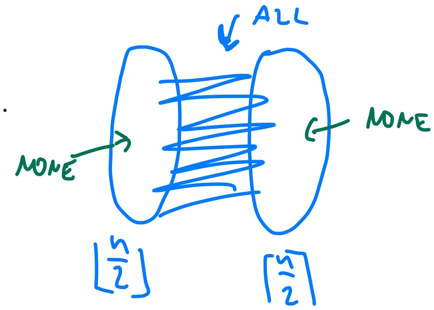
Chapter 7.1 Turán's Theorem

General problem: For a fixed graph F and number n, find an n-vertex graph maximizing the number of edges while avoiding F as a subgraph.

Theorem 1 (Mantel's Theorem, 1907). *The maximum number of edges in a graph on n vertices with no triangle subgraph is $\lfloor \frac{n^2}{4} \rfloor$.*

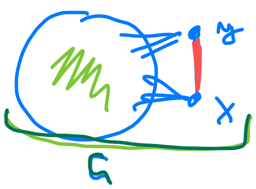
1: Show that the n-vertex complete balanced bipartite graph has $\lfloor \frac{n^2}{4} \rfloor$ edges. It means that the bound in Mantel's theorem is achieved by some graphs.



$$\left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{n^2}{4} \right\rfloor$$

Now we show that there are no triangle-free graphs with more edges than claimed by Mantel's theorem.

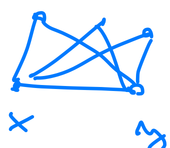
2: Prove Mantel's theorem by induction, where the induction step removes two adjacent vertices.



INDUCTION ON $G - x - y$ (DROP $\lfloor\ \rfloor, \lceil\ \rceil$)

$n=1$ ✓
$n=2$ ✓

$$|E(G)| \le \underbrace{\frac{(n-2)^2}{4}}_{\text{INDUCTION}} + \underbrace{(n-2)}_{N(x) \cup N(y) \setminus \{x,y\}} + \underbrace{1}_{xy} = \frac{n^2 - 4n + 4 + 4n - 8 + 4}{4} = \frac{n^2}{4}$$

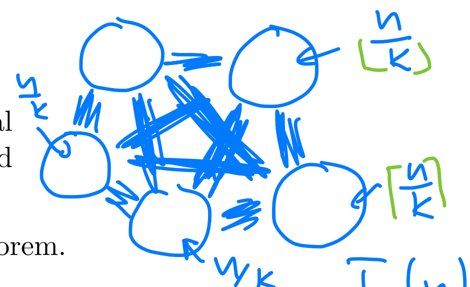


Theorem 2 (weak Turán's Theorem, 1941). *The maximum number of edges in a graph on n vertices with no $(k+1)$-clique subgraph is at most*

$$\left(1 - \frac{1}{k}\right) \frac{n^2}{2}.$$

MANTEL — NO K_3
TURÁN — NO K_{k+1}

Let $T_k(n)$ be a complete k-partite graph on n vertices with parts of as equal sizes as possible, i.e., sizes are $\lfloor \frac{n}{k} \rfloor$ or $\lceil \frac{n}{k} \rceil$. Another way of defining it would be a balanced blow-up of K_k. Such graph is called the *Turán graph.*



3: Show that $T_k(n)$ gives asymptotically tight lower bound for Turán's theorem.

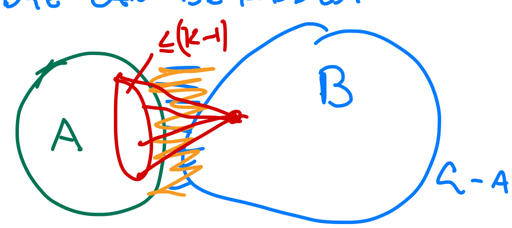
$$|E(T_k(n))| = \left(\frac{n}{k} \cdot \frac{n}{k}\right) \cdot \binom{k}{2} = \frac{n^2}{k^{\cancel{2}}} \cdot \frac{\cancel{k} \cdot (k-1)}{2} = \frac{n^2}{2}\left(\frac{k-1}{k}\right)$$

$T_2(n) =$

CHECK $T_k(n)$ DOES NOT CONTAIN K_{k+1}

4: Prove Turán's theorem by induction. Idea: Find a clique A of size k and remove it for induction.

LET G HAVE MAX # OF EDGES. (AND K_{k+1}-FREE) IF G HAS NO K_k AS SUBGRAPH, ANY ONE EDGE CAN BE ADDED.



$$|E(G)| \le |E(G[A])| + |E(G[B])| + |E(G[A-B])|$$
$$= \binom{k}{2} + \left(1 - \frac{1}{k}\right)\frac{(n-k)^2}{2} + (n-k) \cdot (k-1) =$$

$$= \frac{k\cdot(k-1)}{2} + \left(\frac{k-1}{k}\right)\frac{n^2-2kn+k^2}{2} + (n-k)\cdot(k-1)$$

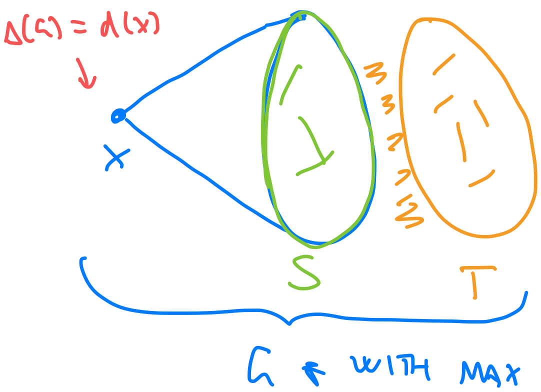
$$= \left(\frac{k-1}{k}\right)\frac{k^2 + n^2-2kn+k^2 + 2\cdot k\cdot(n-k)}{2} = \left(\frac{k-1}{k}\right)\frac{n^2}{2}$$

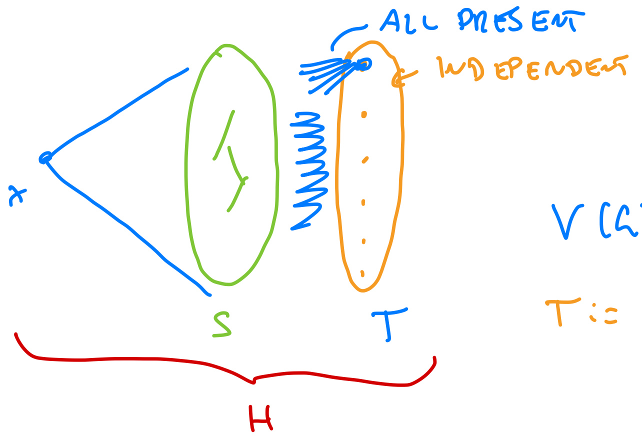
Actually, Turán proved more.

Theorem 3 (Turán's Theorem, 1941)**.** *The maximum number of edges in an n-vertex graph with no $(k+1)$-clique is exactly $e(T_k(n))$. Furthermore, $T_k(n)$ is the unique graph attaining this maximum.*

Note that $T_k(n)$ is a complete multipartite graph and among complete multipartite graphs with no $(k+1)$-clique is it the largest. This leads to a ingenious approach: if we can show that a graph G with no $(k+1)$-clique and the maximum number of edges is complete multipartite then we are done.

5: Prove $T_k(n)$ is the unique extremal graph by using induction on k. Idea: Take a vertex x of maximum degree, use induction on neighbors of x, and maximality arguments on non-neighbors of x.





$V(G) = x \cup S \cup T$

T := NON-NEIGHBORS OF x

H IS OBTAINED FROM G BY 1) DELETING ALL EDGES IN T

2) ADDING ALL S-T EDGES

H IS K_{k+1}-FREE SUPPOSE K_{k+1} EXIST. IT CONTAINS $y \in T$. $\Rightarrow$ x NOT IN K_{k+1}

SWAP x FOR y & GET K_{k+1} IN G (CONTRADICTION)

$|E(H)| \geq |E(G)|$

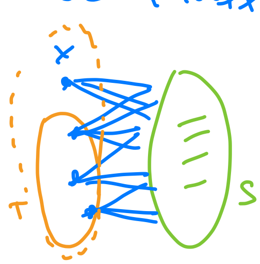
$d(x) = \Delta(G)$ # EDGES INCIDENT WITH VERTICES T IN G : $\sum_{v \in T} d(v) - |E(T)|$

$\leq$

EDGES INCIDENT WITH VTX T IN H : $\sum_{v \in T} \Delta(G)$

$G = H$

SINCE G MAXIMIZING |E|, G = H



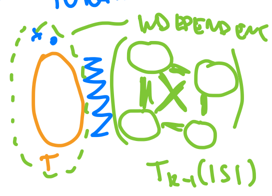
$(T \cup \{x\})$ AND (S) GIVES COMPLETE BIPARTITE GRAPH - NOT INDUCED!

(?) WHAT IS IN S(?)

S IS K_k-FREE & EDGE MAXIMAL SINCE G IS EDGE MAXIMAL

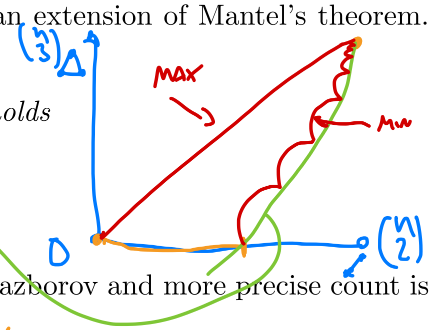
- BY INDUCTION S IS TURAN GRAPH $T_{k-1}(|S|)$

$\Rightarrow$ G IS COMPLETE k-PARTITE



INDEPENDENT, LAST PART TO GET k-PARTITE

Instead of determining the maximum number of edges in a K_{k+1}-free graph we may ask how many copies of K_{k+1} are in a graph with some fixed number of edges. We will only show an extension of Mantel's theorem. Let N_s be the number of copies of K_s in G.



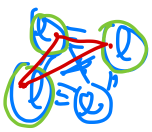
Theorem 4 (Goodman bound)**.** *For every n-vertex graph G with m edges holds*

$$N_3 \geq \frac{m(4m-n^2)}{3n}.$$

The bound is not always tight. Tight asymptotic solution was obtained by Razborov and more precise count is in https://arxiv.org/pdf/1712.00633.pdf.

$n = \ell . k$, NO FLOOR OR CEILING

6: Show that Goodman bound is tight for Turán's graphs $T_k(\ell \cdot k)$.



$$m = \binom{k}{2}\binom{\ell}{1}^2 = \binom{k}{2}\ell^2$$

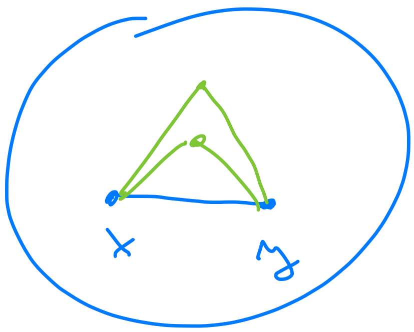
$$\binom{\ell}{1}^3$$

$$\binom{k}{3}\ell^3 = N_3 \overset{?}{\geq} \frac{m(4m-n^2)}{3n} = \frac{\binom{k}{2}\ell^2\left(4\binom{k}{2}\ell^2 - (k.\ell)^2\right)}{3k.\cancel{\ell}}$$

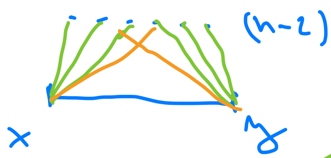
$\Uparrow$

TRIANGLES IN $T_k(\ell . k)$

7: Prove Goodman bound. Outline of the proof: For every edge xy, give a lower bound on the number of triangles containing xy (use $d(x), d(y), n$). Use the bound in $\sum$ over edges and change the $\sum$ to sum over vertices. And then use Cauchy-Schwartz[1].



$d(x) + d(y) - n \leq$ # TRIANGLES ON x,y



HOW MANY TIMES IS $d(x)$ COUNTED IN RESULT ON THE LEFT? $d(x)$ TIMES!

$$\#\Delta \geq \frac{1}{3}\left(\sum_{xy \in E(G)} d(x) + d(y) - n\right) = \frac{1}{3}\left(\sum_{x \in V(G)} d(x)^2 - n.m\right)$$

$$= \frac{1}{3}\left(\sum_{x\in V} d(x)^2 \cdot \sum_{x \in V}\left(\frac{1}{\sqrt{n}}\right)^2 - n.m\right) \overset{CS}{\geq} \frac{1}{3}\left(\sum \frac{d(x)}{\sqrt{n}}\right)^2 - n.m$$

$$= \frac{1}{3}\left(\frac{1}{\sqrt{n}}\sum d(x)\right)^2 - n.m = \frac{1}{3}\left(\frac{1}{n}\cdot 4m^2 - n.m\right)$$

$$= \frac{4m^2 - mn^2}{3n}$$

[1] $(\sum a_i b_i)^2 \leq (\sum a_i^2)(\sum b_i^2)$

Denote by $\mathrm{ex}(n, F)$ the maximum number of edges in an n-vertex graph without any copy of F.

Theorem 5 (Erdős-Stone 1946, Erdős-Simonovits, 1966)**.** *If F is a graph with chromatic number $\chi(F)$, then*

$$\mathrm{ex}(n, F) = \left(1 - \frac{1}{\chi(F) - 1}\right)\frac{n^2}{2} + o(n^2).$$

$\chi(F) = 2 \Rightarrow$

$ex(n, F) = \overbrace{\left(1 - \frac{1}{2-1}\right)\frac{n^2}{2}}^{=0} + o(n^2) = o(n^2)$

The theorem gives asymptotic result if $\chi(F) \geq 3$. It does not say much about bipartite graphs.

Theorem 6 (Erdős; Kővari-Sós-Turán, 1954)**.** *For any naturals $s \leq t$ we have*

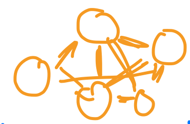
$$\mathrm{ex}(n, K_{s,t}) \leq \frac{1}{2}(t-1)^{1/s} n^{2-1/s} + O(n).$$

$\chi(F) = k$

$\chi(K_k) = k$

Open problem: Determine $\mathrm{ex}(n, C_6)$ or $\mathrm{ex}(n, C_{2k})$.

Open problem: Is $\mathrm{ex}(n, T) = \frac{1}{2}(k-1)n$ for a tree T with $k \geq 2$ edges?



IF $E(G) \gg T_{k-1}(n)$ ONE FINDS $K_{\underbrace{a,a,a,a}_{k}}$

$n^2 - \log n \neq o(n^2)$

$f(n) = o(n^2) \Rightarrow \lim_{n \to \infty} \frac{f(n)}{n^2} = 0$

$n = o(n^2)$

$n^{3/2} = o(n^2)$ $\quad n^{2-0.001} = o(n^2)$ $\quad \frac{n^2}{\log n} = o(n^2)$

$f(n) = O(n)$ IF $\exists n_0 > 0, C > 0$ S.T. $\forall n \geq n_0$ $f(n) \leq C \cdot n$