(4, 2)-choosability of planar graphs with forbidden structures

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Abstract

All planar graphs are 4-colorable and 5-choosable, while some planar graphs are not 4-choosable. Determining which properties guarantee that a planar graph can be colored using lists of size four has received significant attention. In terms of constraining the structure of the graph, for any $\ell \in \{3, 4, 5, 6, 7\}$, a planar graph is 4-choosable if it is ℓ-cycle-free. In terms of constraining the list assignment, one refinement of k-choosability is *choosability with separation.* A graph is (k, s)-*choosable* if the graph is colorable from lists of size k where adjacent vertices have at most s common colors in their lists. Every planar graph is $(4, 1)$-choosable, but there exist planar graphs that are not $(4, 3)$-choosable. It is an open question whether planar graphs are always $(4, 2)$-choosable. A *chorded ℓ-cycle* is an ℓ-cycle with one additional edge. We demonstrate for each $\ell \in \{5, 6, 7\}$ that a planar graph is $(4, 2)$-choosable if it does not contain chorded ℓ-cycles.

1 Introduction

A *proper coloring* is an assignment of colors to the vertices of a graph G such that adjacent vertices are assigned distinct colors. A (k, s)-*list assignment* L is a function that assigns a list $L(v)$ of k colors to each vertex v so that $|L(v) \cap L(u)| \leq s$ whenever $uv \in E(G)$. A proper coloring ϕ of G such that $\phi(v) \in L(v)$ for all $v \in V(G)$ is called an L-*coloring.* We say that a graph G is (k, s)-*choosable* if, for any (k, s)-list assignment L, there exists an L-coloring of G. We call this variation of graph coloring *choosability with separation.* Note that when a graph is (k, k)-choosable, we simply say it is k-*choosable.* Observe that if G is (k, t)-choosable, then G is (k, s)-choosable for all $s \leq t$. A notable result from Thomassen [11] states that every planar graph is 5-choosable, so it follows that all planar graphs are $(5, s)$-choosable for all $s \leq 5$.

Forbidding certain structures within a planar graph is a common restriction used in graph coloring. Theorem 1.2 summarizes the current knowledge on $(3, 1)$-choosability of planar graphs. Škrekovski [13] conjectured that all planar graphs are $(3, 1)$-choosable; this question is still open and is presented below as Conjecture 1.1.

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Conjecture 1.1 (Škrekovski [13])**.** *If G is a planar graph, then G is $(3,1)$-choosable.*

Theorem 1.2. *A planar graph G is $(3,1)$-choosable if G avoids any of the following structures:*

- *3-cycles (Kratochvíl, Tuza, Voigt [9]).*
- *4-cycles (Choi, Lidický, Stolee [4]).*
- *5-cycles and 6-cycles (Choi, Lidický, Stolee [4]).*

In this paper, we focus on 4-choosability with separation. Kratochvíl, Tuza, and Voigt [9] proved that all planar graphs are $(4,1)$-choosable, while Voigt [12] demonstrated that there exist planar graphs that are not $(4,3)$-choosable. It is not known if all planar graphs are $(4,2)$-choosable.

Conjecture 1.3 (Kratochvíl, *et al.* [9])**.** *If G is a planar graph, then G is $(4,2)$-choosable.*

Theorem 1.4 (Kratochvíl, *et al.* [9])**.** *If G is a planar graph, then G is $(4,1)$-choosable.*

Theorem 1.4 was strengthened by Kierstead and Lidický [8], where it is shown that we can allow an independent set of vertices to have lists of size 3 rather than 4.

Theorem 1.5 (Kierstead and Lidický [8])**.** *Let G be a planar graph and $I \subseteq V(G)$ be an independent set. If L assigns lists of colors to $V(G)$ such that $|L(v)| \geq 3$ for every $v \in I$, and $|L(v)| = 4$ for every $v \in V(G) \setminus I$, and $|L(u) \cap L(v)| \leq 1$ for all $uv \in E(G)$, then G has an L-coloring.*

In addition to the work summarized above, there are several results regarding 4-choosability. A graph is *k-degenerate* if each of its subgraphs has a vertex of degree at most k. Euler's formula implies a planar graph with no 3-cycles is 3-degenerate and hence 4-choosable. This and other similar results are listed below in Theorem 1.6. For the last result in Theorem 1.6, note that a *chorded ℓ-cycle* is an ℓ-cycle with an additional edge connecting two of its non-consecutive vertices.

Theorem 1.6. *A planar graph G is 4-choosable if G avoids any of the following structures:*

- *3-cycles (folklore).*
- *4-cycles (Lam, Xu, Liu, [10]).*
- *5-cycles (Wang and Lih [14]).*
- *6-cycles (Fijavz, Juvan, Mohar, and Škrekovski [7]).*
- *7-cycles (Farzad [6]).*
- *Chorded 4-cycles and chorded 5-cycles (Borodin and Ivanova [3]).*

Our main results in this paper are listed below in Theorem 1.7.

Theorem 1.7. *A planar graph G is $(4,2)$-choosable if G avoids any of the following structures:*

- *Chorded 5-cycles.*
- *Chorded 6-cycles.*
- *Chorded 7-cycles.*

We prove each case of Theorem 1.7 separately. In Section 4, we forbid chorded 5-cycles (see Theorem 4.1). In Section 5, we forbid chorded 6-cycles (see Theorem 5.1). In Section 6, we forbid chorded 7-cycles (see Theorem 6.2). There are many features common to all of these proofs, which we detail in Sections 2 and 3.

1.1 Preliminaries and Notation

Refer to [15] for standard graph theory terminology and notation. Let G be a graph with a vertex set $V(G)$ and an edge set $E(G)$; let $n(G) = |V(G)|$. We use K_n, C_n, and P_n to denote the complete graph, cycle graph, and path graph, respectively, each on n vertices. The *open neighborhood* of a vertex, denoted $N(v)$, is the set of vertices adjacent to v in G; the *closed neighborhood*, denoted $N[v]$, is the set $N(v) \cup \{v\}$. The *degree* of a vertex v, denoted $d_G(v)$, is the number of vertices adjacent to v in G; we write $d(v)$ when the graph G is clear from the context. If the degree of a vertex v is k, we call v a *k-vertex*; if the degree of v is at least k (at most k), we call v a *k^+-vertex* (*k^--vertex* respectively). The *length* of a face f, denoted $\ell(f)$, is the length of the face boundary walk. If the length of a face f is k, we call f a *k-face*; if the length of f is at least k, we call f a *k^+-face*.

2 Overview of Method

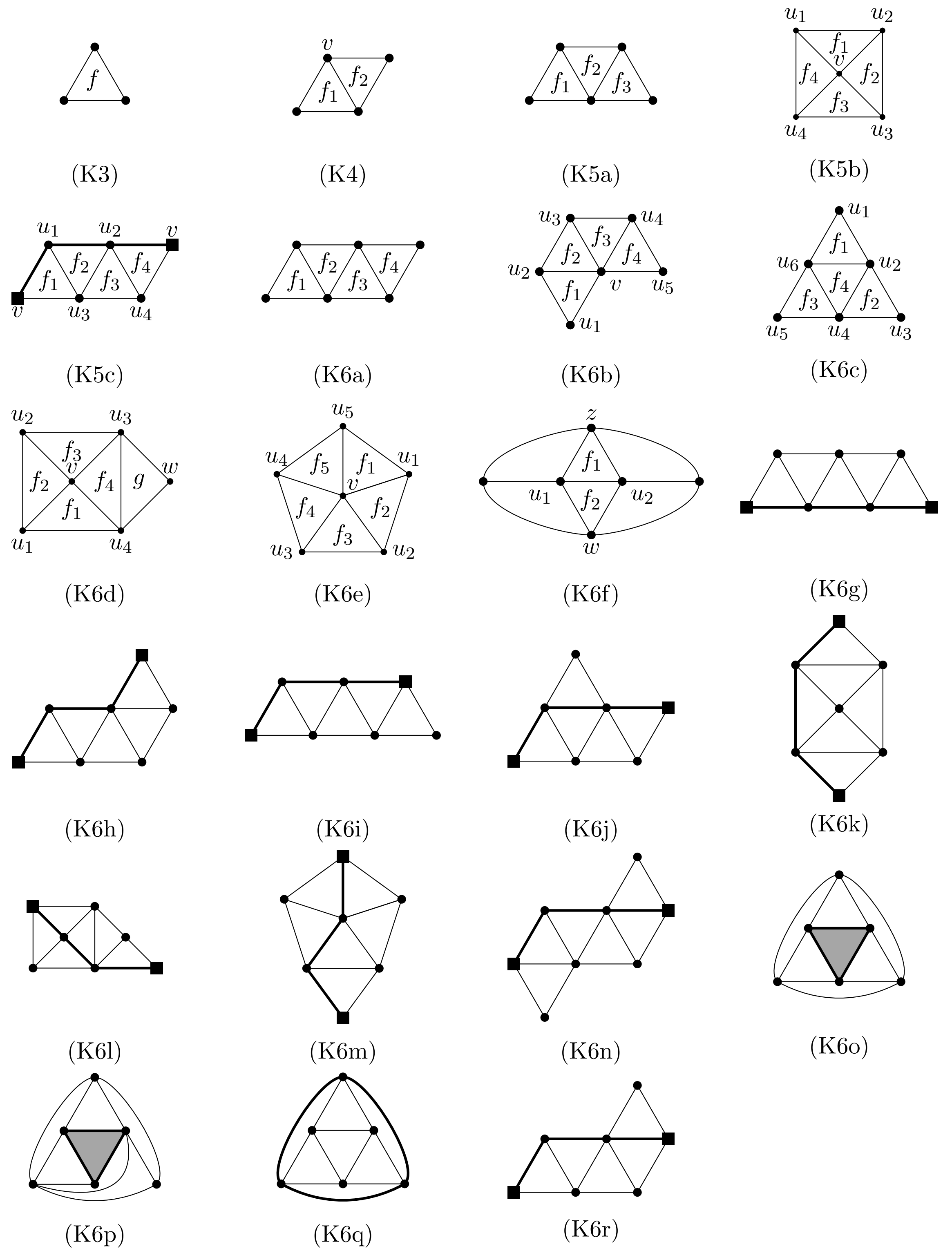
All of our main results use the discharging method. We refer the reader to the surveys by Borodin [2] and Cranston and West [5] for an introduction to discharging, which is a method commonly used to obtain results on planar graphs. For real numbers a_v, a_f, b, we define initial charge values $\mu_0(v) = a_v d(v) - b$ for every vertex v and $\mu_0(f) = a_f \ell(f) - b$ for every face f. If $a_v > 0$, $a_f > 0$, $b > 0$, and $2a_v + 2a_f = b$, then Euler's formula implies that $\sum_v \mu_0(v) + \sum_f \mu_0(f) = -2b$, and the total charge on the entire graph is negative. We then define *discharging rules* that describe a method for moving charge value among vertices and faces while conserving the total charge value. We demonstrate that if G is a "minimal counterexample" to our theorem, then every vertex and face ends with nonnegative charge after the discharging process, which is a contradiction. Intuitively, this process works well when forbidding a structure (such as a short chorded cycle) with low charge.

In Section 3, we concretely define *reducible configurations.* Loosely, a reducible configuration is a structure C in a graph G with $(4, 2)$-list assignment L where any L-coloring of $G-C$ extends to an L-coloring of G. If we are looking for a minimal example of a graph that is not $(4, 2)$-choosable, then none of these reducible configurations appear in the graph. We define a large list of configurations, (C1)–(C21) (see Figure 2), and prove they are reducible using various generic constructions. The configurations (C1)–(C10) are used when forbidding chorded 6- or 7-cycles, while the configurations (C9)–(C21) are used when forbidding chorded 5-cycles. The use of different configurations is due to differences in our discharging arguments.

In Section 4, we forbid chorded 5-cycles and every 3-face is adjacent to at most one other 3-face. Moreover, 3-faces are not adjacent to 4-faces. Thus, our initial charge function in this case guarantees that the only objects with negative initial charge are 4- and 5-vertices.

In Sections 5 and 6, we use a different discharging strategy. Our initial charge values guarantee that the only objects of negative charge are 3-faces. Thus, our discharging rules are designed to send charge from 5^+-faces and 4^+-vertices to 3-faces. However, as we forbid chorded 6-cycles or chorded 7-cycles, there may not be many 3-faces very close to each other.

If G is a plane graph and G^* is its dual, then let F_3 be the set of 3-faces of G and let G_3^* be the induced subgraph of G^* with vertex set F_3. A *cluster* is a maximal set of 3-faces that are connected in G^*, i.e., a connected component of G_3^*. Note that two 3-faces sharing an edge are adjacent in G^*, and two 3-faces sharing only a vertex are not adjacent in G^*. See Figure 1 for a list of the clusters with maximum cycle length six and every internal vertex of degree at least four. In these figures, the outer cycle is not necessarily a facial cycle, any area filled with gray is not a face,



These are all of the possible clusters with longest cycle at most six and minimum degree four. Bold edges demonstrate separating 3-cycles. Gray regions designate cycles that are not faces. We group our clusters by the length of the longest cycle in the cluster. Thus a configuration (Kni) has a maximum cycle length of n.

Figure 1: Clusters with maximum cycle length at most six.

and a pair of square vertices represent a single vertex. Additionally, bold edges describe *separating 3-cycles*, which are cycles in a plane graph whose exterior and interior regions both contain vertices not on the cycle. These figures are based on the list of clusters used by Farzad [6] in the proof that 7-cycle-free planar graphs are 4-choosable.

For $k \in \{1, 2\}$, there is exactly one way to arrange k 3-faces in a cluster. A *triangle* is a cluster containing exactly one 3-face; see (K3). A *diamond* is a cluster containing exactly two 3-faces; see (K4). For $k \geq 3$, there are multiple ways to arrange k facial triangles in a cluster. A *k-fan* is a cluster of k 3-faces all incident to a common vertex of degree at least $k+1$; see (K5a) and (K6b). A *k-wheel* is a cluster of k 3-faces all incident to a common vertex of degree exactly k; see (K5b) and (K6e). Note that the vertex incident to all faces of a 3-wheel has degree 3. A *k-strip* is a cluster of k 3-faces $f_1, \dots, f_k$ where the boundaries of the 3-faces are disjoint except that f_i and f_{i+1} share an edge for $i \in \{1, \dots, k-1\}$ and f_i and f_{i+2} share a vertex for $i \in \{1, \dots, k-2\}$; see (K5a) and (K6a).

If $f_1, \dots, f_k$ are the 3-faces in a cluster, then we will prove that the total charge on $f_1, \dots, f_k$ after discharging is nonnegative. Thus, some of the 3-faces may have negative charge, but this is balanced by other 3-faces in the cluster having positive charge. Hence, our proofs end with a list of all possible cluster types and verifying that each has nonnegative total charge.

While there are 23 total clusters that avoid chorded 7-cycles, we do not have that many cases to check. The clusters (K5c) and (K6g)–(K6r) have three bold edges, demonstrating a separating 3-cycle. We avoid checking these cases by using a strengthened coloring statement (see Theorem 6.2) that allows our minimal counterexample to not contain any separating 3-cycles.

3 Reducible Configurations

In this section, we describe structures that cannot appear in a minimal counterexample to Theorem 1.7. Let G be a graph, $f : V(G) \to \mathbb{N}$, and s be a nonnegative integer. A graph is *f-choosable* if G is L-choosable for every list assignment L where $|L(v)| \geq f(v)$. An *(f, s)-list-assignment* is a list assignment L on G such that $|L(v)| \geq f(v)$ for all $v \in V(G)$, $|L(v) \cap L(u)| \leq s$ for all edges $uv \in E(G)$, and $L(u) \cap L(v) = \emptyset$ if $uv \in E(G)$ and $f(u) = f(v) = 1$. A graph G is *(f, s)-choosable* if G is L-colorable for every (f, s)-list-assignment L.

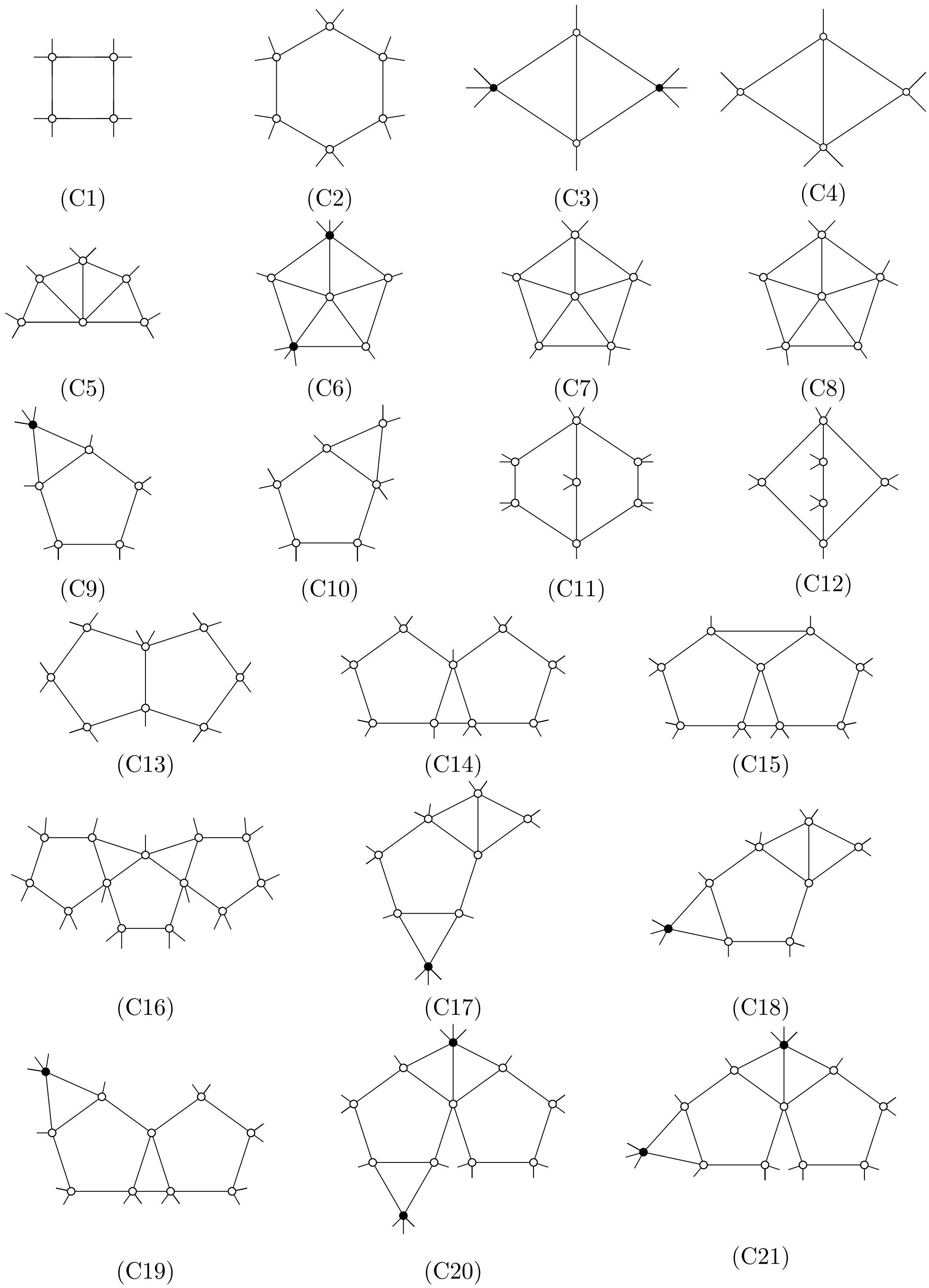
Definition 3.1. A *configuration* is a triple (C, X, ex) where C is a plane graph, $X \subseteq V(C)$, and $\text{ex} : V(C) \to \{0, 1, 2, \infty\}$ is an *external degree* function. A graph G *contains* the configuration (C, X, ex) if C appears as an induced subgraph C' of G, and for each vertex $v \in V(C)$, there are at most $\text{ex}(v)$ edges in G from the copy of v to vertices not in C'. For a triple (C, X, ex), define the *list-size function* $f : V(C) \to \mathbb{N}$ as

$$f(v) = \begin{cases} 4 - \text{ex}(v) & v \in X \\ 1 & v \notin X \end{cases}.$$

A configuration (C, X, ex) is *reducible* if C is $(f, 2)$-choosable.

Note that if a graph G with $(4, 2)$-list assignment L contains a copy of a reducible configuration (C, X, ex) and $G - X$ is L-choosable, then G is L-choosable.

First, we note that if (C, X, ex) is a reducible configuration, then any way to add an edge between distinct vertices of X and lower their external degree by one results in another reducible configuration.



In these configurations, edges with only one endpoint are external edges. Vertices in X are filled with white.

Figure 2: Reducible configurations.

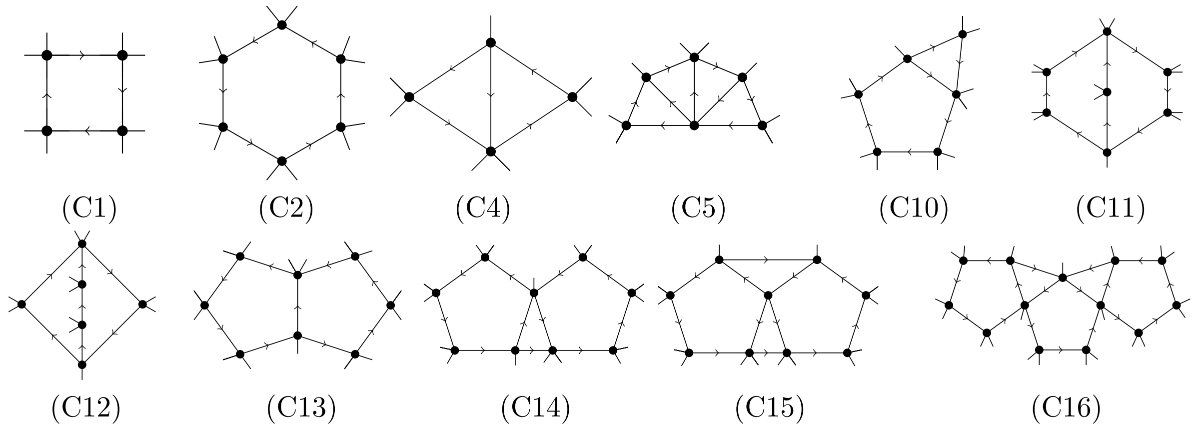


Figure 3: Alon-Tarsi Orientations.

Lemma 3.2. *Let (C, X, ex) be a reducible configuration, and suppose that $x, y \in X$ are nonadjacent vertices with $\mathrm{ex}(x), \mathrm{ex}(y) \geq 1$. Let (C', X', ex') be the configuration where $C' = C + xy$, $X' = X$, and* $\mathrm{ex}'(v) = \begin{cases} \mathrm{ex}(v) & v \notin \{x, y\} \\ \mathrm{ex}(v) - 1 & v \in \{x, y\}, \end{cases}$. *Then the configuration (C', X', ex') is reducible.*

Proof. Let f be the list-size function for C and note that C is $(f, 2)$-choosable. Similarly let f' be the list-size function on the configuration (C', X', ex'), and let L' be an $(f', 2)$-list assignment on $V(C')$. Note that $f'(x) = f(x) + 1$ and $f'(y) = f(y) + 1$. Let $S = L'(x) \cap L'(y)$. If $|S| < 2$, then add at most one element from each of $L'(x)$ and $L'(y)$ to S until $|S| = 2$. Now let $S = \{a, b\}$ such that $a \in L'(x)$ and $b \in L'(y)$, and define a list assignment L on C by removing a from $L'(x)$ and removing b from $L'(y)$. Observe that L is an $(f, 2)$-list assignment and hence there exists an L-coloring of C. Since $L(x) \cap L(y) = \emptyset$, this proper L-coloring of C is also an L'-coloring of C'. $\square$

We will use Lemma 3.2 implicitly by assuming that $C[X]$ appears as an induced subgraph in our minimal counterexample G.

3.1 Reducibility Proofs

In this section, we prove that configurations (C1)–(C21) shown in Figure 2 are reducible.

3.1.1 Alon-Tarsi Theorem

We will use the celebrated Alon-Tarsi Theorem [1] to quickly prove that many of our configurations are reducible. In fact, configurations that are demonstrated in this way are reducible for 4-choosability, not just $(4, 2)$-choosability.

A digraph D is an *orientation* of a graph G if G is the underlying undirected graph of D and D has no 2-cycles; let $d^+_D(v)$ and $d^-_D(v)$ be the out- and in-degree of a vertex v in D. An *Eulerian subgraph* of a digraph D is a subset $S \subseteq E(D)$ such that, for every vertex $v \in V(D)$, the number of outgoing edges of v in S is equal to the number of incoming edges of v in S. Let $EE(D)$ be the number of Eulerian subgraphs of even size and $EO(D)$ be the number of Eulerian subgraphs of odd size.

Theorem 3.3 (Alon-Tarsi Theorem [1]). *Let G be a graph and $f : V(G) \to \mathbb{N}$ a function. Suppose that there exists an orientation D of G such that $d^+_D(v) \le f(v) - 1$ for every vertex $v \in V(G)$ and $EE(D) \ne EO(D)$. Then G is f-choosable.*

We call an orientation an *Alon-Tarsi orientation* if it satisfies the hypotheses of Theorem 3.3. For a configuration (C, X, ex) and the associated list-size function f, it suffices to demonstrate an Alon-Tarsi orientation of C with respect to f. See Figure 3 for a list of Alon-Tarsi orientations of several configurations. One could think that for a vertex v, the outneighbors are vertices that could be colored before v and v could still pick a color not conflicting with them. If there were no cycles in the orientation, the orientation would give an order suitable for the greedy algorithm.

Corollary 3.4. *The following configurations have Alon-Tarsi orientations and hence are reducible:*

(C1), (C2), (C4), (C5), (C10), (C11), (C12), (C13), (C14), (C15), (C16).

3.1.2 Direct Proofs

In the proofs below, we consider a configuration (C, X, ex) with list-size function f and assume that an $(f, 2)$-list-assignment L is given for C. We will demonstrate that each C is L-colorable. Refer to Figure 2 for drawings of the configurations.

First recall the following fact about list-coloring odd cycles.

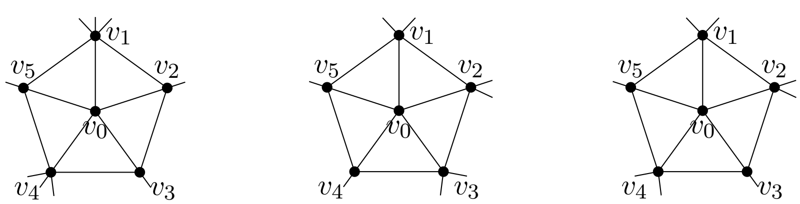
Fact 3.5. *If L is a 2-list assignment of an odd cycle, then there does not exist an L-coloring of the cycle if and only if all of the lists are identical.*

In the proof in this subsection, we use a shorthand notation where for a vertex v_i we denote color $c(v_i)$ by c_i and list $L(v_i)$ by L_i for all i.

Lemma 3.6. (C3) *is a reducible configuration.*

Proof. Let $v_1, \ldots, v_4$ be the vertices of a 4-cycle with chord v_2v_4 and let v_2 and v_4 have external degree 1; the colors c_1 and c_3 are fixed. Each of v_2 and v_4 have at least one color in their lists other than c_1 and c_3. Since $|L_i| \ge 3$ for each $i \in \{2, 4\}$, either one of these vertices has at least two colors available, or $L_2 \cap L_4 = \{c_1, c_3\}$. In either case, we can extend the coloring. □

For the configurations (C6), (C7), and (C8), label the vertices as in Figure 4: label the center vertex v_0 and the outer vertices $v_1, \ldots, v_5$, starting with the vertex directly above v_0, moving clockwise.



(C6) (C7) (C8)

Figure 4: Vertex labels for configurations (C6), (C7), and (C8).

Lemma 3.7. (C6) *is a reducible configuration.*

Proof. The colors c_1 and c_4 are determined. If c_1 and c_4 are both in L_0, then select c_5 from $L_5 \setminus (L_0 \cup \{c_1, c_4\})$; otherwise, select $c_5 \in L_5 \setminus \{c_1, c_4\}$ arbitrarily. Define $L'_0 = L_0 \setminus \{c_1, c_4, c_5\}$, $L'_2 = L_2 \setminus \{c_1\}$, and $L'_3 = L_3 \setminus \{c_4\}$ and note that $|L'_i| \geq 2$ for all $i \in \{0, 2, 3\}$. If $|L'_0| = |L'_2| = 2$, then $L'_0 \neq L'_2$, so the 3-cycle $v_0v_2v_3$ has an L'-coloring by Fact 3.5. □

Lemma 3.8. (C7) *is a reducible configuration.*

Proof. If $L_1 \cap L_2 = \emptyset$, then greedily color v_2 and v_3; what remains is (C4) and the coloring extends. A similar argument works if $L_3 \cap L_2 = \emptyset$.

If $L_1 \cap L_3 = \emptyset$, then $|L_1 \cap L_2| = |L_3 \cap L_2| = 1$. Select $c_1 \in L_1 \setminus L_2$, $c_3 \in L_3 \setminus L_2$. Define $L'_0 = L_0 \setminus \{c_1, c_3\}$, $L'_4 = L_4 \setminus \{c_3\}$, and $L'_5 = L_5 \setminus \{c_1\}$. Observe that we can L'-color the 3-cycle $v_0v_4v_5$ by Fact 3.5 and then select $c_2 \in L_2 \setminus \{c_0\}$.

If there exists a color $a \in L_1 \cap L_3$, start by assigning $c_1 = c_3 = a$ and then assign $c_2 \in L_2 \setminus \{a\}$. Define $L'_0 = L_0 \setminus \{a, c_2\}$, $L'_4 = L_4 \setminus \{a\}$, and $L'_5 = L_5 \setminus \{a\}$. Observe that the 3-cycle $v_0v_4v_5$ has an L'-coloring by Fact 3.5. □

Lemma 3.9. (C8) *is a reducible configuration.*

Proof. If there exists a color $a \in L_1 \cap L_4$, start by assigning $c_1 = c_4 = a$; then greedily color the remaining vertices in the following order: v_2, v_3, v_0, v_5. Otherwise, $L_4 \cap L_1 = \emptyset$.

Suppose that $L_1 \cap L_5 = \emptyset$. Select a color $c_4 \in L_4$. Considering v_4 as an external vertex and ignoring the edges v_1v_5 and v_0v_5, the 4-cycle $v_0v_1v_2v_3$ forms a copy of (C4), which is reducible by Corollary 3.4. Thus, there exists an L-coloring of $v_0, \ldots, v_4$; this coloring extends to v_5 since $L_1 \cap L_5 = \emptyset$. If $L_4 \cap L_5 = \emptyset$, then there exists an L-coloring by a symmetric argument.

Otherwise, there exist colors $a \in L_1 \setminus L_5$ and $b \in L_4 \setminus L_5$; assign $c_1 = a$ and $c_4 = b$. Select $c_2 \in L_2 \setminus \{a\}$. Define $L'_0 = L_0 \setminus \{c_1, c_2, c_4\}$ and $L'_3 = L_3 \setminus \{c_2, c_4\}$. Note that if $|L'_0| = |L'_3| = 1$, then $L_0 \cap L_3 = \{c_2, c_4\}$ and hence $L'_0 \cap L'_3 = \emptyset$. Thus, the coloring extends by greedily coloring v_3, v_0, and v_5. □

Lemma 3.10. (C9) *is a reducible configuration.*

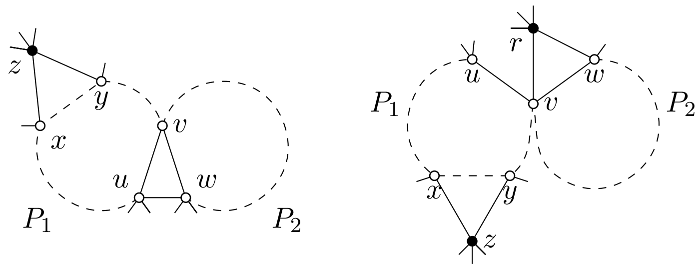
Proof. Consider the vertex v of arbitrary external degree and let $c(v)$ be the color assigned to v. Let u_1 and u_2 be the two neighbors of v in the configuration. If we remove $c(v)$ from the lists on u_1 and u_2, observe that at least two colors remain in every list for every vertex of the 5-cycle. If there is no L-coloring of the configuration, then Fact 3.5 asserts that all lists have size two and contain the same colors; however, this implies that $L(u_1) = L(u_2)$ and $|L(u_1) \cap L(u_2)| = 3$, a contradiction. □

3.1.3 Template Configurations

The configurations (C17)–(C21) are special cases of general constructions called *template constructions.*

Let (C, X, ex) be a configuration with vertices $u, v \in X$. A uv-path P is called a *special uv-path* if all internal vertices of P have degree two in C and external degree two. A uv-path P is called an *extra-special uv-path* if all internal vertices v of P have external degree $\mathrm{ex}(v) = 2$ and degree in C, denoted by $d(v)$, two, except for a consecutive pair xy where $\mathrm{ex}(x) = \mathrm{ex}(y) = 1$, $d(x) = d(y) = 3$, and there is a vertex $z \notin X$ such that z is a common neighbor to x and y, and z is not adjacent to any other vertices in C. Using these special and extra-special paths, we can describe several configurations by the following *templates* (see Figure 5), consisting of

- (B1) a triangle uvw, where $\mathrm{ex}(u) = \mathrm{ex}(w) = 2$, $\mathrm{ex}(v) = 0$, an extra-special uv-path P_1, and a special vw-path P_2, and
- (B2) a triangle vwr, where $\mathrm{ex}(r) = \infty$, $\mathrm{ex}(w) = 1$, $\mathrm{ex}(v) = 0$, a vertex u adjacent to v where $\mathrm{ex}(u) = 2$, an extra-special uv-path P_1, and a special vw-path P_2.



(B1) (B2)

Dotted lines indicate special paths or extra-special paths. Vertices in X are filled with white.

Figure 5: Templates for reducible configurations.

We make some basic observations about special and extra-special paths that will be used to prove that these templates correspond to reducible configurations.

Let P be a special uv-path or an extra-special uv-path. For every color $a \in L(u)$, let $g_P^u(a)$ be the set containing each color $b \in L(v)$ such that assigning $c(u) = a$ and $c(v) = b$ does not extend to an L-coloring of P. Since we can greedily color P starting at u until reaching v, there is at most one color in $g_P^u(a)$. Further, $g_P^u(a) \neq \emptyset$ if and only if this greedy coloring process has exactly one choice for each vertex in P. Thus, if $g_P^u(a) = \{b\}$ then also $g_P^v(b) = \{a\}$.

Since L is an $(f, 2)$-list assignment, adjacent vertices have at most two colors in common. Thus, there are at most two colors $a_1, a_2 \in L(u)$ such that $g_P^u(a_i) \neq \emptyset$. Moreover, observe that if there are two distinct colors $a_1, a_2 \in L(u)$ such that $g_P^u(a_i) \neq \emptyset$, then both a_1 and a_2 are in every list along P and hence $\{a_1, a_2\} \subseteq L(v)$.

If P is an extra-special uv-path with 3-cycle xyz where xy is in the path P, then after a color is assigned to z (as $\mathrm{ex}(z) = \infty$) either one of x or y has three colors available or $|L(x) \cap L(y)| \leq 1$. Therefore, if P is an extra-special uv-path, then there is at most one color $a \in L(u)$ such that $g_P^u(a) \neq \emptyset$.

Lemma 3.11. *All configurations matching the template* (B1) *are reducible.*

Proof. Let (C, X, ex) be a configuration matching the template (B1) and let L be an $(f, 2)$-list assignment.

Let $L(u) = \{a_1, a_2\}$. Since P_1 is an extra-special path, there is at least one $i \in \{1, 2\}$ such that $g_{P_1}^u(a_i) = \emptyset$. Assign $c(u) = a_i$, select $c(w) \in L(w) \setminus \{a_i\}$ and $c(v) \in L(v) \setminus \big(\{c(u), c(w)\} \cup g_{P_1}^w(c(w))\big)$; the coloring extends to P_1 and P_2. $\square$

Corollary 3.12. *The configurations* (C17), (C18)*, and* (C19) *match the template* (B1)*, and hence they are reducible.*

Lemma 3.13. *All configurations matching the template* (B2) *are reducible.*

Proof. Let (C, X, ex) be a configuration matching the template (B2) and let L be an $(f, 2)$-list assignment. Let $c(r)$ be the unique color in the list $L(r)$. Let $L(u) = \{a_1, a_2\}$. Since P_1 is an extra-special path, there is at least one $i \in \{1, 2\}$ such that $g^u_{P_1}(a_i) = \emptyset$. Assign $c(u) = a_i$.

If $c(r) \notin L(v)$, then select $c(w) \in L(w)$, and $L(v) \in L(v) \setminus \left(\{c(u), c(w)\} \cup g^w_{P_2}(c(w))\right)$; the coloring extends to P_1 and P_2.

If $c(r) \in L(v)$, then select $c(w) \in L(w) \setminus L(v)$; observe $c(w) \neq c(r)$. There exists a color $c(v) \in L(v) \setminus \left(\{c(r), c(u)\} \cup g^w_{P_2}(c(w))\right)$; the coloring extends to P_1 and P_2. $\square$

Corollary 3.14. *Using Lemma 3.2, the configurations* (C20) *and* (C21) *match the template* (B2)*, and hence they are reducible.*

4 No Chorded 5-Cycle

In this section we show the case of forbidding chorded 5-cycles from Theorem 1.7.

Theorem 4.1. *If G is a plane graph not containing a chorded 5-cycle, then G is* $(4, 2)$*-choosable.*

Proof. Let G be a counterexample minimizing $n(G)$ among all plane graphs avoiding chorded 5-cycles with a $(4, 2)$-list assignment L such that G is not L-choosable. Observe that $n(G) \geq 4$; in fact, $\delta(G) \geq 4$. Since G is a minimal counterexample, G does not contain any of the reducible configurations (C9)–(C21). If (C, X, ex) is a reducible configuration, then by Lemma 3.2 C does not appear as a subgraph of G where $d_G(x) \leq d_C(x) + \mathrm{ex}(x)$ for all $x \in V(C)$. Further, the configurations (C13)–(C21) are large enough that we must consider configurations that are formed by identifying certain pairs of vertices in these configurations. In Appendix A, we concretely check all vertex pairs that avoid creating a chorded 5-cycle and find that all resulting configurations are reducible.

For each $v \in V(G)$ and $f \in F(G)$ define initial charges $\mu_0(v) = d(v) - 6$ and $\mu_0\mu_0(f) = 2\ell(f) - 6$. By Euler's Formula, the sum of initial charges is -12. After charges are initially assigned, the only elements with negative initial charge are 4-vertices and 5-vertices. Since chorded 5-cycles are forbidden, there is no 3-fan in G and every 4-face is adjacent to only 4^+-faces. The possible arrangements of 3-, 4^+-, or 5^+-faces incident to 4- and 5-vertices are shown in Figure 6.

Sequentially apply the following discharging rules. Note that, for a vertex v and a face f, we define $\mu_i(v)$ and $\mu_i(f)$ to be the charge on v and f, respectively, after applying rule (Ri).

(R1) Let v be a 4-vertex and f be a 4^+-face incident to v. If f is adjacent to a 3-face that is also incident to v, then f sends charge 1 to v; otherwise, f sends charge $\frac{1}{2}$ to v.

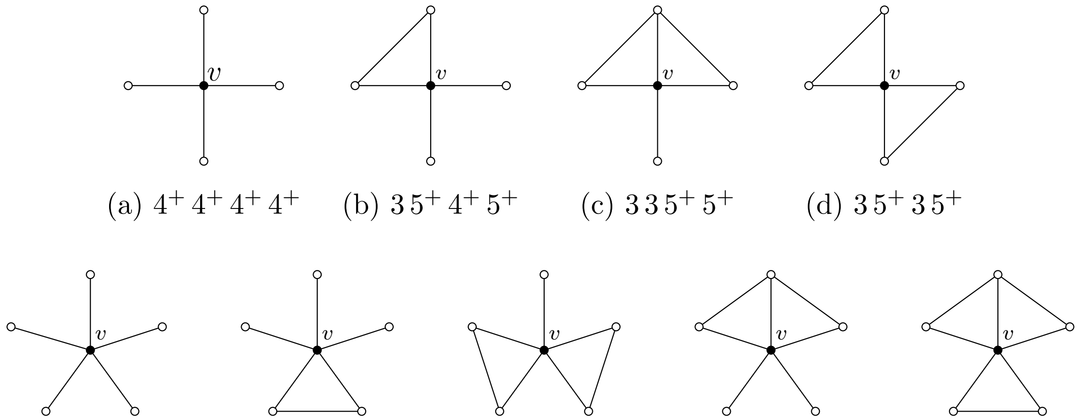
(R2) Let v be a 5-vertex. If f is a 4^+-face incident to v, then f sends charge $\frac{1}{2}$ to v.

A face f is a *needy* face if $\mu_2(f) < 0$; otherwise, f is *non-needy.*

(R3) If v is a 5-vertex incident to a needy 5-face f, then v sends charge $\frac{1}{2}$ to f.

A vertex v is a *needy* vertex if $\mu_3(v) < 0$; otherwise, v is *non-needy.*

(R4) If f is a non-needy 5^+-face incident to a needy 5-vertex v, then f sends charge $\frac{1}{2}$ to v.



(a) $4^+\,4^+\,4^+\,4^+$ (b) $3\,5^+\,4^+\,5^+$ (c) $3\,3\,5^+\,5^+$ (d) $3\,5^+\,3\,5^+$

(e) $4^+\,4^+\,4^+\,4^+\,4^+$ (f) $3\,5^+\,4^+\,4^+\,5^+$ (g) $3\,5^+\,3\,5^+\,5^+$ (h) $3\,3\,5^+\,4^+\,5^+$ (i) $3\,3\,5^+\,3\,5^+$

Figure 6: Possible cyclic arrangements of 3-, 4^+-, and 5^+-faces incident to 4- and 5-vertices

We show that $\mu_4(v) \geq 0$ for each vertex v and $\mu_4(f) \geq 0$ for each face f. Since the total charge was preserved during the discharging rules, this contradicts the negative charge sum from the initial charge values. We begin by considering the charge distribution after applying (R1) and (R2).

Let v be a vertex. If v is a 4-vertex, then $\mu_0(v) = -2$ and v receives total charge at least 2 from its neighboring faces by (R1). Furthermore, v is not affected by any rules after (R1), so $\mu_4(v) \geq 0$. If v is a 6^+-vertex, then $\mu_0(v) \geq 0$ and v is not affected by any other rules, so $\mu_4(v) \geq 0$. If v is a 5-vertex, then $\mu_0(v) = -1$ and v receives total charge at least 1 from its neighboring faces by (R2). Therefore, for any vertex v, $\mu_2(v) \geq 0$.

Let f be a face. If f is a 3-face, then $\mu_0(f) = 0$ and f is not affected by any rule, so $\mu_4(f) = 0$. If f is a 4-face, then $\mu_0(f) = 2$. In (R1) and (R2), the only faces that send charge 1 to a single vertex are adjacent to a 3-face. A 4-face adjacent to a 3-face is a chorded 5-cycle, which is forbidden by assumption, so f sends charge at most $\frac{1}{2}$ to each vertex. Since 4-faces are not affected by rules (R3)–(R4), $\mu_4(f) \geq 0$. If f is a 6^+-face, then f has at least as much initial charge as it has incident vertices. If v is a 4-vertex incident to f, then f sends charge at most 1 to v by (R1) and does not send any charge to v by rules (R2)–(R4). If v is a 5-vertex incident to f, then f sends charge $\frac{1}{2}$ to v by (R1), and possibly another charge $\frac{1}{2}$ by (R4), and does not send charge to v by (R1) or (R3). Thus f sends charge at most 1 to each incident vertex, and $\mu_4(f) \geq 0$.

If f is a 5-face, then $\mu_0(f) = 4$ and f sends charge at most 1 to each incident vertex by (R1) and (R2). Observe that if $\mu_2(f) = -1$, then f is incident to five 4-vertices and f is adjacent to at least one 3-face; this forms (C9), a contradiction. Therefore, we have the following claim about the structure of a needy 5-vertex.

Claim 4.2. *If f is a needy 5-face, then $\mu_2(f) = -\frac{1}{2}$ and f is adjacent to exactly one 5-vertex.*

We now consider the charge distribution after applying (R3). If f is a needy 5-face, then $\mu_2(f) = -\frac{1}{2}$ and f is adjacent to exactly one 5-vertex, so $\mu_3(f) = 0$. No faces lose charge in (R3), therefore $\mu_3(f) \geq 0$ for any face f.

Claim 4.3. *If v is a needy 5-vertex, then v is incident to three 3-faces, two 4^+-faces, and exactly one needy 5-face; hence $\mu_3(v) = -\frac{1}{2}$.*

Proof. Suppose that v is a vertex such that $\mu_3(v) < 0$, and consider the cyclic arrangement of 3- and 4^+-faces about v.

Case 1: v is incident to at least four 4^+-faces (Figures 6(e) and 6(f)). Since $\mu_2(v) \geq 1$ and $\mu_3(v) < 0$, v is incident to at least three needy 5-faces. Hence two of the needy 5-faces are adjacent, forming (C13), a contradiction.

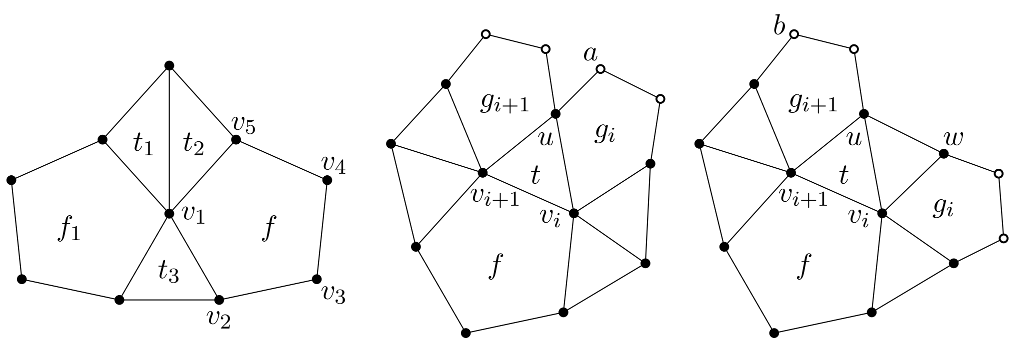
Case 2: v is incident to two non-adjacent 3-faces and three 4^+-faces (Figure 6(g)). Since $\mu_2(v) = \frac{1}{2}$ and $\mu_3(v) < 0$, v is incident to two needy 5-faces, f_1 and f_2. If these two faces are adjacent, then they form (C13), a contradiction. Otherwise, they share a 3-face t as a neighbor and all vertices incident to f_1, f_2, and t other than v are 4-vertices, so the vertices incident to f_1 and t form (C10), a contradiction.

Case 3: v is incident to two adjacent 3-faces and three 4^+-faces (Figure 6(h)). Since $\mu_2(v) = \frac{1}{2}$ and $\mu_3(v) < 0$, v is incident to two needy 5-faces, f_1 and f_2. If f_1 and f_2 are adjacent then they form (C13), a contradiction. Thus, f_1 and f_2 are not adjacent, but they are each adjacent to a 3-face incident to v. Since f_i is needy for each $i \in \{1, 2\}$, f_i sent charge 1 to every 4-vertex incident to f_i. By (R1), every 4-vertex incident to f_i is incident to a 3-face adjacent to f_i. Therefore, f_1 is adjacent to a 3-face that does not share any vertices with the the two 3-faces incident to v, forming one of (C20) or (C21), a contradiction.

Case 4: v is incident to three 3-faces and two 4^+-faces (Figure 6(i)). If v is incident to two needy 5-faces f_1 and f_2, then the 3-face t adjacent to both f_1 and f_2 is incident to two 4-vertices, and the vertices incident to f_1 and t form (C10), a contradiction.

Therefore, v is incident to exactly one needy 5-face, as claimed. □

By (R4), every needy 5-vertex receives charge $\frac{1}{2}$ from its unique incident non-needy 5^+-face, so $\mu_4(v) \geq 0$ for every vertex v. Each needy 5-face has nonnegative charge after (R3), so if $\mu_4(f) < 0$ for some 5-face f, then f sends charge by (R4), and thus is non-needy.



(a) A 5-face f with $\mu_4(f) < 0$. (b) Claim 4.5, Case 1. (c) Claim 4.5, Case 2.

Figure 7: Special cases for a 5-face f with $\mu_4(f) < 0$.

Consider the Figure 7(a), where f is a 5-face with $\mu_4(f) < 0$, f is incident to vertices $v_1, \ldots, v_5$, v_1 is a needy 5-vertex, and f_1 is the needy 5-face incident to v_1. Let t_1 and t_2 be the adjacent pair of 3-faces incident to v_1 with t_1 adjacent to f_1 and t_2 adjacent to f; let t_3 be the other 3-face incident to v_1. We make two basic claims about this arrangement.

Claim 4.4. *The vertex v_2 adjacent to v_1 and incident to t_3 is a 5^+-vertex.*

Proof. If v_2 is a 4-vertex, then the vertices incident to f_1 and t_3 form (C10), a contradiction. □

Claim 4.5. *If v_i and v_{i+1} are consecutive vertices on the border of f, then at most one of v_i and v_{i+1} is needy.*

Proof. Suppose that two consecutive vertices v_i and v_{i+1} are needy 5-vertices. Let g_i and g_{i+1} be the needy 5-faces incident to v_i and v_{i+1}, respectively. Since both v_i and v_{i+1} have three incident 3-faces, f is adjacent to a 3-face t across the edge v_iv_{i+1}. Let u be the third vertex incident to t and consider two cases.

Case 1: t is not in a diamond (Figure 7(b)). Since g_i is needy, the vertex a adjacent to u and incident to g_i (with $a \neq v_i$) is a 4-vertex and is incident to a 3-face t_i such that t_i is adjacent to g_i. The vertices incident to g_i, g_{i+1}, t, and t_i form one of (C15) or (C19), a contradiction.

Case 2: t is in a diamond (Figure 7(c)). Let w be the fourth vertex in the diamond and assume, without loss of generality, that v_i is adjacent to w. Let b be the vertex incident to g_{i+1} that is not adjacent to u or v_{i+1} along the boundary of g_{i+1}; since g_{i+1} is needy, there is a 3-face t_{i+1} incident to b and adjacent to g_{i+1}. The vertices v_i and w and those incident to g_{i+1} and t_{i+1} form one of (C17) or (C18), a contradiction. □

By Claim 4.5, f is incident to at most two needy vertices, and by Claim 4.4, v_2 is non-needy. If f is incident to exactly one needy 5-vertex, then v_3, v_4, and v_5 are 4-vertices since $\mu_2(f) = 0$, but then the vertices incident to f and f_1 form (C14), a contradiction.

Therefore, f is incident to two needy vertices, and since v_2 is a 5^+-vertex by Claim 4.4, f is incident to exactly two 4-vertices. Each of these receives charge 1, so $\mu_4(f) = -\frac{1}{2}$. By Claim 4.5, the needy vertices incident to f consist of v_1 and exactly one of v_3 or v_4. The needy 5-vertex v_i other than v_1 is also incident to three 3-faces t_4, t_5, and t_6, where t_4 and t_5 form a diamond with t_4 adjacent to f. By Claim 4.4, the vertex adjacent to v_i and incident to both f and t_6 is a non-needy 5^+-vertex. The only non-needy 5^+-vertex incident to f is v_2, and hence v_3 is a needy 5-vertex and t_4 is incident to v_4. If v_2 is a 6^+-vertex, then $\mu_4(f) \geq 0$. Therefore, there is a unique arrangement of needy vertices, 4-vertices, and a 5-vertex about a 5-face f with $\mu_4(f) < 0$ (Figure 8). For $i \in \{1, 3\}$, let f_i be the needy 5-face incident to the needy 5-vertex v_i.

The vertices incident to f, f_1, f_3, t_3, and t_6 form (C16), so this arrangement does not appear within G; hence $\mu_4(f) \geq 0$ for all 5-faces f. Therefore, every vertex and face has nonnegative charge after (R4), contradicting the negative initial charge sum. Thus, a minimal counterexample does not exist and every plane graph with no chorded 5-cycle is $(4, 2)$-choosable. □

5 No Chorded 6-Cycle

In this section we show the case of forbidding chorded 6-cycles from Theorem 1.7.

Theorem 5.1. *If G is a plane graph not containing any chorded 6-cycle, then G is $(4, 2)$-choosable.*

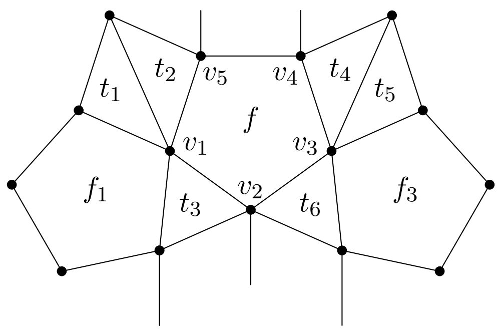


Figure 8: A non-needy 5-vertex v_2 incident to a non-needy 5-face f with $\mu_4(f) < 0$.

We prove the following strengthened statement.

Theorem 5.2. *Let G be a plane graph with no chorded 6-cycle, and let P be a subgraph of G, where P is isomorphic to one of P_1, P_2, P_3, or K_3, and all vertices in $V(P)$ are incident to a common face f. Let L be a $(4,2)$-list assignment of $G - P$ and let c be a proper coloring of P. There exists an extension of c to a proper coloring of G such that $c(v) \in L(v)$ for all $v \in V(G - P)$.*

Proof. Suppose that there exists a counterexample. Select a counterexample (G, P, L, c) by minimizing $n(G) - \frac{1}{4}n(P)$ and subject to that by minimizing the number of edges among all chorded 6-cycle free plane graphs, G, with a subgraph P isomorphic to a graph in $\{P_1, P_2, P_3, K_3\}$, a proper coloring c of P, and a $(4,2)$-list assignment L of $G - P$ such that c does not extend to an L-coloring of G. We will refer to the vertices of P as *precolored vertices.*

Claim 5.3. *G is 2-connected.*

Proof. If G is disconnected, then each connected component can be colored separately by the minimality of G. Suppose that G has a cut-vertex v. Then there exist connected subgraphs G_1 and G_2 where $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{v\}$, $n(G_1) < n(G)$, and $n(G_2) < n(G)$. We can assume without loss of generality that G_1 contains at least one vertex of P, so let S_1 be the subgraph of P contained in G_1. Let $S_2 = \{v\} \cup (V(G_2) \cap V(P))$.

Since (G, P, L, c) is a minimal counterexample, there is an L-coloring c_1 of G_1 that extends the coloring on S_1. Using the color prescribed by c_1 on v, there exists an L-coloring c_2 of G_2 that extends the coloring on S_2. The colorings c_1 and c_2 form an L-coloring of G, a contradiction. □

Claim 5.4. *G has no separating 3-cycles.*

Proof. Suppose that $P' = v_1v_2v_3$ is a separating 3-cycle of G. Let G_1 be the subgraph of G given by the exterior of P' along with P', and let G_2 be the subgraph of G given by the interior of P' along with P'. Since P' is separating, $n(G_1) < n(G)$ and $n(G_2) < n(G)$.

Since the vertices in P share a common face, we can assume without loss of generality that $V(P) \subseteq V(G_1)$. Since (G, P, L, c) is a minimal counterexample, there exists an L-coloring c_1 of G_1. Assign the colors from c_1 to P'. Then there exists an L-coloring of G_2 extending the colors on P', and together c_1 and c_2 form an L-coloring of G, a contradiction. □

Claim 5.5. *If $v \in V(P)$ such that $V(P) \subseteq N[v]$, then the subgraph of G induced by $N(v)$ is not isomorphic to any graph in $\{P_1, P_2, P_3, K_3\}$.*

Proof. Suppose that there exists a vertex $v \in V(P)$ where all precolored vertices are in $N[v]$ and the subgraph $G[N(v)]$ is isomorphic to a subgraph in $\{P_1, P_2, P_3, K_3\}$. Since $|N_G[v]| \le 4$, there exists an L-coloring c' of $G[N[v]]$. Since (G, P, L, c) is a minimal counterexample, c' extends to an L-coloring of G', which in turn extends to an L-coloring of G, a contradiction. □

Claim 5.6. *If $v \in V(P)$ has $d_G(v) \le 2$, then $d_G(v) = 2$ and P is isomorphic to P_1, P_2, or P_3.*

Proof. By Claim 5.3, $d_G(v) \ne 1$. If $d_G(v) = 2$ and $P \cong K_3$, then $G[N_G(v)]$ is isomorphic to P_2, contradicting Claim 5.5. □

Claim 5.7. *P is isomorphic to P_3.*

Proof. Suppose that P is not isomorphic to either P_3 or K_3. If P is isomorphic to P_1, then the vertex v of P has two distinct neighbors u_1 and u_2 that are on the same face as v; let $U = \{u_1, u_2\}$. If P is isomorphic to P_2, then some vertex v in P has a neighbor u_1 not in P that shares a face with the edge in P; let $U = \{u_1\}$. Let P' be induced by $V(P) \cup V(U)$. Notice $|P'| = 3$ hence it is isomorphic to P_3 or K_3. There exists a proper coloring c' of P' that extends the coloring on P. But then (G, P', L, c') has $n(G) - \frac{1}{4}n(P') < n(G) - \frac{1}{4}n(P)$, so there exists an L-coloring of G that extends c', a contradiction.

If P is isomorphic to K_3, we can remove any edge e with both vertices in P. By minimality of G, there exists an L-coloring extending c in $G - e$ but it is also an L-coloring of G since both endpoints of e have different color in c, a contradiction. □

Claim 5.8. *If $v \in V(G - P)$, then $d_G(v) \ge 4$.*

Proof. Suppose that $v \in V(G - P)$ has degree $d(v) \le 3$. Then $G - v$ is a planar graph with no chorded 7-cycle containing a precolored subgraph P and a list assignment L. Since (G, P, L, c) is a minimum counterexample, $G - v$ has an L-coloring. However, v has at most three neighbors and at least four colors in the list $L(v)$. Thus, there is an extension of the L-coloring of $G - v$ to an L-coloring of G, a contradiction. □

Claim 5.4 helps us to prove the following adjacencies of faces.

Claim 5.9. *If a 5-face f_5 is adjacent to a triangle face f_3 then there is a 2-vertex incident to both of them. Moreover, every 5-face is adjacent to at most one triangle face.*

Proof. Let f_5 be a 5-face bounded by a cycle v_1, v_2, v_3, v_4, v_5. Let f_3 be a 3-face with vertices v_1v_2x. Since G has no chorded 6-cycle, $x \in \{v_3, v_4, v_5\}$. If $x = v_4$, then Claim 5.4 implies $v_1v_4v_5$ and $v_2v_3v_4$ are also triangular faces and we obtain a contradiction with Claim 5.8 since G is a graph on 5 vertices and only one 4^+-vertex. By symmetry between v_3 and v_5 suppose that $x = v_3$. Then v_2 is the desired 2-vertex and we are done.

Suppose that f_5 is adjacent to two triangle faces. Each of them has a 2-vertex in common with f_5. By symmetry assume these 2-vertices are v_3 and v_5. Then v_1v_4 and v_2v_4 are edges and $v_1v_2v_4$ is a triangle face adjacent to f_5 not sharing any 2-vertex with f_5, which is a contradiction. □

Claim 5.10. *If two 4-faces are adjacent then they are both incident to the same 2-vertex*

Proof. Let f_1 and f_2 be adjacent 4-faces bounded by cycles v_1, v_2, v_3, v_4 and v_1, v_2, x_2, x_1 respectively. Since G does not contain chorded 6-cycles, f_1 and f_2 must share at least 3 vertices. If they share four vertices, we get a contradiction with Claim 5.8. By symmetry we assume x_1 is v_3 or v_4. If $x_1 = v_3$ then v_2, x_2, v_3 and v_1, v_3, v_4 are triangular faces and we obtain a contradiction with Claim 5.8. Hence $x_1 = v_4$ and v_1 has degree two. □

Claim 5.11. *If a 4-face f shares two or more edges with triangular faces, then it shares edges with exactly two. Moreover, there is a 3-vertex $v \in V(P)$ incident to both triangular faces and to f.*

Proof. Let f be a 4-face bounded by a cycle v_1, v_2, v_3, v_4 and assume that v_1, v_2, x is a triangular face. If $x \in \{v_3, v_4\}$ then G would violate Claim 5.4 or Claim 5.8. Hence x is not a vertex of the cycle.

Suppose for contradiction v_3, v_4, y is also a triangular face. Since G does not contain chorded 6-cycles, $x = y$. By Claim 5.4, G has only 5 vertices and contradicts Claim 5.8. Hence f is adjacent to at most two triangles.

Assume that v_4, v_1, y is a triangular face. Since G does not contain chorded 6-cycles, $x = y$. Then v_1 is the desired 3-vertex since by Claim 5.8, $v_1 \in V(P)$. □

Claim 5.12. *Every* 3*-vertex is adjacent to at most two triangular faces.*

Proof. Let v be a 3-vertex adjacent to three triangular faces. Note that these are all the faces containing v. This contradicts that $P = P_3$. □

Since G is a minimal counterexample, G does not contain any of the reducible configurations. Specifically, we use the fact that G avoids (C3) and (C4) (see Figure 2), where no removed vertex is precolored.

For each $v \in V(G) - V(P)$, $p \in V(P)$, and $f \in F(G)$ define initial charge $\mu_0(v) = d(v) - 4$, $\mu_0(p) = d(p) - 4 + \frac{22}{9}$ and $\mu_0(f) = \ell(f) - 4$. By Euler's Formula, the initial charge sum is $-8 + \frac{22}{3} = -\frac{2}{3}$. Since $\delta(G - P) \geq 4$, the only elements of negative charge are 3-faces. Since a chorded 6-cycle is forbidden, $\delta(G - P) \geq 4$, and Claim 5.4, the clusters (see Figure 1) are triangles (K3), diamonds (K4), 3-fans (K5a), 4-wheels (K5b), and 4-fans with end vertices identified (K5c). Specifically note that the 4-fan (K6b) contains a chorded 6-cycle, so at most three 3-faces in a cluster share a common vertex, unless they form a 4-wheel (K5b) and the common vertex is the 4-vertex in the center of the wheel.

Apply the following discharging rules, as shown in Figure 9.

(R1) If p is a 2-vertex incident with two 4-faces, then p sends charge $\frac{2}{9}$ to each of them.

(R2) If f is a 3-face and e is an incident edge, then let g be the face adjacent to f across e.

- (R2a) If g is a 5^+-face, then f pulls charge $\frac{1}{3}$ from g "through" the edge e.
- (R2b) If g is a 4-face adjacent to one 3-face, then let e_1, e_2, and e_3 be the other edges incident to g. For each $i \in \{1, 2, 3\}$, let h_i be the face adjacent to g across e_i. For each $i \in \{1, 2, 3\}$, the face f pulls charge $\frac{1}{9}$ from the face h_i "through" the edges e and e_i.
- (R2c) If g is a 4-face adjacent to two 3-faces, then let e_1 and e_2 be edges of g not incident to 3-faces. For each $i \in \{1, 2\}$, let h_i be the face adjacent to g across e_i. For each $i \in \{1, 2\}$, the face f pulls charge $\frac{1}{18}$ from the face h_i "through" the edges e and e_i. Let v be the vertex shared by g, f and the other 3-face. Then v send charge $\frac{2}{9}$ to f through e.

(R3) Let v be a 5^+-vertex or precolored, and let f be an incident 3-face.

(R3a) If v is a 5-vertex that is not precolored , then v sends charge $\frac{1}{3}$ to f.

(R3b) If v is a 6^+-vertex or precolored, then v sends charge $\frac{4}{9}$ to f.

(R4) If X is a cluster, then every 3-face in X is assigned the average charge of all 3-faces in X.

Notice that precolored vertices behave similarly to 6^+-vertices.

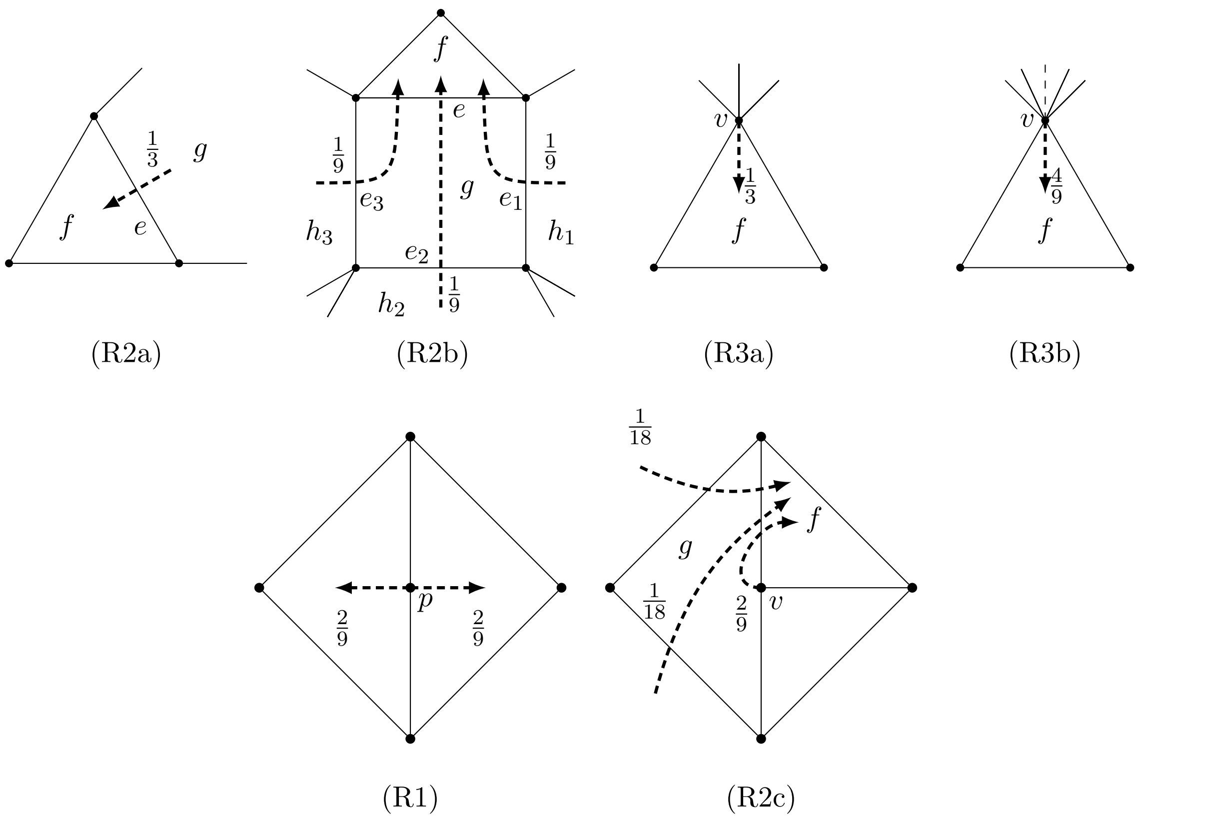


Figure 9: Discharging rules in the proof of Theorem 5.1.

Notice that the rules preserve the sum of the charges. Let $\mu_i(v)$ and $\mu_i(f)$ denote the charge on a vertex v or a face f after rule (Ri). We claim that $\mu_4(v) \geq 0$ for every vertex v and $\mu_4(f) \geq 0$ for every face f; since the total charge sum is preserved by the discharging rules, this contradicts the negative charge sum from the initial charge values.

If v is a 6^+-vertex, then by (R3b) v loses charge $\frac{4}{9}$ to each incident 3-face. Since G avoids chorded 6-cycles, v is incident to at most $\lfloor \frac{3}{4} d(v) \rfloor$ 3-faces. Thus $\mu_4(v)$ satisfies

$$\mu_4(v) \geq d(v) - 4 - \frac{4}{9} \left\lfloor \frac{3}{4} d(v) \right\rfloor \geq d(v) - 4 - \frac{4}{9} \cdot \frac{3}{4} d(v) = \frac{2}{3} d(v) - 4 \geq 0.$$

Let v be a 5^--vertex not in P. If v is a 4-vertex, then v is not involved in any rule, so the resulting charge is 0. If v is a 5-vertex, then by (R3a) v loses charge $\frac{1}{3}$ to each incident 3-face.

Since G avoids chorded 6-cycles, v is incident to at most three 3-faces, so

$$\mu_4(v) \geq d(v) - 4 - \frac{1}{3} \cdot 3 = d(v) - 5 = 0.$$

Therefore, $\mu_4(v) \geq 0$ for every vertex v not in P.

Let v be a 5^--vertex in P. If v is a 5-vertex or 4-vertex then rule (R3b) applies at most $d(v)$ times and

$$\mu_4(v) \geq d(v) - 4 + \frac{22}{9} - \frac{4}{9} \cdot d(v) > 0.$$

If v is a 3-vertex then by Claim 5.12 (R2c) and (R3b) apply at most twice and

$$\mu_4(v) \geq d(v) - 4 + \frac{22}{9} - \frac{6}{9} \cdot 2 > 0.$$

If v is a 2-vertex, then at most one of (R1) and (R3b) apply and if (R3b) applies, it applies only once. Hence

$$\mu_4(v) \geq d(v) - 4 + \frac{22}{9} - \frac{4}{9} = 0.$$

Therefore all vertices $v \in V(G)$ have $\mu_4(v) \geq 0$.

Let f be a 4-face. If (R2b) or (R2c) applies to f then it must be adjacent to another 4-face and by Claim 5.10 and they share a 2-vertex v. Hence (R1) applies to f and v and the charge lost in (R2b) and (R2c) is at most the charge gained in (R1). Thus, $\mu_4(f) \geq 0$ for every 4-face f.

If f is a 6^+-face, then f loses charge at most $\frac{1}{3}$ through each edge by (R2a), (R2b), or (R2c), so

$$\mu_4(f) \geq \ell(f) - 4 - \frac{1}{3}\ell(f) = \frac{2}{3}\ell(f) - 4 \geq 0.$$

Therefore, $\mu_4(f) \geq 0$ for every 6^+-face f.

Let f be a 5-face. If f is not adjacent to a 3-face, f loses no charge by (R2a), but could lose charge using (R2b) and (R2c), so

$$\mu_4(f) \geq \ell(f) - 4 - \frac{1}{9}\ell(f) = \frac{8}{9}\ell(f) - 4 \geq 0.$$

If f is adjacent to a 3-face, by Claim 5.9 it is adjacent to at most one and it shares at most two edges with it, so (R2a) is applies at most twice while at most $\frac{1}{9}$ charge is lost through each of the remaining three edges by (R2b) and (R2c) and we obtain

$$\mu_4(f) \geq \ell(f) - 4 - \frac{1}{9} \cdot 3 - \frac{1}{3} \cdot 2 = 0.$$

Therefore, $\mu_4(f) \geq 0$ if f is a 5-face.

All objects that start with nonnegative charge have nonnegative charge after the discharging process. It remains to show that each cluster of 3-faces receives enough charge to result in a nonnegative charge sum. Observe that the rules (R2a), (R2b), and (R2c) guarantee that if a triangle f is sharing an edge e with a 4^+-face, then f receives total charge $\frac{1}{3}$ trough e.

Case 1: (K3) Let f be an isolated 3-face. The three adjacent faces g_1, g_2, and g_3 are all 4^+-faces. By (R2), f receives charge $\frac{1}{3}$ through each incident edge, so $\mu_4(f) = -1 + 3 \cdot \frac{1}{3} = 0$.

Case 2: (K4) Let f_1 and f_2 be 3-faces in a diamond cluster (K4). Then f_1 is adjacent to two 4^+-faces g_1 and g_2, and f_2 is adjacent to two 4^+-faces h_1 and h_2. By (R2), the cluster receives charge $\frac{1}{3}$ through each of the four edges on the boundary of the diamond. Since $\mu_0(f_1) + \mu_0(f_2) = -2$, the charge value on the diamond after rule (R2) is $-\frac{2}{3}$. Since G contains no (C3), there is a 5^+-vertex v incident to both f_1 and f_2. If v is a 5-vertex, then by (R3a), f_1 and f_2 each receive charge $\frac{1}{3}$, and the resulting charge on the diamond is zero. If v is a 6^+-vertex, then by (R3b), f_1 and f_2 each receive charge $\frac{4}{9}$, and the resulting charge on the diamond is positive.

Case 3: (K5a) Let f_1, f_2, and f_3 be 3-faces in a 3-fan cluster (K5a), where f_2 is adjacent to both f_1 and f_3. The initial charge on this cluster is -3. There are five edges on the boundary of this cluster, so by (R2) the cluster receives charge $\frac{5}{3}$, resulting in charge $-\frac{4}{3}$ after (R2). Note that the face f_2 is adjacent to both f_1 and f_3. Since G contains no (C3), there exists a 5^+-vertex v incident to both f_1 and f_2, and there exists a 5^+-vertex u incident to both f_2 and f_3. If $v \neq u$, then by (R3) v sends charge at least $\frac{1}{3}$ to each of f_1 and f_2 and u sends charge at least $\frac{1}{3}$ to each of f_2 and f_3, resulting in a nonnegative charge on the 3-fan. If $v = u$ and v is a 6^+-vertex, then by (R3b) v sends charge $\frac{4}{9}$ to each face f_1, f_2, and f_3, resulting in a nonnegative charge on the 3-fan. Otherwise, suppose that $v = u$ and v is a 5-vertex. Since G contains no (C4), there exists another 5^+-vertex w incident to at least one of f_1 and f_2. By (R3a) v sends charge $\frac{1}{3}$ to each of f_1, f_2, and f_3, and by (R3) w sends charge at least $\frac{1}{3}$ to at least one of f_1 and f_2, resulting in a nonnegative charge on the 3-fan.

Case 4: (K5b) Let f_1, f_2, f_3, and f_4 be 3-faces in a 4-wheel (K5b). The initial charge on this cluster is -4. There are four edges on the boundary of this cluster, so by (R2) the cluster receives charge $\frac{4}{3}$, resulting in charge $-\frac{8}{3}$ after (R2). Let v be the 4-vertex incident to all four 3-faces. Let u_1, u_2, u_3, and u_4 be the vertices adjacent to v, ordered cyclically such that vu_iu_{i+1} is the boundary of the 3-face f_i for $i \in \{1, 2, 3\}$ and vu_4u_1 is the boundary of f_4. Since G contains no (C3) and $d(v) = 4$, each u_i is a 5^+-vertex. By (R3), each u_i sends charge at least $\frac{2}{3}$ to the cluster, resulting in a nonnegative total charge.

Case 5: (K5c) Let f_1, f_2, f_3, and f_4 be 3-faces in a 4-strip with identified vertices as in (K5c). The initial charge on this cluster is -4. Let v, u_1, u_2, u_3, and u_4 be the vertices in the 4-strip, where v is incident to only f_1 and f_4, u_1 is incident to only f_1 and f_2, u_2 is incident to f_2, f_3, and f_4, u_3 is incident to f_1, f_2, and f_3, and u_4 is incident to only f_3 and f_4. There are six edges on the boundary of this cluster, so by (R2) the cluster receives charge $\frac{6}{3}$, resulting in charge $-\frac{6}{3} = -2$ after (R2).

Since f_2 and f_3 form a diamond, and G contains no (C3), one of u_2 and u_3 is a 5^+-vertex. Without loss of generality, assume u_3 is a 5^+-vertex. Since f_3 and f_4 form a diamond, and G contains no (C3), one of u_2 and u_4 is a 5^+-vertex. If u_2 is a 5^+-vertex, then by (R3), the cluster receives charge at least $\frac{3}{3} + \frac{3}{3}$ from u_2 and u_3, which results in nonnegative total charge. Otherwise, u_2 is a 4-vertex and u_4 is 5^+-vertex. If u_3 is a 6^+-vertex, then by (R3), the cluster receives charge at least $\frac{4}{3} + \frac{2}{3}$ from u_3 and u_4. If u_3 is a 5-vertex, then since f_1 and f_2 form a diamond and G contains no (C4), one of v and u_1 is a 5^+-vertex. By (R3), the cluster receives charge at least $\frac{3}{3} + \frac{2}{3} + \frac{2}{3}$ from u_3 and u_4 and one of v and u_1. In either case, the final charge is nonnegative.

We have verified that the total charge after discharging is nonnegative, contradicting the negative initial charge sum. Thus, a minimal counterexample does not exist and every planar graph with no chorded 6-cycle is $(4,2)$-choosable. □

6 No Chorded 7-Cycle

Theorem 6.1. *If G is a plane graph not containing a chorded 7-cycle, then G is $(4,2)$-choosable.*

We prove the following strengthened statement:

Theorem 6.2. *Let G be a plane graph with no chorded 7-cycle, and let P be a subgraph of G, where P is isomorphic to one of $P_1, P_2, P_3,$ or K_3, and all vertices in $V(P)$ are incident to a common face f. Let L be a $(4,2)$-list assignment of $G - P$ and let c be a proper coloring of P. There exists an extension of c to a proper coloring of G such that $c(v) \in L(v)$ for all $v \in V(G-P)$.*

Proof. Suppose that there exists a counterexample. Select a counterexample (G, P, L, c) by minimizing $n(G) - \frac{1}{4}n(P)$ among all chorded 7-cycle free plane graphs, G, with a subgraph P isomorphic to a graph in $\{P_1, P_2, P_3, K_3\}$, a proper coloring c of P, and a $(4,2)$-list assignment L of $G - P$ such that c does not extend to an L-coloring of G. We will refer to the vertices of P as *precolored vertices.*

Claim 6.3. *G is 2-connected.*

Proof. If G is disconnected, then each connected component can be colored separately. Suppose that G has a cut-vertex v. Then there exist connected subgraphs G_1 and G_2 where $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{v\}$, $n(G_1) < n(G)$, and $n(G_2) < n(G)$. We can assume without loss of generality that G_1 contains at least one vertex of P, so let S_1 be the subgraph of P contained in G_1. Let $S_2 = \{v\} \cup (V(G_2) \cap V(P))$.

Since (G, P, L, c) is a minimal counterexample, there is an L-coloring c_1 of G_1 that extends the coloring on S_1. Using the color prescribed by c_1 on v, there exists an L-coloring c_2 of G_2 that extends the coloring on S_2. The colorings c_1 and c_2 form an L-coloring of G, a contradiction. □

Claim 6.4. *G has no separating 3-cycles.*

Proof. Suppose that $P' = v_1v_2v_3$ is a separating 3-cycle of G. Let G_1 be the subgraph of G given by the exterior of P' along with P', and let G_2 be the subgraph of G given by the interior of P' along with P'. Since P' is separating, $n(G_1) < n(G)$ and $n(G_2) < n(G)$.

Since the vertices in P share a common face, we can assume without loss of generality that $V(P) \subseteq V(G_1)$. Since (G, P, L, c) is a minimal counterexample, there exists an L-coloring c_1 of G_1. Assign the colors from c_1 to P'. Then there exists an L-coloring of G_2 extending the colors on P', and together c_1 and c_2 form an L-coloring of G, a contradiction. □

Claim 6.5. *If $v \in V(P)$ such that $V(P) \subseteq N[v]$, then the subgraph of G induced by $N(v)$ is not isomorphic to any graph in $\{P_1, P_2, P_3, K_3\}$.*

Proof. Suppose that there exists a vertex $v \in V(P)$ where all precolored vertices are in $N[v]$ and the subgraph $G[N(v)]$ is isomorphic to a subgraph in $\{P_1, P_2, P_3, K_3\}$. Then consider the graph $G' = G - v$. Since $|N_G[v]| \leq 4$, there exists an L-coloring c' of $G[N[v]]$. Since (G, P, L, c) is a minimal counterexample, c' extends to an L-coloring of G', which in turn extends to an L-coloring of G, a contradiction. □

Claim 6.6. *If $v \in V(P)$ has $d_G(v) \leq 2$, then $d_G(v) = 2$ and P is isomorphic to P_1, P_2, or P_3.*

Proof. By Claim 6.3, $d_G(v) \neq 1$. If $d_G(v) = 2$ and $P \cong K_3$, then $G[N_G(v)]$ is isomorphic to P_2, contradicting Claim 6.5. □

Claim 6.7. *P is isomorphic to one of P_3 or K_3.*

Proof. Suppose that P is not isomorphic to either P_3 or K_3. If P is isomorphic to P_1, then the vertex p of P has two neighbors u_1 and u_2 that are on the same face as p; let $U = \{u_1, u_2\}$. If P is isomorphic to P_2, then some vertex v in P has a neighbor u_1 not in P that shares a face with the edge in P; let $U = \{u_1\}$. Let P' be induced by $V(P) \cup V(U)$. Notice $|P'| = 3$ hence it is isomorphic to P_3 or K_3. There exists a proper coloring c' of P' that extends the coloring on P. But then (G, P', L, c') has $n(G) - \frac{1}{4}n(P') < n(G) - \frac{1}{4}n(P)$, so there exists an L-coloring of G that extends c', a contradiction. □

Claim 6.8. *If $v \in V(G - P)$, then $d_G(v) \geq 4$.*

Proof. Suppose that $v \in V(G - P)$ has degree $d(v) \leq 3$. Then $G - v$ is a planar graph with no chorded 7-cycle containing a precolored subgraph P and a list assignment L. Since (G, P, L, c) is a minimum counterexample, $G - v$ has an L-coloring. However, v has at most three neighbors and at least four colors in the list $L(v)$. Thus, there is an extension of the L-coloring of $G - v$ to an L-coloring of G, a contradiction. □

Observe that $n(G) \geq 4$. Recall that in a configuration (C, X, ex), an L-coloring of $V(C) \setminus X$ extends to all of C. Because of this fact, if G contains a reducible configuration (C, X, ex), then there is a precolored vertex in the set X, or else $G - X$ has an L-coloring that extends to all of G. Specifically, we will use the fact that G avoids (C2), (C3), (C4), (C5), (C6), (C7), and (C8).

For each $v \in V(G)$ and $f \in F(G)$ define

$$\mu_0(v) = d(v) - 4 + 2\delta(v) \quad \text{and} \quad \mu_0(f) = \ell(f) - 4 + \varepsilon(f),$$

where $\delta(v) \in \{0, 1\}$ has value 1 if and only if $v \in V(P)$, and $\varepsilon(f) \in \{0, 1\}$ has value 1 if and only if the boundary of f is the set of precolored vertices, $V(P)$. By Euler's Formula, the initial charge sum is at most -1. Claims 6.6 and 6.8 assert that the only negatively-charged objects are 3-faces.

For a vertex v, let $t_k(v)$ denote the number of k-faces incident to v. Apply the following discharging rules. Let $\mu_i(v)$ and $\mu_i(f)$ denote the charge on a vertex v or a face f after rule (Ri).

(R0) If v is a precolored vertex and f is an incident 3-face with negative initial charge, then v sends charge $\frac{1}{2}$ to f.

(R1) If f is a 3-face and e is an incident edge, then let g be the face adjacent to f across e.

- (R1a) If g is a 5^+-face, then f pulls charge $\frac{3}{8}$ from g "through" the edge e.
- (R1b) If g is a 4-face and f is the only 3-face adjacent to g, then let e_1, e_2, and e_3 be the other edges incident to g. For each $i \in \{1, 2, 3\}$, let h_i be the face adjacent to g across e_i. For each $i \in \{1, 2, 3\}$, the face f pulls charge $\frac{1}{8}$ from the face h_i "through" the edges e and e_i.

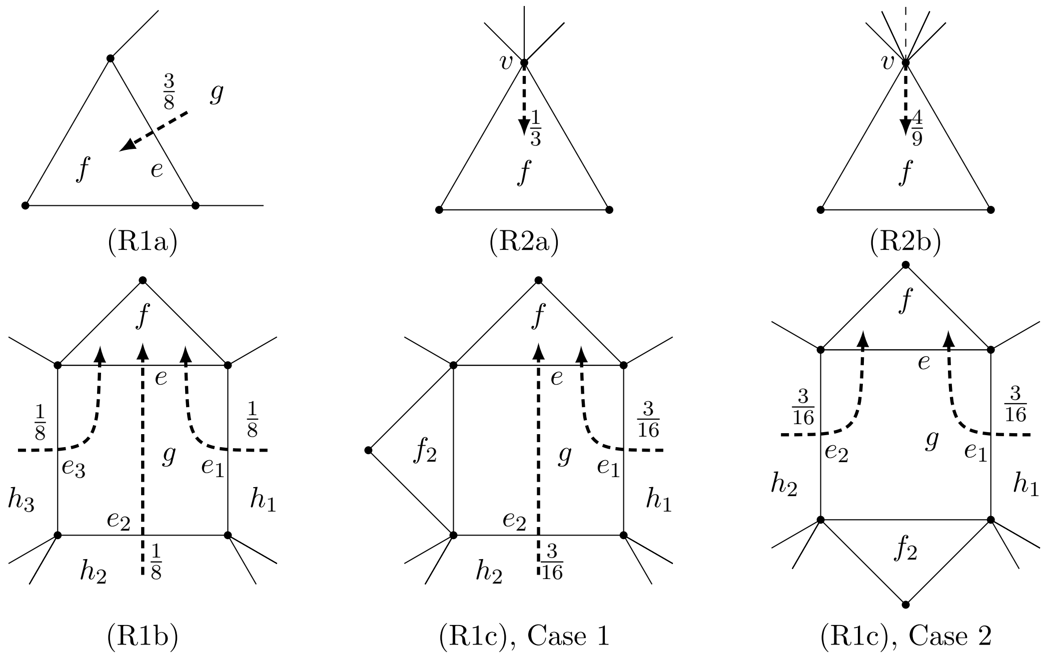


Figure 10: Discharging rules (R1) and (R2) in the proof of Theorem 6.1.

(R1c) If g is a 4-face and g is adjacent to two 3-faces f_1 and f_2 (say $f_1 = f$), then let e_1 and e_2 be the other edges incident to g, where the faces h_1 and h_2 sharing these edges are 6^+-faces. For each $i \in \{1, 2\}$, the face f pulls charge $\frac{3}{16}$ from the face h_i "through" the edges e and e_i.

(R2) Let v be a 5^+-vertex with $v \notin V(P)$ and let f be an incident 3-face.

(R2a) If v is a 5-vertex, then v sends charge $\frac{1}{a}$ to f, when $a = \max\{3, t_3(v)\}$.

(R2b) If v is a 6^+-vertex, then v sends charge $\frac{1}{2}$ to f.

(R3) If f is a 6-face with $\mu_2(f) < 0$ and v is an incident 5^+-vertex or an incident vertex in $V(P)$ with $\mu_0(v) > 0$, then v sends charge $\frac{1}{4}$ to f.

We claim that $\mu_3(v) \geq 0$ for every vertex v and $\mu_3(f) \geq 0$ for every face f. Since the total charge sum was preserved during the discharging rules, this contradicts the negative charge sum from the initial charge values.

Note that 6-faces are not incident to 3-faces since G does not contain a chorded 7-cycle and separating 3-cycles. Observe that a 6-face f has $\mu_1(f) < 0$ if and only if all faces adjacent to f are 4-faces, and each of those 4-faces has two adjacent 3-faces.

Claim 6.9. *Let v be a vertex in $V(P)$. Then $\mu_3(v) \geq 0$. In addition, if v is incident to a 6-face f with $\mu_1(f) < 0$, then $\mu_0(v) > 0$.*

Proof. By Claims 6.6 and 6.7, we have $\mu_0(v) = d(v) - 2 \geq 0$. Note that if $\mu_0(v) \geq \frac{1}{2}t_3(v) + \frac{1}{4}t_6(v)$, then the final charge $\mu_3(v)$ is nonnegative. Since $d(v) \geq t_3(v) + t_6(v)$, it suffices to show that $\mu_0(v) \geq \frac{1}{4}d(v) + \frac{1}{4}t_3(v)$.

Case 1: $P \cong P_3$. Let v_1, v_2, and v_3 be the vertices in the 3-path P. For $i \in \{1,2,3\}$, $\mu_0(v_i) = d(v_i) - 2$. Since P is not isomorphic to K_3, these vertices do not form a cycle, and the face to which all vertices are incident is not a 3-face. Hence $t_3(v_i) \le d(v_i) - 1$. If $d(v_i) \ge 4$, then $\mu_0(v_i) = d(v_i) - 2 \ge \frac{1}{2}d(v_i) > \frac{1}{4}d(v_i) + \frac{1}{4}t_3(v_i)$.

If $d(v_2) = 2$, then $\mu_0(v_i) = 0$. Vertex v_2 is not incident to any 3-faces since v_1 and v_3 are not adjacent. Moreover, v_2 is not incident to any 6-face f with $\mu_1(f) < 0$. If such face f existed, v_2 would be incident also to a 4-face f' that is incident to two triangles. This configuration of faces results in a separating triangle, chorded 7-cycle or contradiction with Claim 6.8.

If $d(v_i) = 2$ for $i \in \{1,3\}$, then $\mu_0(v_i) = 0$. If v_i is adjacent to a 3-face, then let v_i' be the neighbor of v_i not in $V(P)$. Let P' be the subgraph induced by $(V(P) \cup \{v_i'\}) \setminus \{v_i\}$, which forms a copy of P_3 or K_3 in $G - v_i$. For any color $c(v_i') \in L(v_i') \setminus \{c(v_i)\}$, there exists an L-coloring of $G - v_i$ as $(G - v_i, P', L, c)$ is not a counterexample; this coloring extends to an L-coloring of G. Thus, $t_3(v_i) = 0$. If v_i is incident to a 6-face f with $\mu_1(f) < 0$, then the other face incident to v_i is a 4-face that is adjacent to two 3-faces. This results in a chorded 7-cycle, a contradiction; thus (R3) does not apply to v_i.

If $d(v_i) = 3$, Claim 6.4 asserts that G has no separating 3-cycles, so then v_i loses charge at most 1 in (R0). If v_i is incident to a 6-face f with $\mu_1(f) < 0$, then the other two faces incident to v_i are 4-faces and these 4-faces are each adjacent to two 3-faces. This creates a chorded 7-cycle, a contradiction, so (R3) does not apply to v_i and $\mu_3(v_i) \ge 0$.

Case 2: $P \cong K_3$. Let v_1, v_2, and v_3 be the vertices in the 3-cycle P, so $\mu_0(v_i) = d(v_i) - 2$ for each v_i. By Claim 6.4, G has no separating 3-cycle, so the three vertices are incident to a common 3-face f with $\mu_0(f) = 0$. Therefore, each vertex v_i sends charge $\frac{1}{2}$ to at most $d(v_i) - 1$ incident 3-faces by (R0). Recall that $d(v_i) \ge 3$ by Claim 6.6. Suppose that $d(v_i) = 3$. If $t_3(v_i) > 1$, the subgraph of G induced by the neighborhood of v_i is isomorphic to P_3 or K_3, contradicting Claim 6.5. If $d(v_i) \ge 4$, then $\mu_0(v_i) = d(v_i) - 2 \ge \frac{1}{2}d(v_i) \ge \frac{1}{4}d(v_i) + \frac{1}{4}t_3(v_i)$. Therefore, $\mu_3(v_i) \ge 0$.

Thus, in all cases a precolored vertex v has $\mu_3(v) \ge 0$. □

We will now show that all objects that start with nonnegative charge also end with nonnegative charge.

If f is a 4-face, then (R1b) and (R1c) do not pull charge from f, since this would require f to be adjacent to a 4-face g that is adjacent to a 3-face t, but then f, g, and t contain a chorded 7-cycle. Thus, $\mu_3(f) = 0$ for every 4-face f.

If f is a 5-face, then since G contains no chorded 7-cycles, f is not adjacent to two 3-faces and f is not adjacent to a 4-face. Therefore, f loses charge at most $\frac{3}{8}$ by (R1a), but loses no charge using (R1b), so $\mu_3(f) > 0$ for every 5-face f.

If f is a 6-face, then f is not adjacent to a 3-face since G contains no chorded 7-cycle. Observe that by Claim 6.3 the boundary of f is a simple 6-cycle. So if f sends charge through an edge e during (R1), it can send charge $\frac{1}{8}$ through e by (R1b), or it can send charge $\frac{3}{8}$ through e by (R1c). The only way that this will result in a negative charge after (R1) and (R2) is for f to send charge $\frac{3}{8}$ through each of its six edges by (R1c); this will cause $\mu_2(f) = 2 - 6 \cdot \frac{3}{8} = -\frac{1}{4}$. If f has a precolored vertex v on its boundary, then by Claim 6.9, v has positive charge after (R0); by (R3), f receives charge at least $\frac{1}{4}$, resulting in $\mu_3(f) \ge 0$. If f has no incident precolored vertices, then since G contains no (C2), some vertex v on the boundary of f is a 5^+-vertex. By (R3) v sends charge $\frac{1}{4}$ to

f and hence $\mu_3(f) \geq 0$. Observe the following claim concerning the structure about a vertex that loses charge by (R3).

Claim 6.10. *Let v be a 5^+-vertex with the three incident faces f_1, f_2, and f_3, in cyclic order. If v sends charge to f_2 by* (R3)*, then f_1 and f_3 are 4-faces and f_2 is a 6-face.*

If f is a 7^+-face, then by (R1) f loses charge at most $\frac{3}{8}$ through each edge. Thus,

$$\mu_3(f) \geq \ell(f) - 4 - \frac{3}{8}\ell(f) = \frac{5}{8}\ell(f) - 4 > 0.$$

Therefore, $\mu_3(f) > 0$ for every 7^+-face f.

Next, we will consider a vertex v not in $V(P)$.

If v is a 4-vertex, then v does not lose charge by any rule, so the resulting charge is 0.

If v is a 5-vertex, let $a = \max\{3, t_3(v)\}$ and v loses charge $\frac{1}{a}t_3(v)$ to incident 3-faces by (R2a). If (R3) does not apply to v, then v sends charge at most 1 to incident 3-faces and $\mu_3(v) \geq 0$. If (R3) applies to v, then v is incident to faces f_1, f_2, and f_3 where f_1 and f_3 are 4-faces and f_2 is a 6-face. Since $d(v) = 5$ and G has no chorded 7-cycle, the rule (R3) applies at most once. Indeed, if (R3) would apply twice, then v would be incident to two 4-faces sharing an edge and each of these two 4-faces shares two edges with triangles and this gives a chorded 7-cycle. If (R3) applies once, then $t_3(v) \leq 2$ and v loses charge at most $\frac{2}{3}$ by (R2) and charge $\frac{1}{4}$ by (R3), so $\mu_3(v) \geq 0$.

If v is a 6^+-vertex, then let $k = t_3(v)$ and ℓ be the number of times (R3) applies to v. Notice that $k \leq \lfloor \frac{4}{5}d(v) \rfloor$ since G avoids chorded 7-cycles. Further, notice that $k + 2\ell \leq d(v)$, since each 6-face that gains charge from v by (R3) is preceded by a 4-face in the cyclic order of faces around v. By (R2b), v can lose charge $\frac{1}{2}$ to each incident 3-face, and v can lose charge at most $\frac{1}{4}$ to each incident 6-face by (R3). Then v ends with charge

$$\mu_3(v) \geq d(v) - 4 - \frac{1}{2}k - \frac{1}{4}\ell.$$

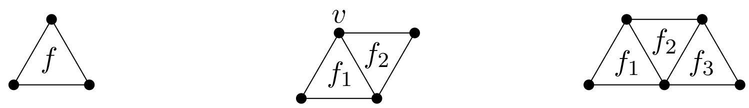
If $d(v) = 6$, then observe $k + \ell \leq 4$ and hence $\mu_3(v) \geq 0$. If $d(v) = d \geq 7$, then d, k, and ℓ satisfy the following linear program with dual on variables a_1, a_2, and a_3:

$$\begin{array}{rrcrcrcl} \min & d & - & \frac{1}{2}k & - & \frac{1}{4}\ell & & \\ \text{s.t.} & d & & & & & \geq & 7 \\ & 4d & - & 5k & & & \geq & 0 \\ & d & - & k & - & 2\ell & \geq & 0 \\ & d, & & k, & & \ell & \geq & 0 \end{array} \qquad \begin{array}{rrcrcrcl} \max & 7a_1 & & & & & & \\ \text{s.t.} & a_1 & + & 5a_2 & + & a_3 & \leq & 1 \\ & & - & 5a_2 & - & a_3 & \leq & -\frac{1}{2} \\ & & & & - & 2a_3 & \leq & -\frac{1}{4} \\ & a_1, & & a_2, & & a_3 & \geq & 0 \end{array}$$

The dual-feasible solution $(a_1, a_2, a_3) = \left(\frac{23}{40}, \frac{1}{20}, \frac{1}{4}\right)$ demonstrates that $d - \frac{1}{2}k - \frac{1}{4}\ell \geq 7 \cdot \frac{23}{40} > 4$, and thus $\mu_3(v) > 0$ for every 7^+-vertex.

It remains to be shown that the clusters receive enough charge to become nonnegative. Since G contains no separating 3-cycle, G does not contain the cluster (K5c) or the clusters (K6g)–(K6r). Observe that there is no precolored vertex v of degree at most three where all faces incident to v have length three. Finally, it is worth noting again that if G contains a reducible configuration (C, X, ex), then there is a precolored vertex in the set X.

If a vertex v is a 5^+-vertex or $v \in V(P)$, we say v is *full*; if v is a 6^+-vertex or $v \in V(P)$, then v is *heavy*. Note that a heavy vertex v sends charge $\frac{1}{2}$ to each incident negatively-charged 3-face by (R0) or (R2b). If $P \cong K_3$, we call P the precolored face.



(K3) (K4) (K5a)

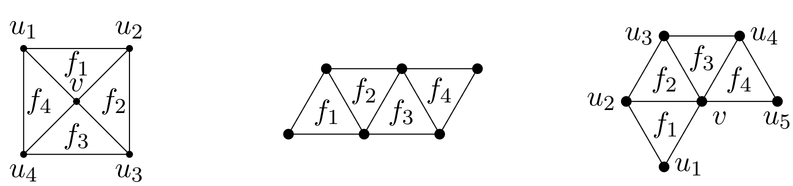
Figure 11: Clusters (K3), (K4), and (K5a)

Case 1: (K3) Let f be the isolated 3-face in (K3). If f is the precolored face, then $\mu_3(f) = \mu_0(f) = 0$. Otherwise, the initial charge on f is -1. By (R1), f receives charge $\frac{9}{8}$ through its boundary edges, resulting in a nonnegative final charge.

Case 2: (K4) Let f_1 and f_2 be 3-faces in a diamond cluster (K4). First, suppose without loss of generality that f_1 is the precolored face. The initial charge of the cluster is -1. Then f_2 receives charge 1 by (R0) and charge $2 \cdot \frac{3}{8}$ by (R1), resulting in a positive final charge. Otherwise, the initial charge on the cluster is -2. By (R1), f_1 and f_2 receive charge $\frac{3}{8}$ through each of the two edges on the boundary of the cluster, resulting in charge $-\frac{1}{2}$. If the cluster contains a precolored vertex u, then it receives charge $\frac{1}{2}$ by (R0). Otherwise, since G contains no (C3), there is a 5^+-vertex v incident to both f_1 and f_2. By (R2), this vertex sends charge at least $\frac{1}{3}$ to each of the faces, resulting in a nonnegative final charge.

Case 3: (K5a) Let f_1, f_2, and f_3 be 3-faces in a 3-fan cluster (K5a), where f_2 is adjacent to both f_1 and f_3. Suppose that the cluster contains a precolored face, so the initial charge on the cluster is -2. If f_2 is precolored, then the cluster receives charge $4 \cdot \frac{1}{2}$ by (R0); if f_1 or f_3 is precolored, then the cluster receives charge $3 \cdot \frac{1}{2}$ by (R0) and charge $3 \cdot \frac{3}{8}$ by (R1). In either case, the final charge is nonnegative.

If $P \ncong K_3$ or the cluster does not contain the precolored face, then the initial charge on the cluster is -3. By (R1), the cluster receives charge $5 \cdot \frac{3}{8}$, resulting in charge $-\frac{9}{8}$. Note that the faces f_1 and f_2 form a diamond and the faces f_2 and f_3 form a diamond. Since G contains no (C3), there exists a full vertex v incident to both f_1 and f_2. Similarly, there exists a full vertex u incident to f_2 and f_3. If $u \neq v$, then by (R0) or (R2), v sends charge at least $\frac{1}{3}$ to each of f_1 and f_2 and u sends charge at least $\frac{1}{3}$ to each of f_2 and f_3, resulting in nonnegative charge on the cluster. If $u = v$ and v is a heavy vertex, then v sends charge $\frac{1}{2}$ to each face f_1, f_2, and f_3, resulting in nonnegative charge on the cluster. Otherwise, suppose that $u = v \notin V(P)$ and v is a 5-vertex. Since G contains no (C4), there exists another full vertex w that is incident to at least one of f_1 and f_2. By (R2a), v sends charge $\frac{1}{3}$ to f_1, f_2, and f_3, and by (R0) or (R2), w sends charge at least $\frac{1}{3}$ to one of f_1 and f_2, resulting in nonnegative charge on the cluster.



(K5b) (K6a) (K6b)

Figure 12: Clusters (K5b), (K6a), and (K6b)

Case 4: (K5b) Let f_1, f_2, f_3, and f_4 be 3-faces in a 4-wheel (K5b). If the cluster contains a precolored face, then the initial charge on the cluster is -3; the cluster receives charge $5 \cdot \frac{1}{2}$ by (R0) and charge $3 \cdot \frac{3}{8}$ by (R1), resulting in a positive final charge. Otherwise, the initial charge on this cluster is -4. By (R1), the cluster receives charge $4 \cdot \frac{3}{8}$, resulting in charge $-\frac{5}{2}$. Let v be the 4-vertex incident to all four 3-faces. Let u_1, u_2, u_3, and u_4 be the vertices adjacent to v, ordered cyclically such that vu_iu_{i+1} is the boundary of the 3-face f_i for $i \in \{1, 2, 3\}$ and vu_4u_1 is the boundary of f_4. Since the cluster does not contain the precolored face, v is not a precolored vertex. Since G contains no (C3), each u_i is a full vertex. When u_i is a 5-vertex, it is incident to two 7^+-faces, so u_i sends charge $\frac{1}{3}$ to each incident 3-face by (R2). Thus, each u_i sends charge at least $2 \cdot \frac{1}{3}$ to the cluster by (R0) or (R2), resulting in a nonnegative final charge.

Case 5: (K6a) Let f_1, f_2, f_3, and f_4 be 3-faces in a 4-strip cluster (K6a). If the cluster contains the precolored face, then the initial charge on the cluster is -3. If f_1 or f_4 is precolored, then the cluster receives charge $3 \cdot \frac{1}{2}$ by (R0) and charge $4 \cdot \frac{3}{8}$ by (R1); if f_2 or f_3 is precolored, then the cluster receives charge $5 \cdot \frac{1}{2}$ by (R0) and charge $5 \cdot \frac{3}{8}$ by (R1). In either case, the resulting final charge is nonnegative. If the cluster does not contain the precolored face, then the initial charge on this cluster is -4. By (R1), the cluster receives charge $6 \cdot \frac{3}{8}$, resulting in charge $-\frac{7}{4}$. Note that for $i \in \{1, 2, 3\}$, the faces f_i and f_{i+1} form a diamond. Since G contains no (C3), there exists a full vertex v incident to both f_i and f_{i+1}. Let u_1 be a full vertex incident to f_2 and f_3. Without loss of generality, u_1 is not incident to f_4, so there is a full vertex u_2 incident to f_1 and f_2. If u_1 is a heavy vertex, the cluster receives charge $3 \cdot \frac{1}{2}$ from u_1 by (R0) or (R2b), and charge at least $2 \cdot \frac{1}{3}$ from u_2 by (R0) or (R2), resulting in a positive final charge. Otherwise, u_1 is a 5-vertex, so u_1 sends charge $3 \cdot \frac{1}{3}$ by (R2a), resulting in charge $-\frac{3}{4}$. If u_2 is incident to f_3, then u_2 sends charge at least $3 \cdot \frac{1}{3}$ by (R0) or (R2), resulting in a positive final charge. Otherwise, u_2 is incident with f_1 and f_2 but not f_3. If u_2 is a large vertex, it sends charge $2 \cdot \frac{1}{2}$ by (R0) or (R2b). Otherwise, since G contains neither a (C3) or a (C4), there is a third full vertex u_3. The cluster receives charge $2 \cdot \frac{1}{3}$ from u_2 by (R2a) and charge at least $\frac{1}{3}$ from u_3 by (R0) or (R2). In each case, the resulting final charge is nonnegative.

Case 6: (K6b) Let f_1, f_2, f_3, and f_4 be 3-faces in a 4-fan cluster (K6b). Let v be the center of the fan, with neighbors u_1, u_2, u_3, u_4, and u_5 where for $i \in \{1, 2, 3\}$, f_i and f_{i+1} are adjacent on the edge vu_{i+1}. If the cluster contains the precolored face, then the initial charge on the cluster is -3. If f_1 or f_4 is precolored, then the cluster receives charge $4 \cdot \frac{1}{2}$ by (R0) and charge $4 \cdot \frac{3}{8}$ by (R1); if f_2 or f_3 is precolored, then the cluster receives charge $5 \cdot \frac{1}{2}$ by (R0) and charge $5 \cdot \frac{3}{8}$ by (R1). In either case, the resulting final charge is positive.

If the cluster does not contain the precolored face, then the initial charge on this cluster is -4. By (R1), the cluster receives charge $6 \cdot \frac{3}{8}$, resulting in charge $-\frac{7}{4}$. If v is a heavy vertex, then by (R0) or (R2b) v sends charge $4 \cdot \frac{1}{2}$ to the cluster, resulting in positive charge. Otherwise, $v \notin V(P)$ and v is a 5-vertex, so v sends charge 1 to the cluster by (R2a), resulting in charge $-\frac{3}{4}$. If there is a heavy vertex in $\{u_2, u_3, u_4\}$, then that vertex contributes charge $2 \cdot \frac{1}{2}$ to the cluster, resulting in a positive charge. If there is no heavy vertex in $\{u_2, u_3, u_4\}$, then there is at least one 5-vertex in $\{u_2, u_3, u_4\}$ since G contains no (C4). If there are multiple 5-vertices in $\{u_2, u_3, u_4\}$, then each sends charge $2 \cdot \frac{1}{3}$ to the cluster by (R2a), resulting in positive charge. If there is only 5-vertex w among u_2, u_3, and u_4, then there is a full vertex $z \in \{u_1, u_5\}$ since G does not contain (C4) or (C5); the cluster receives charge $2 \cdot \frac{1}{3}$ from w by (R2a) and at least $\frac{1}{3}$ from z by (R0) or (R2), resulting in positive final charge.

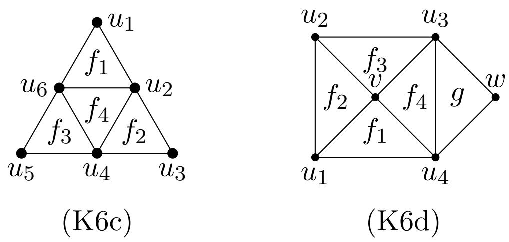


Figure 13: Clusters (K6c) and (K6d).

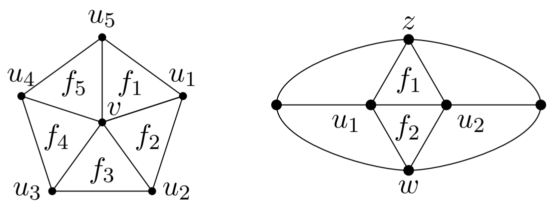
Case 7: (K6c) Let f_1, f_2, f_3, and f_4 be the 3-faces of this cluster (K6c) where f_4 is adjacent to each f_i for $i \in \{1, 2, 3\}$. If the cluster contains the precolored face, then the initial charge on the cluster is -3. If one of f_1, f_2 or f_3 is precolored, the cluster receives charge $4 \cdot \frac{1}{2}$ by (R0) and charge $4 \cdot \frac{3}{8}$ by (R1). If f_4 is precolored, then the cluster receives charge $6 \cdot \frac{1}{2}$ by (R0). In either case, the resulting final charge is nonnegative.

If the cluster does not contain the precolored face, then the initial charge on the cluster is -4. By (R1), the cluster receives charge $6 \cdot \frac{3}{8}$, resulting in charge $-\frac{7}{4}$. Let u_1, u_2, u_3, u_4, u_5, and u_6 be the vertices on the boundary of the cluster ordered such that u_2, u_4, u_6 are the vertices incident to f_1 and f_2, f_2 and f_3, and f_3 and f_1, respectively. Since G contains no (C3), there are at least two full vertices in $\{u_2, u_4, u_6\}$. By (R0) or (R2), these vertices each send charge at least 1 to the cluster, resulting in a positive total charge.

Case 8: (K6d) Let f_1, f_2, f_3, and f_4 be cyclically-ordered 3-faces in a 4-wheel with center vertex v where f_i and f_{i+1} share a common edge for $i \in \{1, 2, 3, 4\}$, where indices are taken modulo 4; let g be a 3-face adjacent to f_4 but not incident to v, completing our cluster (K6d). If the cluster contains the precolored face, then the initial charge on the cluster is -4. If f_1 or f_3 is precolored, then the cluster receives charge $6 \cdot \frac{1}{2}$ by (R0) and charge $4 \cdot \frac{3}{8}$ by (R1). If f_2 is precolored, then the cluster receives charge $5 \cdot \frac{1}{2}$ by (R0) and charge $4 \cdot \frac{3}{8}$ by (R1). If f_4 is precolored, then the cluster receives charge $7 \cdot \frac{1}{2}$ by (R0) and charge $5 \cdot \frac{3}{8}$ by (R1). In each of the above cases, the final charge is nonnegative. If g is precolored, then the cluster receives charge $4 \cdot \frac{1}{2}$ by (R0) and charge $3 \cdot \frac{3}{8}$ by (R1), resulting in charge $-\frac{7}{8}$. Let $N(v) = \{u_1, u_2, u_3, u_4\}$ where u_i is incident to f_i and f_{i+1} for all $i \in \{1, 2, 3, 4\}$. Since G does not contain (C3), u_1 and u_2 are full vertices. Each of u_1 and u_2 sends charge at least $2 \cdot \frac{1}{3}$ to the cluster by (R2), resulting in nonnegative charge.

If the cluster does not contain the precolored face, then the initial charge on this cluster is -5 and $v \notin V(P)$. By (R1), the cluster receives charge $5 \cdot \frac{3}{8}$, resulting in charge $-\frac{25}{8}$. Since G does not contain (C3), u_1, u_2, u_3, and u_4 are full vertices. By (R0) or (R2), the cluster receives charge at least $2 \cdot \frac{1}{3}$ from each of u_1 and u_2 and charge at least $3 \cdot \frac{1}{3}$ from each of u_3 and u_4, resulting in a positive final charge.

Case 9: (K6e) Let f_1, f_2, f_3, f_4, and f_5 be the cyclically-ordered 3-faces in a 5-wheel with center vertex v where f_i and f_{i+1} share a common edge for $i \in \{1, 2, 3, 4, 5\}$, where indices are taken modulo 5. Let $N(v) = \{u_1, u_2, u_3, u_4, u_5\}$ where u_i is incident to f_i and f_{i+1} for $i \in \{1, 2, 3, 4, 5\}$. If the cluster contains the precolored face, then the initial charge on the cluster is -4. The cluster receives charge $6 \cdot \frac{1}{2}$ by (R0) and charge $4 \cdot \frac{3}{8}$ by (R1), resulting in a positive final charge.



(K6e) (K6f)

Figure 14: Clusters (K6e) and (K6f).

If the cluster does not contain the precolored face, then the initial charge is -5 and $v \notin V(P)$. By (R1), the cluster receives charge $5 \cdot \frac{3}{8}$, and by (R2), the cluster receives charge 1 from v, resulting in charge $-\frac{17}{8}$. Since G does not contain (C4) or (C6), there are at least three full vertices in $N(v)$. If $N(v)$ contains at least three heavy vertices, then the cluster receives charge at least $6 \cdot \frac{1}{2}$ by (R0) or (R2b), resulting in a positive final charge. If $N(v)$ contains exactly two heavy vertices, then the cluster receives charge $4 \cdot \frac{1}{2}$ by (R0) or (R2b) and charge $2 \cdot \frac{1}{3}$ from a full vertex by (R2a), resulting in positive charge. If $N(v)$ contains exactly one heavy vertex, then the cluster receives charge $2 \cdot \frac{1}{2}$ by (R0) or (R2b) and charge $2 \cdot \frac{1}{3}$ from each of two full vertices by (R2a), resulting in positive final charge.

If $N(v)$ contains no heavy vertices, then there are at least three full vertices in $N(v)$. Since G does not contain (C4), there are at least two nonadjacent 5-vertices in $N(v)$. Further, since G does not contain (C6), (C7), or (C8), there are at least four 5-vertices in $N(v)$. The cluster receives charge $2 \cdot \frac{1}{3}$ from each of these vertices by (R2a), resulting in a positive final charge.

Case 10: (K6f) Let f_1 and f_2 be the interior 3-faces in the two overlapping 4-wheels that make up the cluster (K6f). Let u_1 and u_2 be the shared vertices of f_1 and f_2 and let z and w be the vertices incident with f_1 and f_2, respectively, that have not yet been labeled. Since G contains no (C3), at least one of u_1 and u_2 is in $V(P)$. Then since all the precolored vertices lie on a common face, the cluster contains the precolored face, so the initial charge is -5. If f_1 or f_2 is precolored, then the cluster receives charge $8 \cdot \frac{1}{2}$ by (R0) and charge $4 \cdot \frac{3}{8}$ by (R1), resulting in a positive final charge. If one of the other 3-faces is precolored, then the cluster receives charge $6 \cdot \frac{1}{2}$ by (R0) and charge $3 \cdot \frac{3}{8}$ by (R1), resulting in charge $-\frac{7}{8}$. Since G contains no (C3), one of w and z is a non-precolored 5^+-vertex. This vertex sends charge at least $3 \cdot \frac{1}{3}$ to the cluster by (R2), resulting in a positive final charge.

We have verified that the total charge after discharging is nonnegative, contradicting the negative initial charge sum. Thus, a minimal counterexample does not exist and every planar graph with no chorded 7-cycle is $(4, 2)$-choosable. □

7 Conclusion and Future Work

We proved that, for each $k \in \{5, 6, 7\}$, planar graphs with no chorded k-cycles are $(4, 2)$-choosable. Our methods for proving reducible configurations created several large classes of reducible configurations, such as templates; naturally, there are many more reducible configurations than the ones we explicitly used. Unfortunately, we were unable to extend these results to prove Conjecture 1.3, that all planar graphs are $(4, 2)$-choosable.

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A Large Reducible Configurations

In the proof of Theorem 4.1, we demonstrated that no minimal counterexample exists by showing that there exists a reducible configuration (C, X, ex) where G contains a copy of $C[X]$ as an induced subgraph (and also the copy agrees with the external degrees). In this appendix, we provide the details that clarify this assumption. By Lemma 3.2, we can relax the condition that $C[X]$ is an induced subgraph. We will demonstrate that the configurations that appear after some vertices in X are merged (while also preserving the face lengths, vertex degrees, and lack of chorded 5-cycle) result in reducible configurations.

Let (C, X, ex) be a reducible configuration and let $\{x_1, x'_1\}, \ldots, \{x_t, x'_t\}$ be a list of vertex pairs in X. For these configurations, we may identify some 3-cycles and 5-cycles that are required to be 5-faces (in the context of the proof of Theorem 4.1). The resulting configuration (C', X', ex) where C' and X' are modified from C and X by merging x_i with x'_i and removing any multiedges or loops that result. We say a list $\{x_1, x'_1\}, \ldots, \{x_t, x'_t\}$ is *valid* for (C, X, ex) if the resulting configuration (C', X', ex) may appear in a planar graph of minimum degree at least four containing no chorded 5-cycle. There are three situations that can occur when we perform this action.

Pairs too close: If some pair $\{x_i, x'_i\}$ have $d(x_i, x'_i) \le 2$, then either we create a loop or a multiedge when merging x_i and x'_i. This will reduce the degree of the resulting vertex, in addition to possibly shortening known 3- and 5-cycles. Since distances only decrease as vertices are merged, a pair failing this property will not appear in any valid list of pairs.

Pairs creating chord: If merging x_i and x'_i creates a chorded 5-cycle, then this configuration would not appear in the minimal counterexample from Theorem 4.1. Since distances only decrease as vertices are merged, a pair failing this property will not appear in any valid list of pairs.

Reducible pairs: If merging x_i and x'_i does not fit in the above two cases, then we will demonstrate that the resulting configuration is reducible. Even if merging one pair of vertices creates a reducible configuration, we need to check all possible lists of pairs that contain that pair.

After considering all pairs that could be identified, observe that in each case there is no set of three or more vertices where every pair can be identified.

In the following tables, we list one of the configurations (C10)–(C21), label the vertices, and list all pairs of vertices into the three categories above. In the case of reducible pairs, we present the contracted graph. Most of these contracted graphs contain a copy of (C1), (C2), (C10), (C11), or (C12). The only exceptions are the contracted graphs derived from (C16), but each of these configurations has an Alon-Tarsi orientation and hence is reducible.

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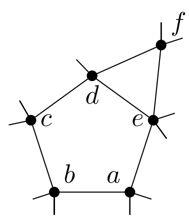
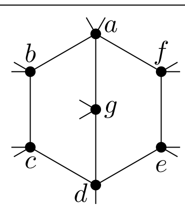
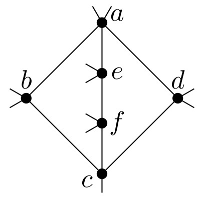
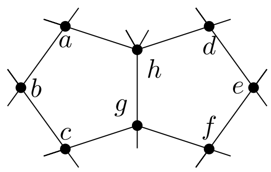
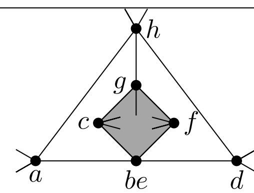
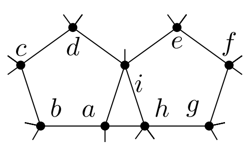
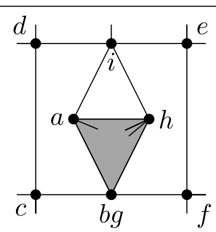
<tr><td rowspan="3">
</td><td>**Pairs too close:** ab, ac, ad, ae, af, bc, bd, be, cd, ce, cf, de, df, ef.</td></tr>
<tr><td>**Pairs creating chord:** bf</td></tr>
<tr><td>**Reducible pairs:** *None remain.*</td></tr>
</table>

<table>
<tr><th colspan="2">(C11)</th></tr>
<tr><td rowspan="3">
</td><td>**Pairs too close:** ab, ac, ad, ae, af, ag, bc, bd, bf, bg, cd, ce, cg, de, df, dg, ef, eg, fg.</td></tr>
<tr><td>**Pairs creating chord:** be, cf</td></tr>
<tr><td>**Reducible pairs:** *None remain.*</td></tr>
</table>

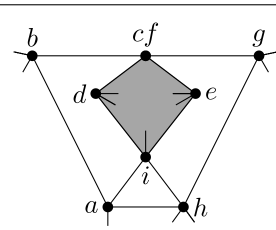
(C12)	
	**Pairs too close:** *ab*, *ac*, *ad*, *ae*, *af*, *bc*, *bd*, *be*, *bf*, *cd*, *ce*, *cf*, *de*, *df*, *ef*.
	Pairs creating chord: *None remain.*
	Reducible pairs: *None remain.*

(C13)	
	**Pairs too close:** *ab*, *ac*, *ad*, *ag*, *ah*, *bc*, *bg*, *bh*, *cf*, *cg*, *ch*, *de*, *df*, *dg*, *dh*, *ef*, *eg*, *eh*.
	Pairs creating chord: *ae*, *af*, *bf*, *bd*, *cd*. *ce*.
	Reducible pairs: *be* (contains (C1))
 Contains (C1) on 4-cycle be, f, g, c.	

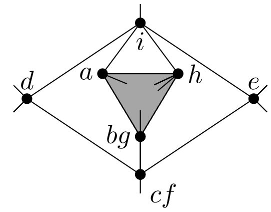
(C14)	
	**Pairs too close:** *ab*, *ac*, *ad*, *ae*, *ag*, *ah*, *ai*, *bc*, *bd*, *bh*, *bi*, *cd*, *ci*, *de*, *dh*, *di*, *ef*, *eg*, *eh*, *ei*, *fg*, *fg*, *fi*, *gh*, *gi*, *hi*.
	Pairs creating chord: *af*, *be*, *ce*, *ch*, *df*, *dg*.
	Reducible pairs: *bg* (contains (C11)), *cf* (contains (C11)), *bg* and *cf* (contains (C12)).



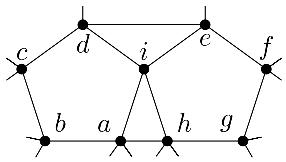
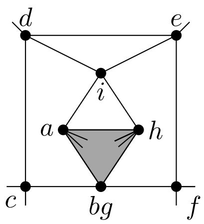
Contains (C11) after deleting vertex h.

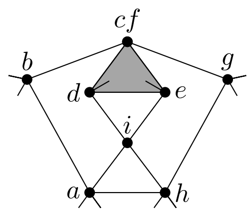


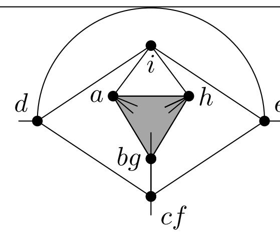
Contains (C11) after deleting vertex d.



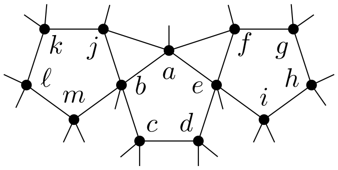
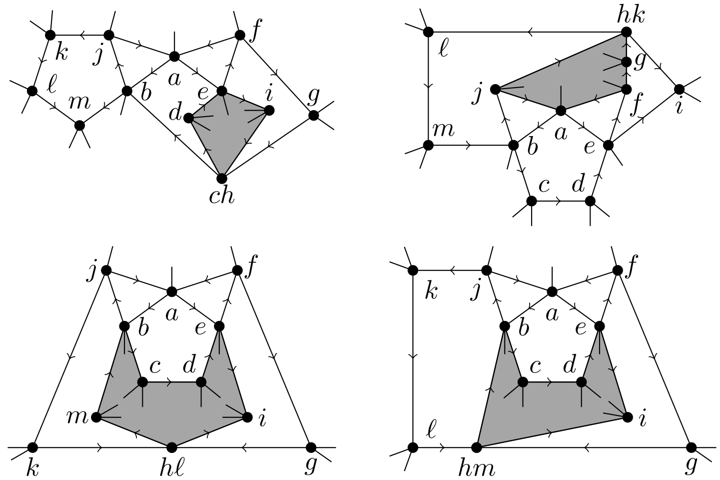
Contains (C12) after deleting vertex h.

<table>
<tr><td colspan="3">(C15)</td></tr>
<tr><td rowspan="3">
</td><td colspan="2">Pairs too close: ab, ac, ad, ae, ag, ah, ai, bc, bd, bh, bi, cd, ce, ci, de, df, dh, di, ef, eg, eh, ei, fg, fh, fi, gh, gi, hi.</td></tr>
<tr><td colspan="2">Pairs creating chord: af, ag, be, bf (bf, a, i, h, g, bf), bh, cg (cg, d, i, e, f, cg), ch, dg.</td></tr>
<tr><td colspan="2">Reducible pairs: bg (contains (C2)), cf (contains (C1)), bg and cf (contains (C1)).</td></tr>
<tr><td>

Contains (C2)
on 6-cycle bf, f, e, i, d, c.</td><td>

Contains (C1)
on 4-cycle cf, e, i, d.</td><td>

Contains (C1)
on 4-cycle cf, e, i, d.</td></tr>
</table>

<table>
<tr><td colspan="2">(C16)</td></tr>
<tr><td rowspan="3"></td><td>Pairs too close: ab, ac, ad, ae, af, ag, ai, aj, ak, am, bc, bd, be, bf,bj, bk, $b\ell$, bm, cd, ce, cj, cm, de, df, di, ef, eg, eh, ei, ej, fg, fh, fi, fj, gh, gi, hi, jk, $j\ell$, jm, $k\ell$, km, ℓm.</td></tr>
<tr><td>Pairs creating chord: ah, $a\ell$, bh, bg, bi, cf, ck, cg, ci, $c\ell$, dg, dh, dj, dk, dm, ek, $e\ell$, em, fk, $f\ell$, fm, gj,gk, gm, hj, ij, ik, im.</td></tr>
<tr><td>Reducible pairs: ch (has Alon-Tarsi orientation), $d\ell$ (symmetric to ch), hk (has Alon-Tarsi orientation), hm (has Alon-Tarsi orientation), $h\ell$ (has Alon-Tarsi orientation), $g\ell$ (symmetric to hk), $i\ell$ (symmetric to hm).</td></tr>
<tr><td colspan="2"></td></tr>
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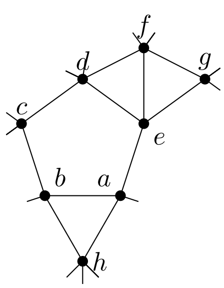
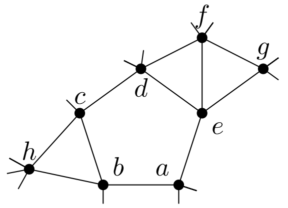
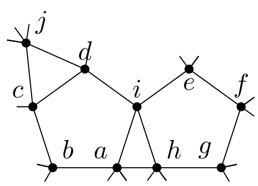
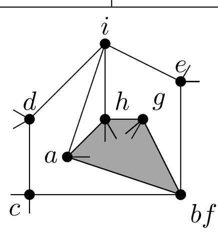
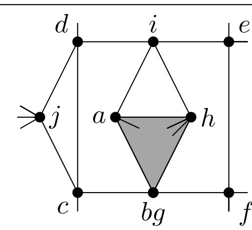
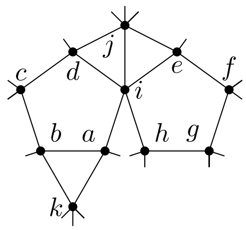
<table>
<tr><td colspan="2">(C17)</td></tr>
<tr><td rowspan="3"></td><td>Pairs too close: ab, ac, ad, ae, af, ag, ah, bc, bd, be, bh, cd, ce, cf, ch, de, df, dg, ef, eg, fg.</td></tr>
<tr><td>Pairs creating chord: bf, bg, cg, dh, fh, gh.</td></tr>
<tr><td>Reducible pairs: None remaining.</td></tr>
</table>

(C18)	
	**Pairs too close:** ab, ac, ad, ae, af, ag, ah, bc, bd, be, bh, cd, ce, cf, ch, de, df, dg, dh, ef, eg.
	Pairs creating chord: bf, bg, cg, eg.
	Reducible pairs: *None remaining.*

(C19)	
	**Pairs too close:** ab, ac, ad, ae, ag, ah, ai, bc, bc, bh, bi, bj, cd, ci, cj, de, dh, di, dj, ef, eg, eh, ei, fg, fh, fi, gh, gi, hi.
	Pairs creating chord: af, aj, be, ce, cf (cf, e, i, d, j), cg (cg, h, i, d, j), ch, df, dg.
	Reducible pairs: bf (contains (C10)), bg (contains (C2)).
 Contains (C10) on 5-cycle h, g, bf, e, i and vertex a.	 Contains (C2) on 6-cycle bg, f, e, i, d, c.

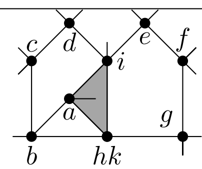
(C20)



Pairs too close: *ab*, *ac*, *ad*, *ae*, *ah*, *ai*, *aj*, *ak*, *bc*, *bd*, *bi*, *bk*, *cd*, *ci*, *cj*, *ck*, *de*, *dh*, *di*, *dj*, *ef*, *eg*, *eh*, *ei*, *ej*, *fg*, *fh*, *fi*, *fj*, *gh*, *gi*, *hi*, *hj*, *ij*, *ik*.

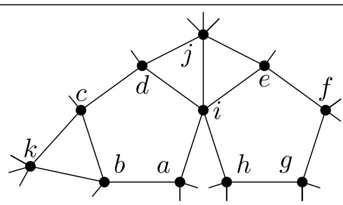
Pairs creating chord: *af* (af, e, j, d, i), *ag* (ag, f, e, j, i), *be*, *bf* (bf, a, i, j, e), *bg* (bg, h, i, a, k), *bh*, *bj*, *ce*, *cf* (cf, d, i, j, e), *cg* (cg, h, i, j, d), *ch*, *df*, *dg*, *dk*, *ek* (ek, j, d, i, a), *fk* (fk, e, j, i, a), *gj*, *gk* (gk, h, i, a, b), *jk* (jk, d, c, b, a).

Reducible pairs: *hk* (contains (C10)).



Contains (C10) on 5-cycle hk, g, f, e, i and vertex a.

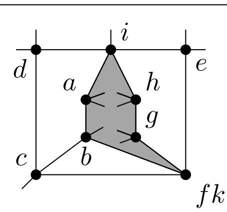
(C21)



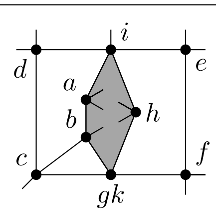
Pairs too close: *ab*, *ac*, *ad*, *ae*, *ah*, *ai*, *aj*, *ak*, *bc*, *bd*, *bi*, *bk*, *cd*, *ci*, *cj*, *ck*, *de*, *dh*, *di*, *dj*, *dk*, *ef*, *eg*, *eh*, *ei*, *ej*, *fg*, *fh*, *fi*, *fj*, *gh*, *gi*, *hi*, *hj*, *ij*.

Pairs creating chord: *af* (af, e, j, d, i), *ag* (ag, f, e, j, i), *be*, *bf* (bf, a, i, j, e), *bh*, *bj*, *ce*, *cf* (cf, d, i, j, e), *cg* (cg, h, i, j, d), *ch*, *df*, *dg*, *dk*, *ek* (ek, j, i, d, c), *gj*, *hk* (hk, i, a, b, c), *ik*, *jk* (jk, i, a, b, c).

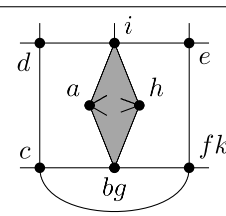
Reducible pairs: *fk* (Contains (C11)), *gk* (Contains (C11)), *bg* and *fk* (Contains (C12)). (Note: if we identify only *bg*, then *k* must be identified with *f* in order to preserve that *g* has total degree four.)



Contains (C11) after deleting vertices *c* and *d*.



Contains (C11) after deleting vertices *c* and *d*.



Contains (C12) after deleting vertices *c* and *d*.