Chapter 3.3: Differentiation Rules

Basic Functions (compute them once and for all)

$\frac{d}{dx}\left[c\right] = 0$

$$\frac{d}{dx}\left[c\right] = \lim_{h\to 0}\frac{c-c}{h} = \lim_{h\to 0}\frac{0}{h} = 0.$$

$\frac{d}{dx}\left[x\right] = 1$

$$\frac{d}{dx}\left[x\right] = \lim_{z\to x}\frac{z-x}{z-x} = 1.$$

$\frac{d}{dx}\left[x^n\right] = nx^{n-1}$
works for any $n \in \mathbb{R}$

$$\frac{d}{dx}\left[x^n\right] = \lim_{z\to x}\frac{z^n-x^n}{z-x} =$$

$$\lim_{z\to x}\frac{(z-x)(z^{n-1}+z^{n-2}x+\ldots+zx^{n-2}+x^{n-1})}{z-x}$$

$$= \lim_{z\to x} z^{n-1}+z^{n-2}x+\ldots+zx^{n-2}+x^{n-1}$$

$$= \underbrace{x^{n-1}+x^{n-1}+\ldots+x^{n-1}}_{n \text{ times}} = nx^{n-1}$$

$\frac{d}{dx}\left[e^x\right] = e^x$

Fact: $\lim_{h\to 0}\frac{e^h-1}{h} = 1$; $e = 2.718281828459\ldots$

$$\frac{d}{dx}\left[e^x\right] = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}$$

$$= \lim_{h\to 0}\frac{e^x\cdot e^h-e^x}{h} = e^x\lim_{h\to 0}\frac{e^h-1}{h} = 1$$

Combining Functions

Pulling out constants

$$\frac{d}{dx}\left[c \cdot f(x)\right] = c \cdot \frac{d}{dx}\left[f(x)\right]$$

$$\frac{d}{dx}\left[cf(x)\right] =$$
$$= \lim_{h\to 0} \frac{cf(x+h) - cf(x)}{h}$$
$$= c \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= c\frac{d}{dx}\left[f(x)\right].$$

Example: $\frac{d}{dx}\left[3x^2\right] = 6x$

Separating over sums

$$\frac{d}{dx}\left[f(x) + g(x)\right] = \frac{d}{dx}\left[f(x)\right] + \frac{d}{dx}\left[g(x)\right]$$

$$\frac{d}{dx}\left[f(x) + g(x)\right] =$$
$$= \lim_{h\to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h}$$
$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$
$$= \frac{d}{dx}\left[f(x)\right] + \frac{d}{dx}\left[g(x)\right].$$

Example: $\frac{d}{dx}\left[x^3 + x\right] = 3x^2 + 1$

Examples

Example: Find $\frac{d}{dx}\left[\frac{2}{x}+e^{100}\right]$

$$= \frac{d}{dx}\left[2x^{-1}+e^{100}\right] = 2\frac{d}{dx}\left[x^{-1}\right] + \frac{d}{dx}\left[e^{100}\right] = 2(-1)x^{-1-1} + 0 = -2x^{-2}$$

Example: Find tangent line at $x = 1$ to function $f(x) = x^3 + \sqrt{x} - \frac{2}{e}e^x$.
Tangent line has equation $y = ax + b$ where $a = f'(1)$. Compute $f'(x) = 3x^2 + \frac{1}{2}x^{-1/2} - \frac{2}{e}e^x$. Then $a = 3 + 1/2 - 2 = 1.5$. Compute b from $f(1) = 1.5 \cdot 1 + b$. Hence $b = f(1) - 1.5 = -1.5$. Solution is $y = 1.5x - 1.5$.

Example: Find all x so that the tangent lines to $y = x^3 - 12x + 17$ are horizontal.
Horizontal means slope 0. The slope at x is the same as the derivative. Hence we look for x such that $0 = \frac{d}{dx}\left[x^3 - 12x + 17\right] = 3x^2 - 12 = 3(x^2 - 4)$. Solutions are $0 = x^2 - 4$ and $x = +2$ and $x = -2$.

Product Rule

$$\frac{d}{dx}\Big[f(x)g(x)\Big] = \frac{d}{dx}\Big[f(x)\Big]g(x) + f(x)\frac{d}{dx}\Big[g(x)\Big]$$

$$\begin{aligned}
\frac{d}{dx}\Big[f(x)g(x)\Big] &= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)\overbrace{-f(x)g(x+h) + f(x)g(x+h)}^{=0}-f(x)g(x)}{h} \\
&= \lim_{h\to 0}\left(\underbrace{\frac{f(x+h)-f(x)}{h}}_{=f'(x)}g(x+h) + f(x)\underbrace{\frac{g(x+h)-g(x)}{h}}_{=g'(x)}\right) \\
&= f'(x)g(x) + f(x)g'(x)
\end{aligned}$$

Example: Find $\frac{d}{dx}\left(x^{2/3}e^x\right) = \frac{2}{3}x^{-1/3}e^x + x^{2/3}e^x$

Reciprocal and Quotient Rules

Reciprocal rule

$$\frac{d}{dx}\left[\frac{1}{f(x)}\right] = \frac{-f'(x)}{f(x)^2}$$

Will be obvious after Chain rule in Section 3.6.

Quotient rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{d}{dx}\left[f(x)\cdot\frac{1}{g(x)}\right] = \frac{d}{dx}\left[f(x)\right]\cdot\frac{1}{g(x)} + f(x)\frac{d}{dx}\left[\frac{1}{g(x)}\right]$$

$$= f'(x)\cdot\frac{1}{g(x)} + f(x)\cdot\frac{-g'(x)}{g(x)^2} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Examples for Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Example: Find $\frac{d}{dx}\left[\frac{x^2}{x+2}\right] = \frac{2x(x+2) - x^2}{(x+2)^2} = \frac{x^2+4x}{(x+2)^2}$

Example: Find $\frac{d}{dx}\left[\frac{\sqrt{x}+x^2}{3x^3+x\cdot e^x}\right]$

$$= \frac{\left(\frac{1}{2}x^{-\frac{1}{2}}+2x\right)\cdot\left(3x^3+x\cdot e^x\right) - \left(\sqrt{x}+x^2\right)\cdot\left(9x^2+e^x+x\cdot e^x\right)}{\left(3x^2+x\cdot e^x\right)^2}$$

Higher Order Derivatives

Derivatives are functions so we can take derivatives of derivatives, and so on.

$$y = f(x) = \text{ function}$$

$$y' = \frac{dy}{dx} = f'(x) = \text{ first derivative}$$

$$y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d^2y}{dx^2} = f''(x) = \text{ second derivative}$$

$$y^{(n)} = \frac{d^ny}{dx^n} = f^{(n)}(x) = n^{\text{th}} \text{ derivative}$$

Example: Compute derivatives of $y = 2x^3 - x^2 + 4x + 3$

$$\frac{dy}{dx} = 6x^2 - 2x + 4$$

$$\frac{d^2y}{dx^2} = 12x - 2$$

$$\frac{d^3y}{dx^3} = 12$$

$$\frac{d^4y}{dx^4} = 0$$

$$\frac{d^5y}{dx^5} = 0$$

Notice the degree of the polynomial is decreasing and eventually it is 0

More Examples

Example: Find derivatives of

$f(x) = (7-2x)\cdot(5+x^3)^{-1} = \frac{7-2x}{5+x^3}$

$f'(x) = \frac{-2(5+x^3)-3x^2(7-2x)}{(5+x^3)^2}$

$f(x) = e^{-x} = \dfrac{1}{e^x}$

$f'(x) = \frac{-e^x}{e^{2x}} = -e^x$

$f(x) = e^{2x} = e^x \cdot e^x$

$f'(x) = e^x \cdot e^x + e^x \cdot e^x = 2e^{2x}$

$f(x) = \dfrac{1-2x+4\sqrt{x}}{x} = x^{-1} - 2 + 4x^{1/2}$

$f'(x) = -x^{-2} + 2x^{-1/2}$ for $x \neq 0$. Notice we did not need the reciprocal rule.

Example: Find second derivative of $f(x) = \frac{x^3+7}{x}$.

$f'(x) = \frac{3x^2\cdot x-((x^3+7)1)}{x^2} = \frac{2x^3-7}{x^2}$ and $f''(x) = \frac{6x^2\cdot x^2-(2x\cdot(2x^3-7))}{(x^2)^2} = \frac{2x^4+14x}{x^4}$

Chapter 3.3 Recap

$$\frac{d}{dx}\left[x^r\right] = rx^{r-1} \qquad \frac{d}{dx}\left[e^x\right] = e^x$$

$$\frac{d}{dx}\left[c \cdot f(x)\right] = c \cdot \frac{d}{dx}\left[f(x)\right]$$

$$\frac{d}{dx}\left[f(x) + g(x)\right] = \frac{d}{dx}\left[f(x)\right] + \frac{d}{dx}\left[g(x)\right]$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

$\frac{d^2y}{dx^2}\left[f(x)\right]$ is the second derivative