Short proofs of coloring theorems on planar graphs

Oleg V. Borodin* Alexandr V. Kostochka†
Bernard Lidický‡ Matthew Yancey§

February 28, 2013

Abstract

A recent lower bound on the number of edges in a k-critical n-vertex graph by Kostochka and Yancey yields a half-page proof of the celebrated Grötzsch Theorem that every planar triangle-free graph is 3-colorable. In this paper we use the same bound to give short proofs of other known theorems on 3-coloring of planar graphs, among whose is the Grünbaum-Aksenov Theorem that every planar with at most three triangles is 3-colorable. We also prove the new result that every graph obtained from a triangle-free planar graph by adding a vertex of degree at most four is 3-colorable.

*Sobolev Institute of Mathematics and Novosibirsk State University, Novosibirsk 630090, Russia E-mail: brdnoleg@math.nsc.ru. Research of this author is supported in part by grants 12-01-00448 and 12-01-00631 of the Russian Foundation for Basic Research.

†University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA and Sobolev Institute of Mathematics, Novosibirsk 630090, Russia. E-mail: kostochk@math.uiuc.edu. Research of this author is supported in part by NSF grant DMS-0965587.

‡University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA, E-mail: lidicky@illinois.edu

§University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA, E-mail: yancey1@illinois.edu. Research of this author is partially supported by the Arnold O. Beckman Research Award of the University of Illinois at Urbana-Champaign and from National Science Foundation grant DMS 08-38434 "EMSW21-MCTP: Research Experience for Graduate Students."

1 Introduction

Graphs considered in this paper are simple, i.e., without loops or parallel edges. For a graph G, the set of its vertices is denoted by $V(G)$ and the set of its edges by $E(G)$.

An *embedding* σ of a graph $G = (V, E)$ in a surface Σ is an injective mapping of V to a point set P in Σ and E to non-self-intersecting curves in Σ such that (a) for all $v \in V$ and $e \in E$, $\sigma(v)$ is never an interior point of $\sigma(e)$, and $\sigma(v)$ is an endpoint of $\sigma(e)$ if and only if v is a vertex of e, and (b) for all $e, h \in E$, $\sigma(h)$ and $\sigma(e)$ can intersect only in vertices of P. A graph is *planar* if it has an embedding in the plane. A graph with its embedding in the (projective) plane is a *(projective) plane* graph. A cycle in a graph embedded in Σ is *contractible* if it splits Σ into two surfaces where one of them is homeomorphic to a disk.

A *(proper) coloring* φ of a graph G is a mapping from $V(G)$ to a set of colors C such that $\varphi(u) \neq \varphi(v)$ whenever $uv \in E(G)$. A graph G is *k-colorable* if there exists a coloring of G using at most k colors. A graph G is *k-critical* if G is not $(k-1)$-colorable but every proper subgraph of G is $(k-1)$-colorable. By definition, if a graph G is not $(k-1)$-colorable then it contains a k-critical subgraph.

Dirac [12] asked to determine the minimum number of edges in a k-critical graph. Ore conjectured [22] that an upper bound obtained from Hajós' construction is tight. More details about Ore's conjecture can be found in [18][Problem 5.3] and in [20]. Recently, Kostochka and Yancey [20] confirmed Ore's conjecture for $k = 4$ and showed that the conjecture is tight in infinitely many cases for every $k \geq 5$. In [19] they gave a 2.5-page proof of the case $k = 4$:

Theorem 1 ([19]). *If G is a 4-critical n-vertex graph then*

$$|E(G)| \geq \frac{5n-2}{3}.$$

Theorem 1 yields a half-page proof [19] of the celebrated Grötzsch Theorem [14] that every planar triangle-free graph is 3-colorable. This paper presents short proofs of some other theorems on 3-coloring of graphs close to planar. Most of these results are generalizations of Grötzsch Theorem.

Examples of such generalizations are results of Aksenov [2] and Jensen and Thomassen [17].

Theorem 2 ([2, 17])**.** *Let G be a triangle-free planar graph and H be a graph such that $G = H - h$ for some edge h of H. Then H is 3-colorable.*

Theorem 3 ([17])**.** *Let G be a triangle-free planar graph and H be a graph such that $G = H - v$ for some vertex v of degree 3. Then H is 3-colorable.*

We show an alternative proof of Theorem 2 and give a strengthening of Theorem 3.

Theorem 4. *Let G be a triangle-free planar graph and H be a graph such that $G = H - v$ for some vertex v of degree 4. Then H is 3-colorable.*

Theorems 2 and 4 yield a short proof of the following extension theorem that was used by Grötzsch [14].

Theorem 5. *Let G be a triangle-free planar graph and F be a face of G of length at most 5. Then each 3-coloring of F can be extended to a 3-coloring of G.*

An alternative statement of Theorem 2 is that each coloring of two vertices of a triangle-free planar graph G by two different colors can be extended to a 3-coloring of G. Aksenov et al. [3] extended Theorem 2 by showing that each proper coloring of each induced subgraph on two vertices of G extends to a 3-coloring of G.

Theorem 6 ([3])**.** *Let G be a triangle-free planar graph. Then each coloring of two non-adjacent vertices can be extended to a 3-coloring of G.*

We show a short proof of Theorem 6.

Another possibility to strengthen Grötzsch's Theorem is to allow at most three triangles.

Theorem 7 ([1, 4, 15])**.** *Let G be a planar graph containing at most three triangles. Then G is 3-colorable.*

The original proof by Grünbaum [15] was incorrect and a correct proof was provided by Aksenov [1]. A simpler proof was given by Borodin [4], but our proof is significantly simpler.

Youngs [30] constructed triangle-free graphs in the projective plane that are not 3-colorable. Thomassen [25] showed that if G is embedded in the projective plane without contractible cycles of length at most 4 then G is 3-colorable. We slightly strengthen the result by allowing two contractible 4-cycles or one contractible 3-cycle.

Theorem 8. *Let G be a graph embedded in the projective plane such that the embedding has at most two contractible cycles of length 4 or one contractible cycle of length three such that all other cycles of length at most 4 are non-contractible. Then G is 3-colorable.*

It turned out that restricting the number of triangles is not necessary. Havel conjectured [16] that there exists a constant c such that if every pair of triangles in a planar graph G is at distance at least c then G is 3-colorable. The conjecture was proven true by Dvořák, Král' and Thomas [13].

Without restriction on triangles, Steinberg conjectured [23] that every planar graph without 4- and 5-cycles is 3-colorable. Erdős suggested to relax the conjecture and asked for the smallest k such that every planar graphs without cycles of length 4 to k is 3-colorable. The best known bound for k is 7 [9]. A cycle C is *triangular* if it is adjacent to a triangle other than C. In [6], it is proved that every planar graph without triangular cycles of length from 4 to 7 is 3-colorable, which implies all results in [7, 8, 9, 10, 11, 21, 26, 27, 28, 29].

We present the following result in the direction towards Steinberg's conjecture with a Havel-type constraint on triangles. As a free bonus, the graph can be in the projective plane instead of the plane.

Theorem 9. *Let G be a 4-chromatic projective planar graph where every vertex is in at most one triangle. Then G contains a cycle of length 4,5 or 6.*

There are numerous other results on the Three Color Problem in the plane. See a recent survey [5] or a webpage maintained by Montassier `http://janela.lirmm.fr/~montassier/index.php?n=Site.ThreeColorProblem`.

The next section contains proofs of the presented theorems and Section 3 contains constructions showing that some of the theorems are best possible.

2 Proofs

Identification of non-adjacent vertices u and v in a graph G results in a graph G' obtained from $G - \{u, v\}$ by adding a new vertex x adjacent to every vertex that is adjacent to at least one of u and v.

The following lemma is a well-known tool to reduce the number of 4-faces. We show its proof for the completeness.

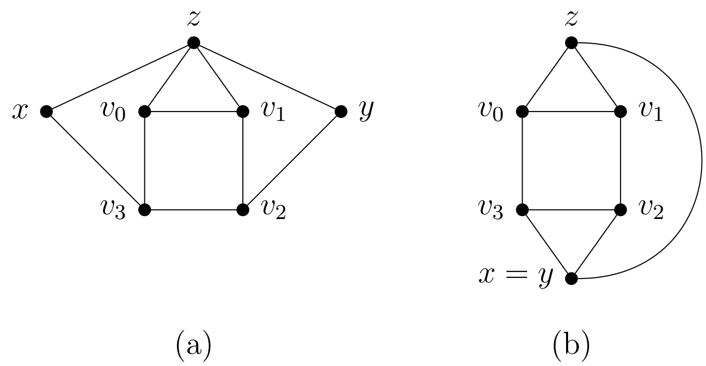

Figure 1: Triangle adjacent to a 4-face in Lemma 10.

Lemma 10. *Let G be a plane graph and $F = v_0v_1v_2v_3$ be a 4-face in G such that $v_0v_2, v_1v_3 \notin E(G)$. Let G_i be obtained from G by identifying v_i and v_{i+2} where $i \in \{0, 1\}$. If the number of triangles increases in both G_0 and G_1 then there exists a triangle $v_iv_{i+1}z$ for some $z \in V(G)$ and $i \in \{0, 1, 2, 3\}$. Moreover, G contains vertices x and y not in F such that $v_{i+1}zxv_{i-1}$ and v_izyv_{i+2} are paths in G. Indices are modulo 4. See Figure 1.*

Proof. Let G, F, G_0 and G_1 be as in the statement of the lemma. Since the number of triangles increases in G_0 there must be a path v_0zyv_2 in G where $z, y \notin F$. Similarly, a new triangle in G_1 implies a path v_1wxv_3 in G where $w, x \notin F$. By the planarity of G, $\{z, y\}$ and $\{w, x\}$ are not disjoint. Without loss of generality assume $z = w$. This results in triangle v_0v_1z and paths v_1zxv_3 and v_0zyv_2. Note that x and y do not have to be distinct. See Figure 1(b). □

Proof of Theorem 2. Let H be a smallest counterexample and G be a plane triangle-free graph such that $G = H - h$ for some edge $h = uv$. Let H have n vertices and e edges and G have f faces. Note that G has n vertices and $e - 1$ edges. By the minimality of H, H is 4-critical. So Theorem 1 implies $e \geq \frac{5n-2}{3}$.

CASE 1: G has at most one 4-face. Then $5f - 1 \leq 2(e - 1)$ and hence $f \leq (2e - 1)/5$. By this and Euler's Formula $n - (e - 1) + f = 2$ applied on G we have $5n - 3e + 1 \geq 5$, i.e., $e \leq \frac{5n-4}{3}$. This contradicts Theorem 1.

CASE 2: Every 4-face of G contains both u and v and there are at least two such 4-faces $F_x = ux_1vx_2$ and $F_y = uy_1vy_2$. If there exists $z \in \{x_1, x_2\} \cap \{y_1, y_2\}$ then z has degree two in G which contradicts the 4-criticality of G.

Let G' be obtained from G by identification of x_1 and x_2 into a new vertex x. If G' is not triangle-free then there is a path $P = x_1q_1q_2x_2$ in G where $q_1, q_2 \notin F_x$. Since P must cross uy_1v and uy_2v, we may assume that $y_1 = q_1$ and $y_2 = q_2$. However, $y_1y_2 \notin E(G)$. This contradicts the existence of P. Hence G' is triangle-free. Let $H' = G' + h$. By the minimality of H, there exists a 3-coloring φ of H'. This contradicts that H is not 4-colorable since φ can be extended to H by letting $\varphi(x_1) = \varphi(x_2) = \varphi(x)$.

CASE 3: G has a 4-face F with vertices $v_0v_1v_2v_3$ in the cyclic order where h is neither v_0v_2 nor v_1v_3. Since G is triangle-free, neither v_0v_2 nor v_1v_3 are edges of G. Lemma 10 implies that either v_0 and v_2 or v_1 and v_3 can be identified without creating a triangle. Without loss of generality assume that G', obtained by from G identification of v_0 and v_2 to a new vertex v, is triangle-free. Let $H' = G' + h$. By the minimality of H, there is a 3-coloring φ of H'. The 3-coloring φ can be extended to H by letting $\varphi(v_0) = \varphi(v_2) = \varphi(v)$ which contradicts the 4-criticality of H. □

Proof of Theorem 4. Let H be a smallest counterexample and G be a plane triangle-free graph such that $G = H - v$ for some vertex v of degree 4. Let H have n vertices and e edges and G have f faces. Then G has $n-1$ vertices and $e - 4$ edges. By minimality, H is 4-critical. So Theorem 1 implies $e \geq \frac{5n-2}{3}$.

CASE 1: G has no 4-faces. Then $5f \leq 2(e-4)$ and hence $f \leq 2(e-4)/5$. By this and Euler's Formula $(n-1) - (e-4) + f = 2$ applied to G, we have $5n - 3e - 8 \geq -5$, i.e., $e \leq \frac{5n-3}{3}$. This contradicts Theorem 1.

CASE 2: G has a 4-face F with vertices $v_0v_1v_2v_3$ in the cyclic order. Since G is triangle-free, neither v_0v_2 nor v_1v_3 are edges of G and Lemma 10 applies. Without loss of generality assume that G_0 obtained from G by identification of v_0 and v_2 is triangle-free.

By the minimality of H, the graph obtained from H by identification of v_0 and v_2 satisfies the assumptions of the theorem and hence has a 3-coloring. Then H also has a 3-coloring, a contradiction. □

Proof of Theorem 5. Let the 3-coloring of F be φ.

CASE 1: F is a 4-face where $v_0v_1v_2v_3$ are its vertices in cyclic order.

CASE 1.1: $\varphi(v_0) = \varphi(v_2)$ and $\varphi(v_1) = \varphi(v_3)$. Let G' be obtained from G by adding a vertex v adjacent to v_0, v_1, v_2 and v_3. Since G' satisfies the assumptions of Theorem 4, there exists a 3-coloring ϱ of G'. In any such 3-coloring, $\varrho(v_0) = \varrho(v_2)$ and $\varrho(v_1) = \varrho(v_3)$. Hence by renaming the colors in ϱ we obtain an extension of φ to a 3-coloring of G.

By symmetry, the other subcase is the following.

CASE 1.2: $\varphi(v_0) = \varphi(v_2)$ and $\varphi(v_1) \neq \varphi(v_3)$. Let G' be obtained from G by adding the edge v_1v_3. Since G' satisfies the assumptions of Theorem 2, there exists a 3-coloring ϱ of G'. In any such 3-coloring, $\varrho(v_1) \neq \varrho(v_3)$ and hence $\varrho(v_0) = \varrho(v_2)$. By renaming the colors in ϱ we obtain an extension of φ to a 3-coloring of G.

CASE 2: F is a 5-face where $v_0v_1v_2v_3v_4$ are its vertices in cyclic order. Observe that up to symmetry there is just one coloring of F. So without loss of generality assume that $\varphi(v_0) = \varphi(v_2)$ and $\varphi(v_1) = \varphi(v_3)$.

Let G' be obtained from G by adding a vertex v adjacent to v_0, v_1, v_2 and v_3. Since G' satisfies the assumptions of Theorem 4, there exists a 3-coloring ϱ of G'. Note that in any such 3-coloring $\varrho(v_0) = \varrho(v_2)$ and $\varrho(v_1) = \varrho(v_3)$. Hence by renaming the colors in ϱ we can extend φ to a 3-coloring of G. □

Proof of Theorem 6. Let G be a smallest counterexample and let $u, v \in V(G)$ be the two non-adjacent vertices colored by φ. If $\varphi(u) \neq \varphi(v)$ then the result follows from Theorem 2 by considering graph obtained from G by adding the edge uv. Hence assume that $\varphi(u) = \varphi(v)$.

CASE 1: G has at most two 4-faces. Let H be a graph obtained from G by identification of u and v. Any 3-coloring of H yields a 3-coloring of G where u and v are colored the same. By this and the minimality of G we conclude that H is 4-critical. Let G have e edges, $n+1$ vertices and f faces.

Since G is planar $5f - 2 \leq 2e$. By this and Euler's formula,

$$2e + 2 + 5(n+1) - 5e \geq 10$$

and hence $e \leq (5n-3)/3$, a contradiction to Theorem 1.

CASE 2: G has at least three 4-faces. Let F be a 4-face with vertices $v_0v_1v_2v_3$ in the cyclic order. Since G is triangle-free, neither v_0v_2 nor v_1v_3 are edges of G. Hence Lemma 10 applies.

Without loss of generality let G_0 from Lemma 10 be triangle-free. By the minimality of G, G_0 has a 3-coloring φ where $\varphi(u) = \varphi(v)$ unless $uv \in E(G_0)$. Since $uv \notin E(G)$, without loss of generality $v_0 = u$ and $v_2v \in E(G)$. Moreover, the same cannot happen to G_1 from Lemma 10, hence G_1 contains a triangle. Thus G contains a path $v_1q_1q_2v_3$ where $q_1, q_2 \notin F$, and G also

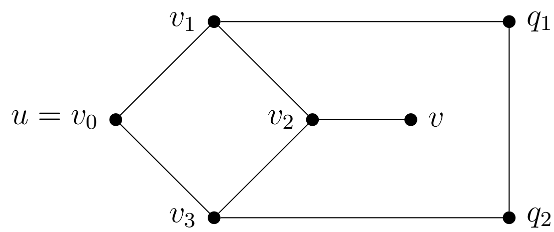

Figure 2: Configuration from Theorem 6.

contains a 5-cycle $C = uv_1q_1q_2v_3$ (see Figure 2). By Theorem 5, C is a 5-face. Hence u is a 2-vertex incident with only one 4-face.

By symmetric argument with a different 4-face, v is also a 2-vertex incident with one 4-face and 5-face. However, G has at least one more 4-face where identification of vertices does not result in the edge uv, a contradiction to the minimality of G. □

Proof of Theorem 8. Let G be a minimal counterexample with e edges, n vertices and f faces. By minimality, G is a 4-critical and has at most two 4-faces or one 3-face. From embedding, $5f - 2 \leq 2e$. By Euler's formula, $2e+2+5n-5e \geq 5$. Hence $e \leq (5n-3)/3$, a contradiction to Theorem 1. □

Borodin used in his proof of Theorem 7 a technique called *portionwise coloring*. We avoid it and build the proof on the previous results arising from Theorem 1.

Proof of Theorem 7. Let G be a smallest counterexample. By minimality, G is 4-critical and every triangle is a face. By Theorem 5 for every separating 4-cycle and 5-cycle C, both the interior and exterior of C contain triangles.

CASE 1: G has no 4-faces. Then $5f - 6 \leq 2e$ and by Euler's Formula $3e + 6 + 5n - 5e \geq 10$, i.e., $e \leq \frac{5n-4}{3}$. This contradicts Theorem 1.

CASE 2: G has a 4-face $F = v_0v_1v_2v_3$ such that $v_0v_2 \in E(G)$. By the minimality, $v_0v_1v_2$ and $v_0v_3v_2$ are both 3-faces and hence G has 4 vertices, 5 edges and it is 3-colorable.

CASE 3: For every 4-face $F = v_0v_1v_2v_3$, neither v_0v_2 nor v_1v_3 are edges of G. By Lemma 10, there exist paths v_0zyv_2 and v_1zxv_3.

CASE 3.1: G contains a 3-prism with one of its 4-cycles being a 4-face. We may assume that this face is our F and $x = y$, see Figure 1(b). Theorem 5 implies that one of zv_0v_3x, zv_1v_2x is a 4-face. Without loss of generality assume that zv_1v_2x is a 4-face. Let G_0 be obtained from G by identification of v_0 and v_2 to a new vertex v. Since G_0 is not 3-colorable, it contains a 4-critical subgraph G_0'. Note that G_0' contains triangle xvz that is not in G but v_0v_1z is not in G_0' since $d(v_1) = 2$ in G_0. By the minimality of G, there exists another triangle T that is in G_0 but not in G. By planarity, $x \in T$. Hence there is a vertex $w_1 \neq v_3$ such that v_0 and x are neighbors of w_1.

By considering identification of v_1 and v_3 and by symmetry, we may assume that there is a vertex $w_2 \neq v_0$ such that v_3 and z are neighbors of w_2. By planarity we conclude that $w_1 = w_2$. This contradicts the fact that G has at most three triangles. Therefore G is 3-prism-free.

CASE 3.2: G contains no 3-prism with one of its 4-cycles being a 4-face. Then $x \neq y$, see Figure 1(a). If $v_0x \in E(G)$ then $G - v_0$ is triangle-free and Theorem 3 gives a 3-coloring of G, a contradiction. Similarly, $v_1y \notin E(G)$.

Suppose that zv_0v_3x if a 4-face. Let G' be obtained from $G-v_0$ by adding edge xv_1. If the number of triangles in G' is at most three, then G' has a 3-coloring φ by the minimality of G. Let ϱ be a 3-coloring of G such that $\varrho(v) = \varphi(v)$ if $v \in V(G')$ and $\varrho(v_0) = \varphi(x)$. Since the neighbors of v_0 in G are neighbors of x in G', ϱ is a 3-coloring, a contradiction. Therefore G' has at least four triangles and hence G contains a vertex $t \neq z$ adjacent to v_1 and x. Since $v_1y \notin E(G)$, the only possibility is $t = v_2$. Having edge xv_2 results in a 3-prism being a subgraph of G which is already excluded. Hence zv_0v_3x is not a face and by symmetry zv_1v_2y is not a face either.

Since neither zv_0v_3x nor zv_1v_2y is a face, each of them contains a triangle in its interior. Since we know the location of all three triangles, Theorem 5 implies that zyv_2v_3x is a 5-face. It also implies that the common neighbors of z and v_3 are exactly v_0 and x, and the common neighbors of z and v_2 are exactly v_1 and y. Without loss of generality, let zyv_2v_3x be the outer face of G.

Let H_1 be obtained from the 4-cycle zv_0v_3x and its interior by adding edge zv_3. The edge zv_3 is in only two triangles, and there is only one triangle in the interior of the 4-cycle. Hence by the minimality of G, there exists a 3-coloring φ_1 of H_1.

Let H_2 be obtained from the 4-cycle zv_1v_2y and its interior by adding edge zv_2. By the same argument as for H_1, there is a 3-coloring of φ_2 of H_2.

Rename the colors in φ_2 so that $\varphi_1(z) = \varphi_2(z)$, $\varphi_1(v_0) = \varphi_2(v_2)$ and $\varphi_1(v_3) = \varphi_2(v_1)$. Then $\varphi_1 \cup \varphi_2$ is a 3-coloring of G, a contradiction. □

Proof of Theorem 9. Let G be a 4-chromatic projective plane graph where

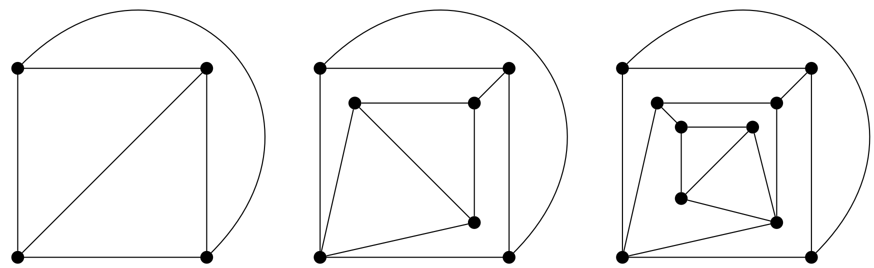

Figure 3: First three 4-critical graphs from the family described by Thomas and Walls [24].

every vertex is in at most one triangle and let G be 4-,5- and 6-cycle free. Then G contains a 4-critical subgraph G'. Let G' have e edges, n vertices and f faces. Since G' is also 4-,5- and 6-cycle-free and every vertex is in at most one triangle, we get $f \leq \frac{n}{3}+\frac{2e-n}{7}$. By Eulers formula, $7n+6e-3n+21n-21e \geq 21$. Hence $e \leq 5n/3 - 21/15$, a contradiction to Theorem 1. $\square$

3 Tightness

This section shows examples where Theorems 2,4,5,6,7, and 8 are tight.

Theorem 2 is best possible because there exists an infinite family [24] of 4-critical graphs that become triangle-free and planar after removal of just two edges. See Figure 3. Moreover, the same family shows also the tightness of Theorem 7, since the construction has exactly four triangles.

Aksenov [1] showed that every plane graph with one 6-face F and all other faces being 4-faces has no 3-coloring in which the colors of vertices of F form the sequence $(1, 2, 3, 1, 2, 3)$. This implies that Theorem 5 is best possible. It also implies that Theorems 4 and 6 are best possible. See Figure 4 for constructions where coloring of three vertices or an extra vertex of degree 5 force a coloring $(1, 2, 3, 1, 2, 3)$ of a 6-cycle.

Theorem 8 is best possible because there exist embeddings of K_4 in the projective plane with three 4-faces or with two 3-faces and one 6-face.

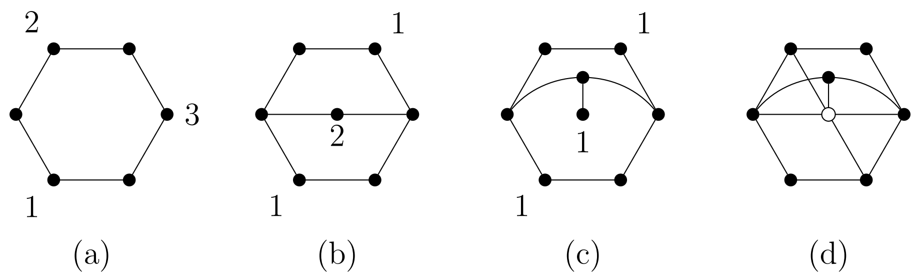

Figure 4: Coloring of three vertices by colors 1, 2 and 3 in (a), (b) and (c) or an extra vertex of degree 5 in (d) forces a coloring of the 6-cycle by a sequence $(1, 2, 3, 1, 2, 3)$ in cyclic order.

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