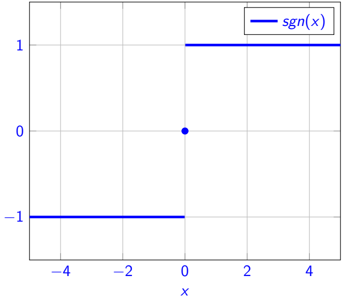
Chapter 2.4 - One-sided limits

Motivation for one-sided limit



Recall: $\lim_{x \to 0} sgn(x)$ is not defined.

If we look at $\lim_{x \to 0} sgn(x)$ *only* for $x > 0$ then the limit *could* be 1.

We will write $\lim_{x \to 0^+} sgn(x) = 1$.

If we look at $\lim_{x \to 0} sgn(x)$ *only* for $x < 0$ then the limit *could* be -1.

We will write $\lim_{x \to 0^-} sgn(x) = 1$.

One-sided limit definition

If $f(x)$ approaches L as x approaches a from *the right (i.e. $x > a$)* we write

$$\boxed{\lim_{x \to a^+} f(x) = L}$$

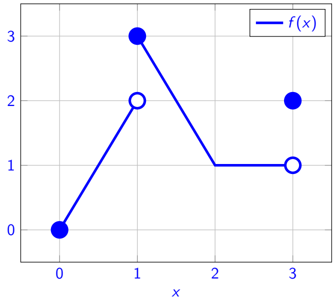
If $f(x)$ approaches L as x approaches a from *the left (i.e. $x < a$)* we write

$$\boxed{\lim_{x \to a^-} f(x) = L}$$

$\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^+} f(x) = L$ and $\lim_{x \to a^-} f(x) = L$

Simple example for limits

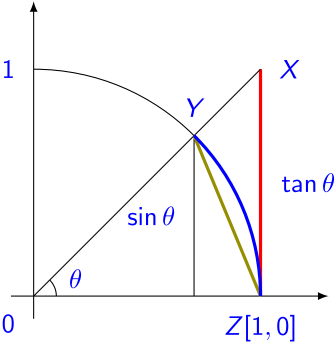
Example: Compute limits and values for the $f(x)$ defined on $[0, 3]$.



- $\lim_{x \to 1^-} f(x) = 2$
- $\lim_{x \to 1^+} f(x) = 3$
- $f(1) = 3$
- $\lim_{x \to 2} f(x) = 1$
- $f(3) = 2$
- $\lim_{x \to 3^-} f(x) = 1$
- $\lim_{x \to 3^+} f(x) =$ not defined

$\sin(x)$ behaves like the line $y = x$ around 0.

$$\boxed{\lim_{x \to 0^+} \frac{\sin x}{x} = 1}$$



Assume $\pi/2 > \theta > 0$. Then

$$\text{area } \Delta 0ZY < \text{area sector } 0ZY < \text{area } \Delta 0ZX$$

area $\Delta 0ZY = \frac{1}{2} \cdot 1 \cdot \sin\theta$
area sector $0ZY = \frac{1}{2} r^2 \theta = \theta/2$
area $\Delta 0ZX = \frac{1}{2} \tan\theta$

$$\frac{1}{2}\sin\theta < \frac{\theta}{2} < \frac{1}{2}\tan\theta = \frac{1}{2}\frac{\sin\theta}{\cos\theta}$$
$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$
$$1 > \frac{\sin\theta}{\theta} > \cos\theta$$

Now 2cops theorem does it for $\theta \to 0^+$.

Examples using $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$

- $\displaystyle \lim_{h\to 0} \frac{h}{\sin(3h)} = \lim_{h\to 0} \frac{1}{3} \cdot \frac{3h}{\sin(3h)} = \frac{1}{3}$

- $\displaystyle \lim_{x\to 0} \frac{\tan(2x)}{x} = \lim_{x\to 0} \frac{\sin(2x)}{\cos(2x)} \cdot \frac{1}{x} = \lim_{x\to 0} \frac{\sin(2x)}{2x} \cdot \frac{1}{\cos(2x)} \cdot 2 = 1 \cdot 1 \cdot 2 = 2.$

- $\displaystyle \lim_{\theta\to 0} \frac{\cos(\theta) - 1}{\theta} = \lim_{\theta\to 0} \frac{\cos^2(\theta) - 1}{\theta(\cos(\theta) + 1)} = \lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} \cdot \frac{\sin(\theta)}{\cos(\theta) + 1} = 1 \cdot \frac{0}{2} = 0$

- $\displaystyle \lim_{\theta\to 0} \frac{\cos(\theta) - 1}{\sin(2\theta)} = \lim_{\theta\to 0} \frac{\cos(\theta) - 1}{\theta} \cdot \frac{1}{2} \cdot \frac{2\theta}{\sin(2\theta)} = 0 \cdot \frac{1}{2} \cdot 1 = 0$

Chapter 2.4 Recap

- One-sided limit is approaching x only from one side
- One-sided limits are same if and only if the limit exists
- $\lim_{x \to 0} \frac{\sin x}{x} = 1$