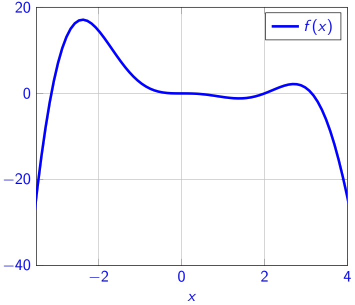
Chapter 2.5 - Continuous Functions

Motivation for Continuous Function

Recall: For polynom $P(x)$ holds $\lim_{x \to a} P(x) = P(a)$.

What should happen at $x \to a$ happens at $x = a$.



This is a nice property and behavior, called *continuous*.

$f(x)$ is *continuous at $x = a$* if $\lim_{x \to a} f(x) = f(a)$

Some Continuous Functions

If f is continuous at a, the following three MUST ALL happen:

- $\lim_{x \to a} f(x)$ exists
- $f(a)$ exists
- $\lim_{x \to a} f(x) = f(a)$

If any of them fail, f is *not* continuous at a.

The following are continuos at all points

- Polynomials
- Rational functions where denominator $\neq 0$.
- $\sin x$, $\cos x$, e^x,
- $\ln x$ for $x > 0$

Intuition: If you can draw the graph of f around a with one stroke, then f is continuous at a.

Types of Discontinuity

Removable by (re)defining the function at $x = a$, we can fix the problem.
Example: Is $f(x) = (x^2 - 1)/(x - 1)$ continusous at $x = 1$?
It is not defined at $x = 1$, but

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$$

Removable discontinuities are great because we can "fix" them. The piecewise-defined function

$$F(x) = \begin{cases} \frac{x^2-1}{x-1} & x \neq 1 \\ 2 & x = 1 \end{cases}$$

is basically the function f, but with the open hole at $x = 1$ filled in.

Jump, left and right hand limits exist but disagree.
Example:

$$f(x) = \begin{cases} x + 3 & x < 1 \\ 2 & x = 1 \\ 2x - 2 & x > 1 \end{cases}$$

at $x = 1$.

$$\lim_{x \to 1^-} f(x) = 4$$
$$\lim_{x \to 1^+} f(x) = 0$$

Hence $\lim_{x \to 1} f(x)$ does not exist.

More Types of Discontinuity

Oscilating if limit does not exist due to oscilating.

Example:

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x > 1 \\ 0 & x \leq 0 \end{cases}$$

at $x = 0$.

$$\lim_{x \to 0^-} f(x) = 0$$

$$\lim_{x \to 0^+} f(x) \text{ does not exist}$$

Hence $\lim_{x \to 0} f(x)$ does not exist.

Infinite Discontinuity appears when asymptote from one (or both) sides. More in next section.

Example:

$$f(x) = \begin{cases} \frac{1}{x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

at $x = 0$.

$$\lim_{x \to 0^-} f(x) = \infty$$

$$\lim_{x \to 0^+} f(x) = \infty$$

Hence one could say $\lim_{x \to 0} f(x) = \infty$ but we cannot have $f(x) = \infty$.

Continuous Function

A function is *continuous* if it is continuous at *every* point.

Intuition: If you could draw the graph of f with one stroke.

Example: Find a and b so that $f(x)$ is continuous and

$$f(x) = \begin{cases} X & x < -1 \\ ax + b & -1 \leq x \leq 1 \\ Y & x > 1 \end{cases}$$

We need $\lim_{x \to -1} f(x)$ to exist. Hence $f(-1) = -1$. This means $a(-1) + b = X$. Same, $\lim_{x \to 1} f(x)$ gives $a + b = Y$. One can add them together and get $2b = (X + Y)$ hence $b = (X + Y)/2$. Now $a = Y - (X + Y)/2 = (Y - X)/2$.

Combining Continuous Functions

If $f(x)$ and $g(x)$ are continuous then so are

- $f(x) + g(x)$
- $Kf(x)$ for $K \in \mathbb{R}$
- $f(x)g(x)$
- $\frac{f(x)}{g(x)}$ if $g(x) \neq 0$
- $(f(x))^q$ for reasonable q
  $q \in \mathbb{N}^+$
- $f(g(x))$ composition

Example: Is $f(x)$ continuous?

$$f(x) = \frac{\sin(e^x + \pi - 1) + x^3 - 16\cos(\sin(x))}{x^4 + 2 + sin^2(e^x - e^{-x}) + \ln(x^2 + 1)}$$

Yes. The functions $\sin x, e^x, \cos x$ are continuous. Also $\ln x$ is continuous for $x > 0$ and notice that $x^2 + 1 \geq 1$. In particular $\ln x^2 + 1 \geq 0$. Hence the denominator is > 0 and $f(x)$ is continuous.

Nice fact about continuous functions:

$$\lim_{x \to a} f(x) = f(a)$$

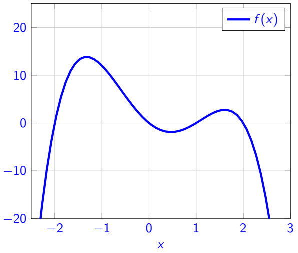
Example: Given $f(x)$ above find

$\lim_{x \to 0} f(x) =$

$$\frac{\sin(\pi) + 0 - 16\cos(0)}{0 + 2 + \sin^2(0) + \ln(1)} = -8$$

Intermediate Value Theorem

If $f(x)$ is *continuous* on $[a, b]$, then for any y between $f(a)$ and $f(b)$ there is a c in $[a, b]$ with $f(c) = y$.



Example: $f(x) = 18x^3 - 63x^2 + 67x - 20$
Does $f(x)$ have a root for $0 \leq x \leq 1$? $f(x)$ is continuous. Also $f(0) = -20$ and $f(1) = 2$. The IVT implies there is $c \in [0, 1]$ such that $f(c) = 0$. Actually, $c = \frac{1}{2}$.
Does $f(x)$ have a root for $1 \leq x \leq 2$? $f(1) = 2$ and $f(2) = 6$. The IVT does not say anything about the roots! There are two roots $\frac{4}{3}$ and $\frac{5}{3}$.

Example: Show exists $0 \leq x$ that $\cos(x) = x$. Let $f(x) = \cos(x) - x$. Using $x = 0$ and $x = \frac{\pi}{2}$ yields

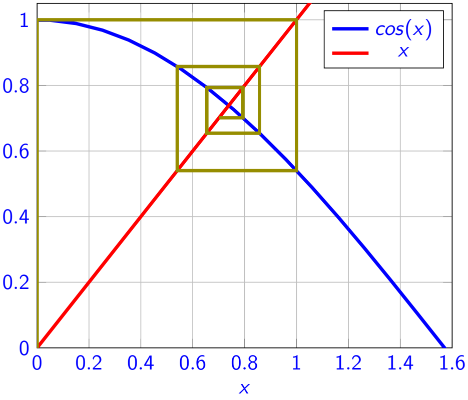
$$f(0) = \cos(0) - 0 = 1 \quad f(\frac{\pi}{2}) = \cos(\frac{\pi}{2}) - \frac{\pi}{2} = -\frac{\pi}{2}$$

As 0 in $[-\frac{\pi}{2}, 1]$, the intermediate value theorem implies that there exists a c in $[0, \frac{\pi}{2}]$ such that $0 = f(c) = \cos(c) - c$ and thus $\cos(c) = c$.

$\cos(\cos(\cos(\ldots\cos(x)\ldots)))$

What happens when you type to calculator 0 and keep pressing cos button? (in Rad)

Let's move between points
$(0,0), (0,\cos(0)), (\cos(0),\cos(0)), (\cos(0),\cos(\cos(0))), (\cos(\cos(0)),\cos(\cos(0))), \ldots$



Notice the green spiral is behaving like a fractal.

Chapter 2.5 Recap

- $f(x)$ is *continuous at* $x = a$ if $\lim_{x \to a} f(x) = f(a)$
- f is continuous if it is continuous at every point
- Removable discontinuity, jump, oscillation, asymptotes (next time)
- combining continuous functions gives continuous function
- Intermediate Value Theorem