) contains a vertex v of degree $d(v) < \Delta(G)$. Then

$$\chi(G) \leq \Delta(G). \tag{2}$$



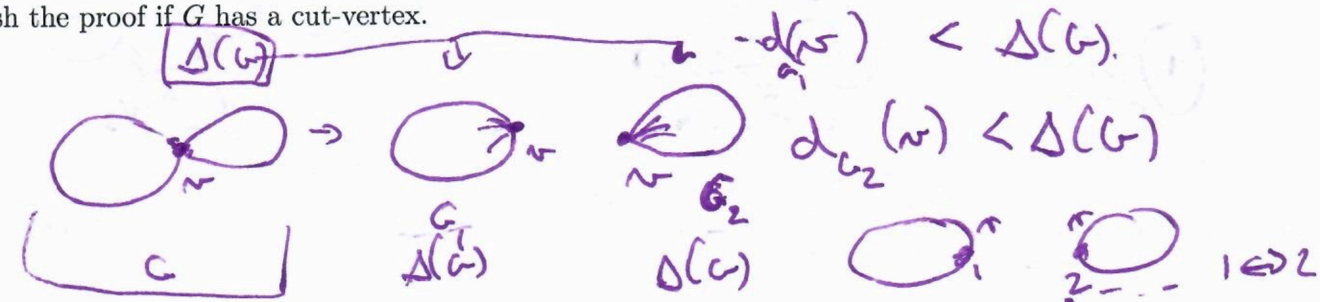
Notice that in (1), we have equality for odd cycles and complete graphs. The classical theorem of Brooks assures that these is no other such graph.

Theorem 1 (Brooks). *If G is not an odd cycle nor complete graph, then*

$$\chi(G) \leq \Delta(G).$$

Proof. Let $\Delta = \Delta(G)$. The claim obviously holds for $\Delta \leq 2$, so assume in what follows that $\Delta \geq 3$.

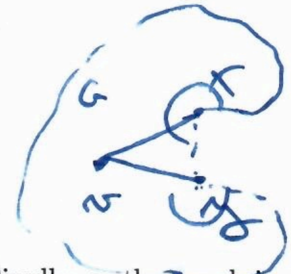
6: Finish the proof if G has a cut-vertex.

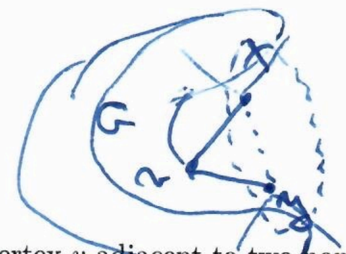


Suppose now G has a 2-cut $\{u, v\}$ such that u and v are not adjacent. Define G_1 and G_2, whose union is G and intersection is comprised of vertices u and v. Note that these vertices are of degree $\leq \Delta - 1$.

7: Assume that for each G_1 and G_2 exists $x_i \in \{u, v\}$ such that $d_{G_i}(x) \leq \Delta - 2$.

8: So we may assume that in one of these graphs, say G_1, vertices u and v are of degree precisely $\Delta - 1$. Finish the proof.





Finally, as the graph is not complete we have a vertex v adjacent to two non-adjacent vertices x and y. Above arguments assure that $G - x - y$ is connected.

9: Finish the proof by cleverly ordering the vertices and using greedy coloring.

USE Δ COLORS

$\square$

1.1 Two conjectures

Most trivial bounds are $\omega(G) \le \chi(G)$ and $\chi(G) \le \Delta(G) + 1$.

ODD CYCLE CHECK

$\left\lceil \frac{2+2+1}{2} \right\rceil = 3 = \chi(C_5)$

Conjecture 2 (Reed). *For every graph G, it holds*

$$\chi(G) \le \left\lceil \frac{\omega(G) + \Delta(G) + 1}{2} \right\rceil .$$

To support his conjecture, Reed proved that there exists $\varepsilon > 0$ such that for every graph G,

$$\chi(G) \le \varepsilon\omega(G) + (1 - \varepsilon)(\Delta(G) + 1).$$

$\varepsilon = \frac{1}{2}$

A significant improvement to $\varepsilon = \frac{1}{13}$ was obtained by Delcourt and Postle.

$\frac{1}{13} \varepsilon\omega + \frac{12}{13} (\Delta(G)+1)$

10: Prove the conjecture for $\chi(G) = \Delta(G) + 1$

USE BROOKS THM $\Rightarrow \omega(G) = \Delta(G)+1$

In case of $\chi(G) = \Delta(G)$, the conjecture claims that $\omega(G) \in [\Delta(G) - 2, \Delta(G)]$. This is still open, but for large $\Delta(G)$, Reed proved that $\omega(G) = \Delta(G)$. The next conjecture of Borodin and Kostochka claims that this holds whenever $\Delta(G) \ge 9$. And, it is known for each smaller $\Delta(G)$ that it does not hold.

Conjecture 3 (Borodin and Kostochka). *Let G be a graph with $\chi(G) = \Delta(G) \ge 9$. Then,*

$$\omega(G) = \Delta(G) .$$

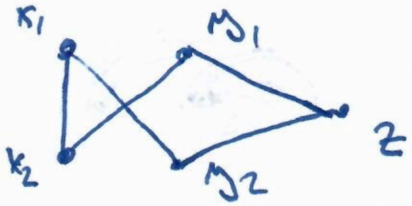
A weakening by Cranston and Rabern is that that if $\chi(G) = \Delta(G) \ge 13$, then $\omega(G) \ge \Delta(G) - 3$.

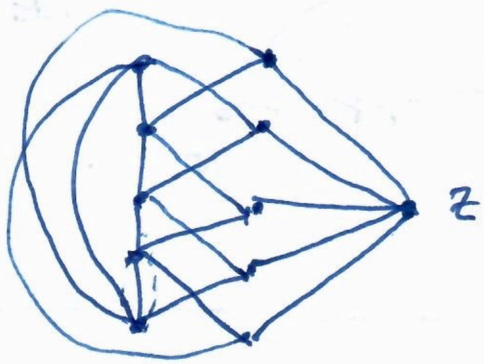
1.2 Triangle-free graphs with large chromatic number

Let G be a graph with $V(G) = \{x_1, x_2, \ldots, x_n\}$. *Mycielski graph* $M(G)$ for G is the graph defined in the following way:

- $V(M(G)) = V(G) \cup \{y_1, y_2, \ldots, y_n\} \cup \{z\}$, and
- $E(M(G)) = E(G) \cup \{y_i x_j ; x_i x_j \in E(G)\} \cup \{zy_1, zy_2, \ldots, zy_n\}$.

11: Find $M(K_2)$ and $M(C_5)$.

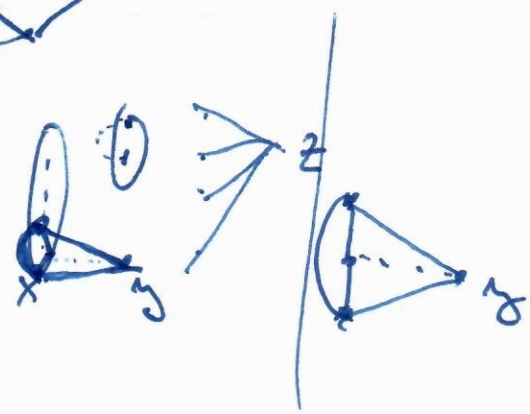




K_2 ..

$M(K_2) = C_5$

$\chi(K_2) + 1 = \chi(C_5)$



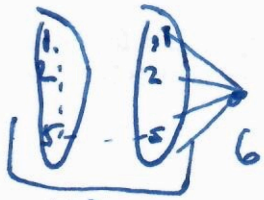
The following property is obvious.

Observation 4. *If G is traingle-free, then $M(G)$ is triangle-free.*

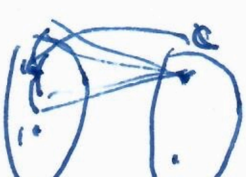
Proposition 5. *For every graph G, it holds*

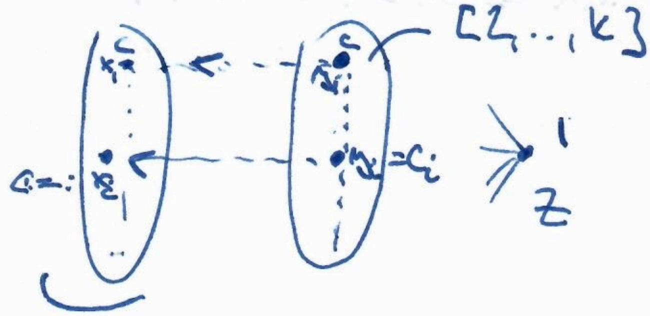
$$\chi(M(G)) = \chi(G) + 1\,.$$

12: Prove the proposition. First find $(\chi(G)+1)$-coloring of $M(G)$. Then assume $M(G)$ has $\chi(G)$-coloring and find a contradiction.



$\chi(G)$-coloring





$\in [2..k]$ $\chi(G)$-coloring

$\chi(G) = k$

Theorem 6. *For every k, there is a triangle-free k-chromatic number.*

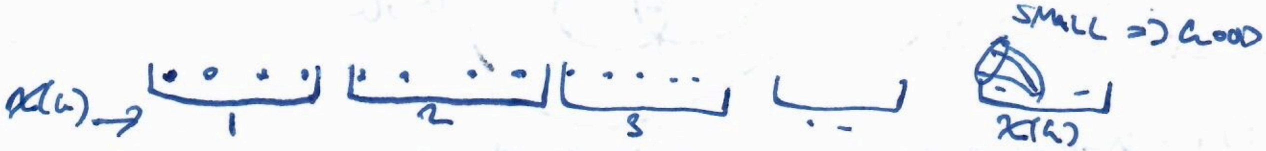
13: Prove the proposition.

$M(M(M(M(M(K_2)))))$

1.3 Degenericity

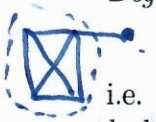
A greedy coloring of a graph G is a coloring of vertices one by one in some order and always using the smallest available color from colors $1, 2, 3, \ldots$.

14: Show that there exists an ordering of vertices of G, such that greedy coloring uses only $\chi(G)$ colors.



One ordering that is considered to be "good" (although does not always arise an optimal coloring) and that can be easily constructed is related with the graph degenericity.

Degenericity $d(G)$ of a graf G is



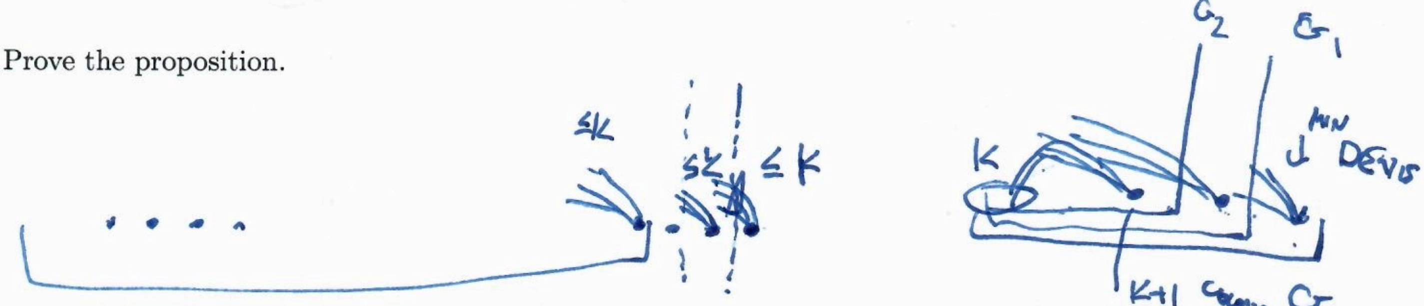
$$d(G) = \max_{H \subseteq G} \delta(H),$$

i.e. maximum of the minimum degrees of its subgraphs. And, if $d(G) \leq k$, then we say that the graph G is *k-degenerated.*

$n.\ d(v) \leq d(G)$

Proposition 7. *If G is k-degenerated, then we can order its vertices $v_1, v_2, \ldots, v_n$ such that each vertex v_i has at most k neighbors with smaller index (i.e. among $v_1, v_2, \ldots, v_{i-1}$).*

15: Prove the proposition.



Now, the greedy coloring applied on the ordering from the proposition above gives the following result.

Proposition 8. *Every k-degenerated graph is $(k+1)$-colorable.*

Because of this, the value $d(G)+1$ is known as the *coloring number* of a graph G, and it is denoted by col(G). Thus, $\chi(G) \leq \text{col}(G)$. Notice that it is easy to calculate col(G) unlike $\chi(G)$. Another classical result is the Nordhaus–Gaddum Theorem. We will use degenericty in its proof.

Theorem 9 (Nordhaus and Gaddum). *For a graph G on n vertices, and its complement $\bar{G}$, the following statements hold:*

(a) $\lceil 2\sqrt{n} \rceil \overset{1}{\leq} \chi(G) + \chi(\bar{G}) \overset{2}{\leq} n + 1;$

(b) $n \underset{3}{\leq} \chi(G) \cdot \chi(\bar{G}) \underset{4}{\leq} \lfloor (\frac{n+1}{2})^2 \rfloor.$

Proof. Suppose that c and $\bar{c}$ are optimal colorings of G and $\bar{G}$. By assigning the pair $(c(u), \bar{c}(u))$ to every vertex u, we obtain a proper coloring of K_n.

16: Show this coloring is proving one if the inequalities.

2 ...

4 ...

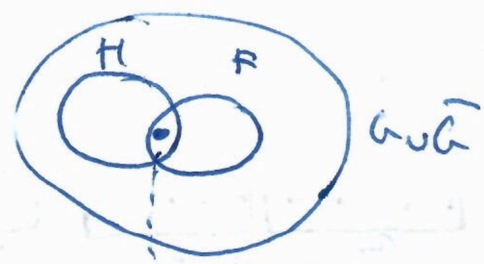
3 .. $n = \chi(K_n) \leq \chi(G) . \chi(\bar{G})$

DEGENERACY
↓

17: Suppose that $d(G) = k$. Show that $d(\bar{G}) \le n - k - 1$.
Hint: One option is to suppose otherwise for contradiction and count the number of vertices.

FOR CONTRADICTION:

- $\exists\, H \subseteq \bar{G}$ S.T. $\delta(H) \ge n-k$
- $\exists\, F \subseteq G$ S.T. $\delta(F) \ge k$



$|G| \ge |V(H)| + |V(F)| = n-k+1+k+1 = n+2$ $\quad \underline{(n-k)+k} = n$

$|V_G| = n$

18: Finish the proof of the theorem.

2 $\quad \chi(G) + \chi(\bar{G}) \le col(G) + col(\bar{G}) \le k+1+n-k+1-1 =$

$= n+1$

□

Finale notes. For the end, let us mention, another interesting strengthening of the inequality (1) is the following theorem of Hajnal and Szemerédi from 1970.

Theorem 10 (Hajnal and Szemerédi). *A graph G can be colored with $\Delta(G)+1$ colors such that all color classes are of equal or almost equal size.*

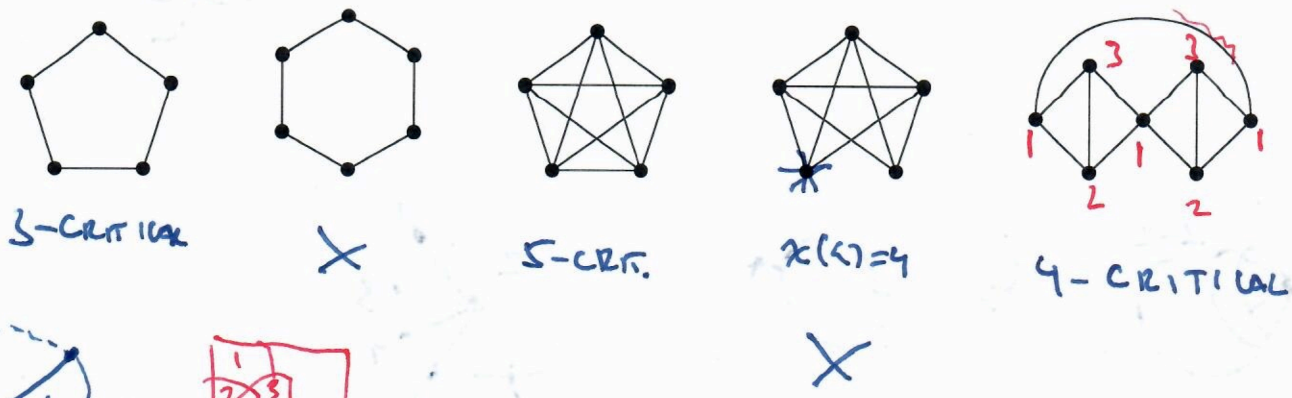
This theorem initiated the study of so called *equitable* colorings that require color classes to be of almost same size. The proof of this theorem is not simple enough to be considered here.

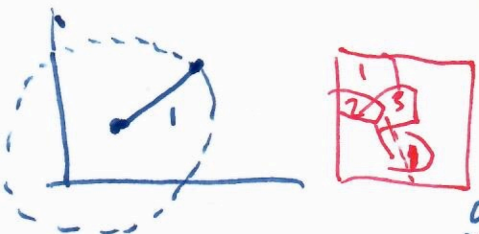
It is naturally to consider a Brook's-type theorem for triangle free graphs, i.e. finding as good as possible upper bound of χ in term of Δ for triangle free graphs. One can go even more general asking about K_r-free graphs. Interested reader can find more about this topic in the book of Jensen and Toft.

Lecture 2 - Critical Graphs

A graph G is k-*critical* if $\chi(G) = k$ but every proper subgraph of G is $(k-1)$-colorable.
Note: Sometimes there are critical graphs also for other properties.

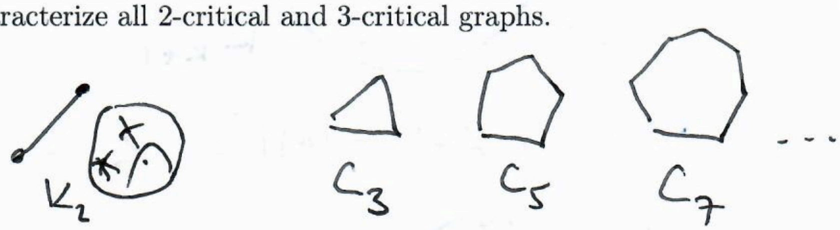
1: Are the depicted graph critical for some k?





2: Show that every k-chromatic GRAPH contains a subgraph that is k-critical.

3: Characterize all 2-critical and 3-critical graphs.



No good characterization of k-critical for $k \geq 4$ is known. Knowing the k-critical graphs helps a lot with deciding if a graph is $(k-1)$-colorable. For particular classes of graphs, it is worth it to be able to enumerate k-critical graphs if possible.

We will see below, that a critical graph has minimum degree $k-1$. Here is an interesting special case.

4: Show that if all vertices of a k-critical graph are of degree $k-1$, then the graph is K_k, or $k = 3$ and it is an odd cycle.
Hint: Recall previous lecture.

$\Delta(G) = k-1$
$\chi(G) = k$ $\underset{\text{BROOKS}}{\Longrightarrow}$ G IS K_k OR $k=3$ & G IS $C_{2\ell+1}$

Observation 1. *The following hold:*

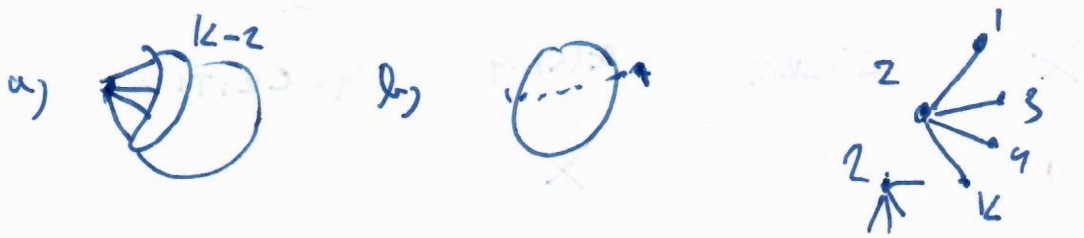
(a) *The minimum degree of a k-critical graphs is $\geq k-1$.*

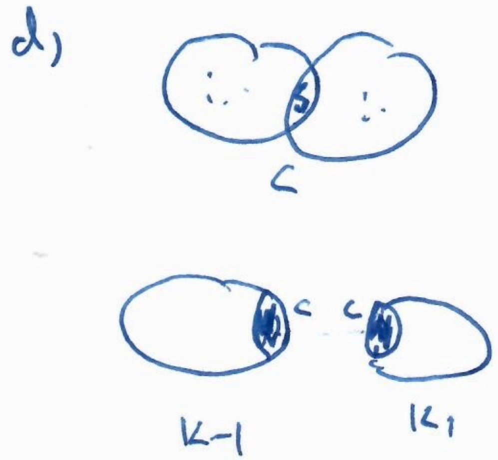
(b) *K_k is the smallest k-critical graph.*

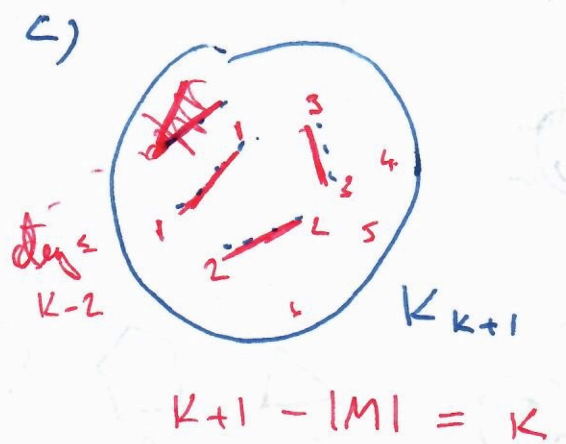
(c) *There is no k-critical graph on $k+1$ vertices.*

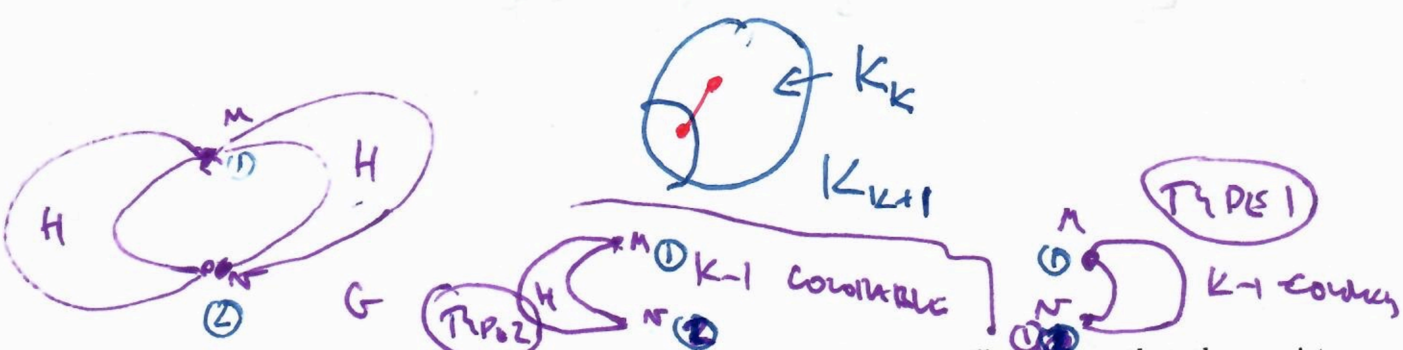
(d) *No k-critical graph has a clique-cut. In particular, for $k \geq 3$, they are 2-connected.*

5: Prove Observation 1. Recall that a clique-cut in a graph G is a subset of vertices C, such that $G[C]$ is a clique and $G - C$ is not connected.









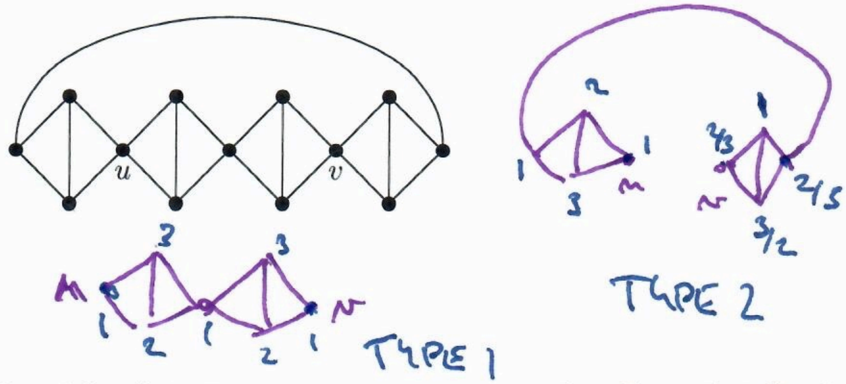
Regarding the last observation, the 2-connectivity cannot be increased as we will see soon that there exist critical graphs of connectivity two, and Dirac characterized them. Next we state his result but leave its proof for an exercise to the reader.

Let G be a graph of vertex-connectivity 2, and $\{u, v\}$ its 2-cut, and fix some integer k. A $\{u, v\}$-*component* of G is a subgraph H induced by the vertices of a component C of $G - u - v$ union $\{u, v\}$, i.e. $H = G[V(C) \cup \{u, v\}]$. A $\{u, v\}$-component G_1 of G is of *type 1* if it is $(k-1)$-colorable and every $(k-1)$-coloring assigns a same color to u and v, and a $\{u, v\}$-component G_2 of G is of *type 2* if it is $(k-1)$-colorable and every $(k-1)$-coloring assigns distinct colors to u and v.

6: Determine which components are of which type for u and v.

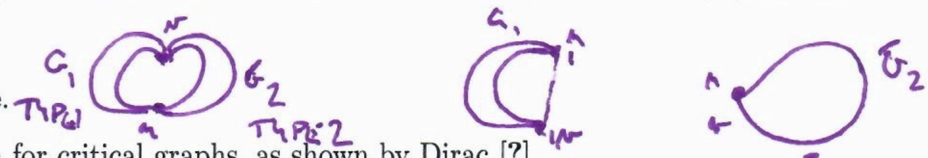
4-CRITICAL

2-COLORING



Theorem 2 (Dirac)**.** *Let G be a graph and $\{u,v\}$ its 2-vertex-cut, $e = uv$ a new edge. Prove that G is k-critical if and only there are two $\{u,v\}$-components G_1 and G_2 such that $G = G_1 \cup G_2$, each G_i is of type i, and $G_1 + e$ and G_2/e are k-critical.*

Proof will be as an in-home exercise.

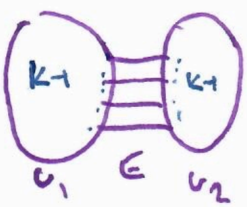


Edge-connectivity turned to be high for critical graphs, as shown by Dirac [?].

Theorem 3 (Dirac)**.** *For $k \geq 2$, every k-critical graph is $(k-1)$-edge-connected.*

Proof. Suppose that G is a k-critical graph and $E = [U_1, U_2]$ is an edge-cut of size $\leq k-2$. We will obtain a contradiction by $(k-1)$-coloring of G. We can $(k-1)$-color both components of $G - E$ independently but to obtain a coloring of G, we may need to permute the colors on one side (in order to avoid monochromatic edges of E). We will achieve this as it is described bellow by the use of Hall's theorem: *A balanced bipartite graph with bipartition A, B has a perfect matching if only if for every subset $S \subseteq A$ holds $|N(S)| \geq |S|$, i.e., S has as many neighbours in B as its size is.*

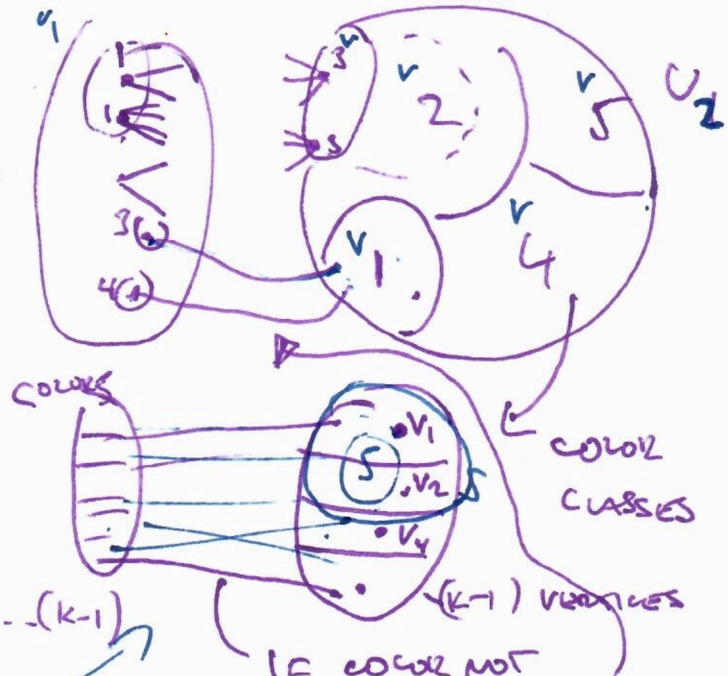
7: Finish the proof by creating creating an auxiliary bipartite graph with vertices on one side and available colors on the other. Verify Hall's condition.



$|E| \leq k-2$

FIND (k-1)COLORING & PERMUTE

FIX COLORING (k-1) OF U_1



GOAL

$(k-1)|S| - (k-2) \leq |E_S| \leq N(S)(k-1)$

$|S| - \frac{k-2}{k-1} \leq |N(S)|$

< 1

1 .. (k-1)

IF COLOR NOT FORBIDDEN BY

$\forall$ (k-1)(k-1) MISSING AT MOST (k-2) EDGES

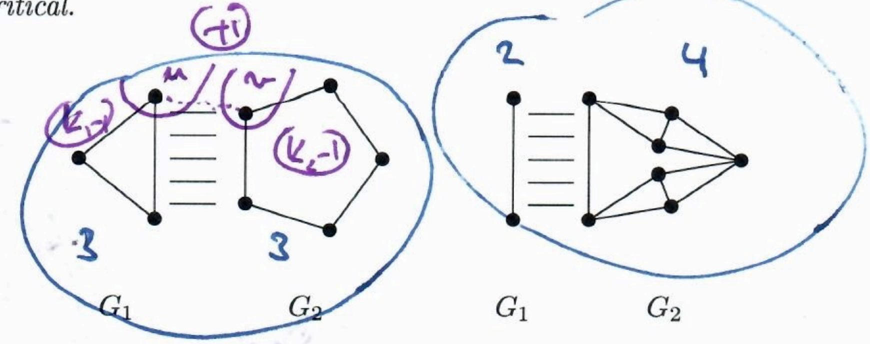
□

Lecture 2 - Critical Graphs II

1 Construction of critical graphs

Proposition 1 (Dirac construction). *Let G_1 be a k_1-critical graph and G_2 a k_2-critical graph. Then, the join $G_1 + G_2$ is $(k_1 + k_2)$-critical.*

6-CRITICAL



CRITICAL

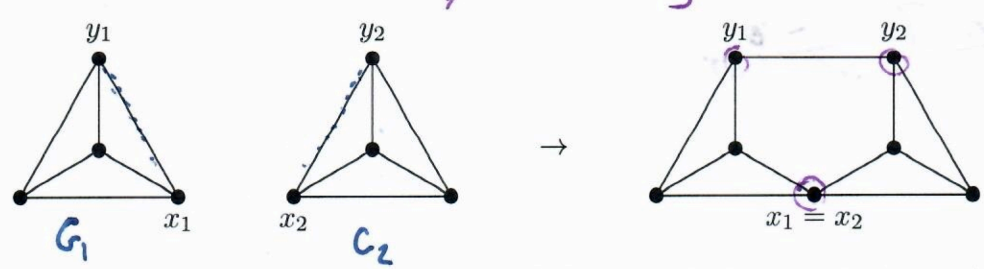
Two 6-critical graphs embeddable in the Klein bottle obtained using Dirac's construction.

Proposition 2 (Mycielski construction). *If G is a k-critical graph, then Mycielski graph $M(G)$ is $(k+1)$-critical.*

Proposition 3 (Hajós construction). *Let G_1 and G_2 be disjoint k-critical graphs, and $x_i y_i$ an edge in G_i for $i = 1, 2$. Then, the graph H obtained by the following three steps is k-critical:*

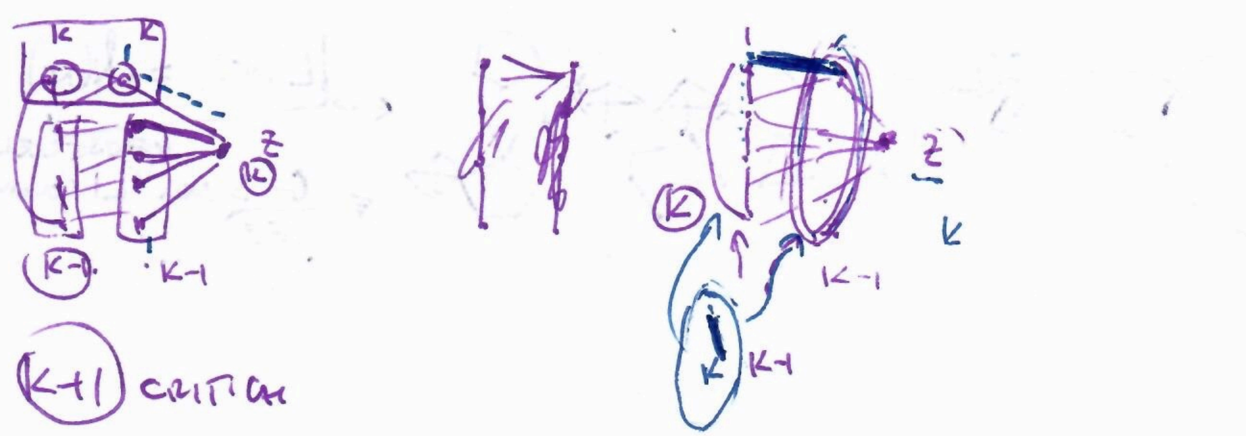
- *identify x_1 in x_2,*
- *remove the edges x_1y_1 and x_2y_2, and*
- *connect vertices y_1 and y_2.*

NOT (k-1) COLORABLE



Hajós construction performed on a pair of K_4s gives the Moser spindle graph.

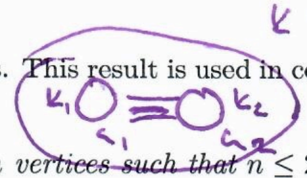
1: Prove the propositions.



(k+1) CRITICAL

k critical

Gallai succeeded to characterize the "small" critical graphs as join graphs. This result is used in coloring graphs on surfaces, since there we often deal with small critical graphs.

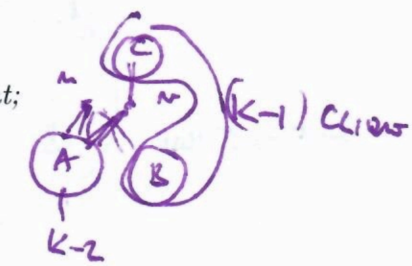


Theorem 4 (Gallai). *Let G be a k-critical graph with $k \geq 4$ and with n vertices such that $n \leq 2k-2$. Then, there exists a k_1-critical graph G_1 and k_2-critical graph G_2 such that G is $G_1 + G_2$ and $k = k_1 + k_2$.*

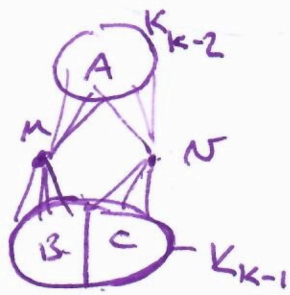
The following construction of critical graphs on $2k-1$ vertices shows that the bound $2k-2$ is optimal. Moreover, these graphs are example of critical graphs with connectivity two, and for more such graphs, one can use the Hajós construction.

Proposition 5. *Let G be the graph obtained in the following way:*

- *$V(G) = A \cup B \cup C \cup \{u, v\}$, where these sets are nonempty and disjoint;*
- *$G[A]$ is $(k-2)$-clique and $G[B \cup C]$ is $(k-1)$-clique;*
- *$N(u) = A \cup B$ and $N(v) = A \cup C$.*



Then G is k-critical.



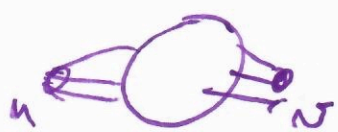
2: Prove the Proposition above.

Solution:

FIND $(k-1)$-coloring?

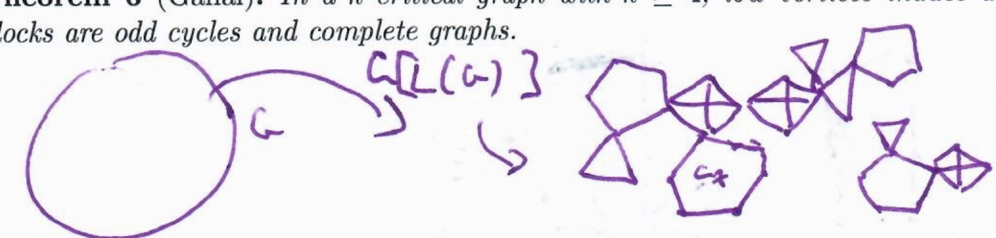
$(k-1)$ ON $B \cup C$ .. u, v HAVE DIFFERENT COLORS

ONLY $(k-3)$ LEFT FOR A



In a k-critical graph, vertices of degree $k-1$ are called *low* vertices, and are denote by $L(G)$. Similarly, the vertices of degree $\geq k$ are called *high* vertices, and are denote by $H(G)$. An interesting generalization of Brooks theorem is the following result of Gallai.

Theorem 6 (Gallai). *In a k-critical graph with $k \geq 4$, low vertices induce a forest (possibly empty) whose blocks are odd cycles and complete graphs.*



$|L(v)| = d(v)$

NECESSARILY

G IS NOT L-COLORABLE (IFF)

For now, we leave its proof as an exercise, but later when we consider list-colorings we will derive it as a corollary. Trees/forests with blocks described in the above theorem, nowdays are called *Gallai trees/forests*.

2 Minimum number of edges in critical graphs

Let G be a k-critical graph on n vertices and with m edges.

3: Give a lower bound on m.
Hint: Minimum degree. Can you improve by using Brook's Theorem?

n VERTICES I K-CRITICAL GRAPH K-CRITICAL GRAPH $\delta(G) \geq k-1$

$\frac{(k-1)n}{2} \leq m$ $(k-1) \cdot n \leq 2m$ IS TIGHT? C_{2k+1}

K_k

IF $k \geq 4$ & NOT K_k THEN

$2m \geq (k-1) \cdot n + 1$ $(k-1)n = 2m$

$\forall v$ $d(v) = (k-1)$

We give two classical improvements one is due to Gallai and the other due to Dirac. Nowadays these bounds are improved.

Theorem 7 (Gallai). *Let $G \neq K_k$ be a k-critical graph on n vertices and with m edges. Then*

$$2m \geq (k-1)n + \frac{k-3}{k^2-3}n + \frac{2(k-1)}{k^2-3}. \tag{1}$$

Notice that the above inequality is a better one for graphs with large number of vertices, and for dealing with graphs of small number of vertices is more appropriate the following one of Dirac.

Theorem 8 (Dirac). *Let $G \neq K_k$ be a k-critical graph on n vertices and with m edges. Then*

$$2m \geq (k-1)n + k - 3. \tag{2}$$

Kostochka and Yancey give a further improvement.

Theorem 9 (Kostochka and Yancey). *If $k \geq 4$ and G is a k-critical graph on n vertices and with m edges. Then*

$$m \geq \left\lceil \frac{(k+1)(k-2)n - k(k-3)}{2(k-1)} \right\rceil. \tag{3}$$

The bound in Theorem 9 is tight for $k = 4$ and every $n \geq 6$. For $k \geq 5$, the bound is exact for every $n \equiv 1($ mod $k-1)$, $n \neq 1$.

4: Evaluate all three bounds for several values of k. Can you find a construction matching Kostochka and Yancey bound?