5-choosability of graphs with crossings far apart[*]

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Abstract

We give a new proof of the fact that every planar graph is 5-choosable, and use it to show that every graph drawn in the plane so that the distance between every pair of crossings is at least 15 is 5-choosable. At the same time we may allow some vertices to have lists of size four only, as long as they are far apart and far from the crossings.

Thomassen [1] showed that every planar graph is 5-choosable. We prove a strengthening of this result, allowing some crossings. Suppose that a graph G is drawn in the plane with some crossings, and for $i \in \{1, 2\}$, let $u_i v_i$ and $x_i y_i$ be two edges of G crossing each other. The *distance* between the crossing of $u_1 v_1$ with $x_1 y_1$ and the crossing of $u_2 v_2$ with $x_2 y_2$ is the length of the shortest path in G with one end in the set $\{u_1, v_1, x_1, y_1\}$ and the other end in the set $\{u_2, v_2, x_2, y_2\}$. Our main result is the following.

Theorem 1. *Every graph drawn in the plane so that the distance between every pair of crossings is at least* 15 *is* 5-*choosable.*

Let us recall that a *list assignment* L for G is a function that assigns to each vertex of G a set $L(v)$, called the *list of admissible colors* for v. An *L-coloring* is a choice of a color $\varphi(v) \in L(v)$ for each $v \in V(G)$ such that no two adjacent vertices receive the same color. The graph is *k-choosable* if it admits an L-coloring for every list assignment L with $|L(v)| \geq k$ for every $v \in V(G)$.

The main idea of Thomassen's beautiful proof of 5-choosability of planar graphs is to establish the following more general statement.

Theorem 2 (Thomassen [1])**.** *Let G be a plane graph with the outer face F, xy an edge of F, and L a list assignment such that $|L(v)| \geq 5$ for $v \in V(G) \setminus V(F)$, $|L(v)| \geq 3$ for $v \in V(F) \setminus \{x, y\}$, $|L(x)| = |L(y)| = 1$ and $L(x) \neq L(y)$. Then G is L-colorable.*

Let us note that the lists of x and y of size 1 give a precoloring of a path of length 1 in the outer face of G. Naturally, one might try to prove Theorem 1 by showing a variant of Theorem 2 allowing distant crossings. Unfortunately, almost any attempt to alter the statement of Theorem 2 (e.g., by allowing more than two vertices to be precolored, allowing lists of size 2 subject to some constraints, allowing some crossings in the drawing, etc.) fails with infinitely many counterexamples. To overcome this obstacle, we give a new proof of 5-choosability of planar graphs, see Theorem 6 in Section 1, which turns out to be more robust with respect to some strengthenings of the planar 5-choosability theorem. The proof of Theorem 6 is inspired by Thomassen's proof [2] of 3-choosability of planar graphs of girth 5.

In the course of the proof of Theorem 1, it is convenient to allow in addition to crossings also some other irregularities, such as vertices with fewer than 5 available colors. Hence, we actually obtain the following stronger statement (the distance between a vertex z and a crossing of edges uv and xy is the length of the shortest path from z to $\{u, v, x, y\}$).

Theorem 3. *Let G be a graph drawn in the plane with some crossings and let $N \subseteq V(G)$ be a set of vertices such that the distance between any pair of crossed edges is at least* 15, *the distance between any crossing and a vertex in N is at least* 13, *and the distance between any two vertices in N is at least* 11. *Then G is L-colorable for any list assignment L such that $|L(v)| = 4$ for $v \in N$ and $|L(v)| \geq 5$ for $v \in V(G) \setminus N$.*

Some distance condition on the crossings in Theorem 1 is necessary, even if we allow only three crossings, as shown by K_6. On the other hand, it was proved in [3] and independently also in [4] that the distance requirement is not needed, if we have at most two crossings. The inductive proof of Theorem 3 involves a stronger inductive hypothesis that is stated later as Theorem 7 and in particular also implies the above-mentioned result from [3, 4], see Section 3.

Theorem 4 ([3, 4]). *Every graph whose crossing number is at most two is 5-choosable.*

The proof of Theorem 4 is given at the end of Section 3. Another special case of Theorem 3 is the following.

Theorem 5. *Let G be a planar graph and $N \subseteq V(G)$ a set of vertices such that the distance between any pair of vertices in N is at least 11. Then G is L-colorable for any list assignment L such that $|L(v)| = 4$ for $v \in N$ and $|L(v)| \geq 5$ for $v \in V(G) \setminus N$.*

The last result is motivated by the result of Voigt [5] showing that not all planar graphs are 4-choosable. Furthermore, it is related to the following problem of Albertson [6]:

Problem 1. *Does there exist a constant d such that whenever G is a planar graph with list assignment L that gives a list of size one or five to each vertex and the distance between any pair of vertices with list of size one is at least d, then G is L-colorable?*

Building upon the ideas of this paper, we were able to give a positive answer to this problem, which we present in a separate paper [7].

We start with giving the new proof of 5-choosability of planar graphs in Section 1. After introducing some terminology in Section 2, we formulate a strengthening of Theorem 3 suitable for the inductive argument in Section 3. We give a proof of this more general statement in Section 4.

1 Planar graphs

Let P be a path or a cycle. The *length $\ell(P)$* of P is the number of its edges, i.e., a path of length l has $l + 1$ vertices and a cycle of length l has

l vertices. Given a graph G and a cycle $K \subseteq G$, an edge uv of G is a *chord* of K if $u, v \in V(K)$, but uv is not an edge of K. For an integer $k \geq 2$, a path $v_0 v_1 \ldots v_k$ is a *k-chord* if $v_0, v_k \in K$ and $v_1, \ldots, v_{k-1} \notin V(K)$. We define a chord to be a 1-*chord*. If G is a plane graph, let $\mathrm{Int}_K(G)$ be the subgraph of G consisting of the vertices and edges drawn inside the closed disc bounded by K, and $\mathrm{Ext}_K(G)$ the subgraph of G obtained by removing all vertices and edges drawn inside the open disc bounded by K. In particular, $K = \mathrm{Int}_K(G) \cap \mathrm{Ext}_K(G)$. Note that each k-chord of K belongs to exactly one of $\mathrm{Int}_K(G)$ and $\mathrm{Ext}_K(G)$. We say that a cycle K is *separating* if $V(\mathrm{Int}_K(G)) \neq V(K) \neq V(\mathrm{Ext}_K(G))$. If the cycle K bounds the outer face of G and Q is a k-chord of K, let C_1 and C_2 be the two cycles in $K \cup Q$ that contain Q. Then the subgraphs $G_1 = \mathrm{Int}_{C_1}(G)$ and $G_2 = \mathrm{Int}_{C_2}(G)$ are the *Q-components* of G.

As we mentioned earlier, Thomassen's Theorem 2 does not extend to the case when we have a precolored path of length two. However, if we strengthen the condition on the list sizes of the other vertices on the outer face, such an extension is possible.

Theorem 6. *Let G be a plane graph with the outer face F, P a subpath of F of length at most two and L a list assignment such that the following conditions are satisfied:*

(i) *$|L(v)| \geq 5$ for $v \in V(G) \setminus V(F)$,*

(ii) *$|L(v)| \geq 3$ for $v \in V(F) \setminus V(P)$,*

(iii) *$|L(v)| = 1$ for $v \in V(P)$,*

(iv) *no two vertices with lists of size three are adjacent in G,*

(v) *L gives a proper coloring to the subgraph induced by $V(P)$, and*

(vi) *if $P = uvw$ has length two and x is a common neighbor of u, v and w, then $L(x) \neq L(u) \cup L(v) \cup L(w)$.*

Then G is L-colorable.

Proof. Suppose for a contradiction that the claim is false, and let G be a counterexample with $|V(G)| + |E(G)|$ the smallest possible, and subject to that, with the longest path P and with the minimum size of the lists (while satisfying (i)–(vi)). It is clear that G is connected and that every vertex $v \in V(G)$ satisfies $\deg(v) \geq |L(v)|$.

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Furthermore, G is 2-connected: otherwise, let v be a cut-vertex and let G_1 and G_2 be subgraphs of G such that $G_1 \cup G_2 = G$, $V(G_1) \cap V(G_2) = \{v\}$ and $|V(G_1)|, |V(G_2)| > 1$. If $v \in V(P)$, then by the minimality of G there exist L-colorings of G_1 and G_2, and these colorings together give an L-coloring of G. Otherwise, we may assume by symmetry that $P \subseteq G_1$. Consider an L-coloring φ of G_1. Let L_2 be the list assignment for G_2 such that $L_2(u) = L(u)$ for $u \neq v$ and $L_2(v) = \{\varphi(v)\}$. By the minimality of G, G_2 is L_2-colorable, and this coloring together with φ gives an L-coloring of G.

Every triangle T in G bounds a face: otherwise, first color the subgraph $\text{Ext}_T(G)$ and then extend the coloring to $\text{Int}_T(G)$. A similar argument shows that G contains no separating 4-cycles; otherwise, consider such a 4-cycle $K = k_1 k_2 k_3 k_4$, and let φ be an L-coloring of $\text{Ext}_K(G)$. Let $G' = \text{Int}_K(G)$. Since K is separating, we have $V(G') \neq V(K)$, and since every triangle bounds a face, we conclude that K has no chord in G'. Let L' be the list assignment for $G' - k_1$ such that $L'(z) = \{\varphi(z)\}$ for $z \in \{k_2, k_3, k_4\}$, $L'(z) = L(z) \setminus \{\varphi(k_1)\}$ if $z \notin \{k_2, k_4\}$ is a neighbor of k_1 and $L'(z) = L(z)$ if z is any other vertex. By the minimality of G, the graph $G' - k_1$ is L'-colorable, and this coloring together with φ gives an L-coloring of G.

Since G is 2-connected, its outer face is bounded by a cycle, which we denote by F as well. Next, we show that F has no chords. Otherwise, let uv be a chord of F and let G_1 and G_2 be the uv-components of G. If $P \subseteq G_1$, then we first color G_1 and then extend the coloring to G_2. The case that $P \subseteq G_2$ is symmetric. It follows that P has length two and all the chords of F are incident with its middle vertex. Let $P = z_1 u z_2$, where $z_i \in V(G_i)$ for $i \in \{1, 2\}$. Let φ be an L-coloring of G_1 and let L_2 be the list assignment for G_2 such that $L_2(z) = L(z)$ for $z \neq v$ and $L_2(v) = \{\varphi(v)\}$. Since G is not L-colorable, G_2 is not L_2-colorable. By the minimality of G, either v is adjacent to z_2, or u, v and z_2 have a common neighbor w with list of size three (which means, in particular, that $w \in V(F)$). Since every chord of G is incident with u, the edge vz_2 or vw belongs to F. Since every triangle bounds a face, we conclude that v has degree two in G_2. By symmetry, v has degree two in G_1 as well, and thus v has degree three in G. It follows that $|L(v)| = 3$, and thus v cannot be adjacent to any other vertex with list of size three. In particular, we cannot have the case with the vertex w. We conclude that v is adjacent to z_1 and z_2 and $V(G) = \{u, v, z_1, z_2\}$. However, $L(v) \neq L(u) \cup L(z_1) \cup L(z_2)$ by (vi), and thus G is L-colorable. This contradiction proves that F has no chords.

Similarly, we have the following property:

(1) *Let uvw be a 2-chord of F and let G_1 and G_2 be uvw-components of G. If $P \subseteq G_1$, then either u and w are adjacent and G_2 is equal to the triangle uvw, or there exists a vertex x such that $V(G_2) = \{u, v, w, x\}$, $|L(x)| = 3$ and x is adjacent to u, v and w.*

If $\ell(P) < 2$, then it is easy to see that we can precolor $2 - \ell(P)$ more vertices of F without violating (vi). Thus, we may assume that $\ell(P) = 2$. Let $P = p_0 p_1 p_2$. Suppose that p_0, p_1 and p_2 have a common neighbor v. If $v \in V(F)$, then $V(G) = \{p_0, p_1, p_2, v\}$ and G is L-colorable. If $v \notin V(F)$, then v has degree at most four in G by (1) and thus $\deg(v) < |L(v)|$, which is a contradiction. Therefore, p_0, p_1 and p_2 have no common neighbor.

Furthermore, $\ell(F) \geq 6$: If $\ell(F) = 3$, then we remove one vertex of F and remove its color from the lists of all its neighbors, and observe that the resulting graph is a smaller counterexample to Theorem 6. In the case when $\ell(F) = 4$, then similarly color and remove the vertex of $V(F) \backslash V(P)$. Finally, suppose that $\ell(F) = 5$. Let φ be an arbitrary L-coloring of $F = p_2 p_1 p_0 v_1 v_2$. Remove v_1 and v_2 from G and remove their colors according to φ from the lists of their neighbors, obtaining a graph G' with the list assignment L'. Since every triangle in G bounds a face, at most one vertex in G' has list of size three. Since p_0, p_1 and p_2 have no common neighbor and p_0 is not adjacent to p_2, G' with the list assignment L' is a smaller counterexample to Theorem 6, which is a contradiction.

Let $F = p_2 p_1 p_0 v_1 v_2 v_3 v_4 \ldots$ If $\ell(F) = 6$, then we set $v_4 = p_2$. We may assume that $|L(v_1)| = 3$ or $|L(v_2)| = 3$, since otherwise we can remove a color from the list of v_1. Let us consider a set $X \subseteq V(F) \backslash V(P)$ and a partial L-coloring φ of X that are defined as follows:

(X1) If $|L(v_1)| = 3$ and $|L(v_3)| \neq 3$, then $X = \{v_1\}$ and $\varphi(v_1) \in L(v_1) \backslash L(p_0)$ is chosen arbitrarily.

(X2) If $|L(v_1)| = 3$ and $|L(v_3)| = 3$, then $X = \{v_1, v_2\}$ and φ is chosen so that $\varphi(v_2) \in L(v_2) \backslash L(v_3)$ and $\varphi(v_1) \in L(v_1) \backslash (L(p_0) \cup \{\varphi(v_2)\})$.

(X3) If $|L(v_2)| = 3$, and either $|L(v_4)| \neq 3$ or $|L(v_3)| \geq 5$, then $X = \{v_2\}$ and $\varphi(v_2) \in L(v_2)$ is chosen arbitrarily.

(X4) If $|L(v_2)| = 3$, $|L(v_3)| = 4$ and $|L(v_4)| = 3$, then:

(X4a) If v_1, v_2 and v_3 do not have a common neighbor or $|L(v_1)| \geq 5$, then $X = \{v_2, v_3\}$ and φ is chosen so that $\varphi(v_3) \in L(v_3) \backslash L(v_4)$ and $\varphi(v_2) \in L(v_2) \backslash \{\varphi(v_3)\}$.
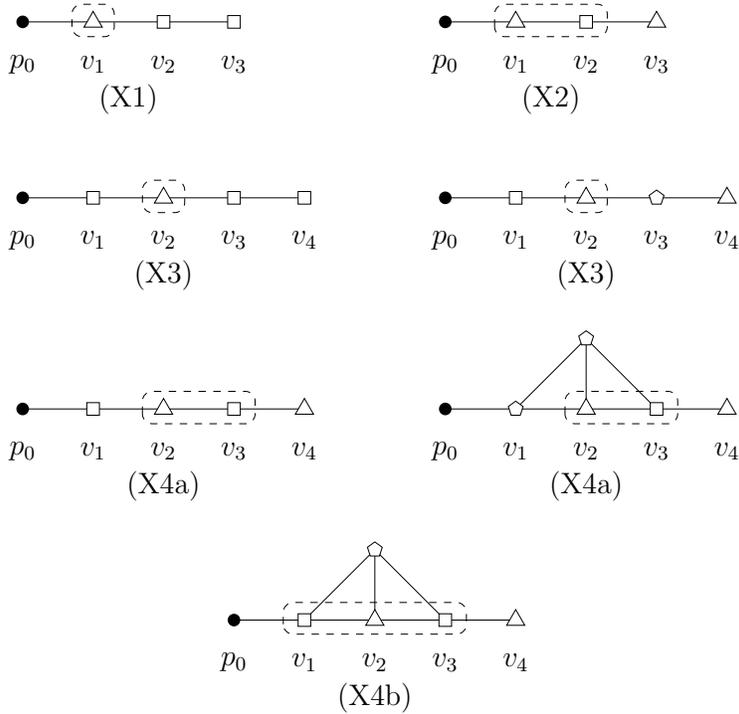
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Figure 1: Subcases in the definition of X. Triangle vertices represent lists of size 3, square vertices list of size ≥ 4. Encircled vertices are in X.

(X4b) If v_1, v_2 and v_3 have a common neighbor and $|L(v_1)| = 4$, then $X = \{v_1, v_2, v_3\}$ and φ is chosen so that $\varphi(v_3) \in L(v_3) \setminus L(v_4)$, $\varphi(v_1) \in L(v_1) \setminus L(p_0)$ and either at least one of $\varphi(v_1)$ and $\varphi(v_3)$ does not belong to $L(v_2)$, or $\varphi(v_1) = \varphi(v_3)$. The vertex v_2 is left uncolored.

For later reference, Figure 1 shows the subcases used in the definition of X and φ.

Let $G' = G - X$ and let L' be the list assignment obtained from L by removing the colors of the vertices of X according to φ from the lists of their neighbors (if a vertex of X is not colored according to φ, we do not remove any colors for it). Observe that any L'-coloring of G' can be extended to an L-coloring of G, thus G' is not L'-colorable. By the minimality of G, this implies that G' violates the assumptions of Theorem 6. Since F has no chords,

the choice of X and φ implies that every vertex of $V(G') \setminus V(P)$ has list of size at least three. Since p_0 is not adjacent to p_2, and since p_0, p_1, and p_2 do not have a common neighbor in G, it follows that the conditions (v) and (vi) are satisfied by G' with the list assignment L'. We conclude that (iv) is false, i.e., G' contains adjacent vertices u and v such that $|L'(u)| = |L'(v)| = 3$.

Since F has no chords, the choice of X ensures that at most one of u and v belongs to $V(F)$; hence, we can assume that $v \notin V(F)$ and v has two neighbors in $\mathrm{dom}(\varphi)$. In particular, X and φ were chosen according to the cases (X2) or (X4). Since G contains no separating cycles of length at most 4, we conclude that u has at most one neighbor in $\mathrm{dom}(\varphi)$. Since $|L'(u)| = 3$, it follows that $u \in V(F)$. Let $x \in X$ be the neighbor of v such that the distance between u and x in $F - P$ is as large as possible. By (1) applied to the 2-chord xvu, we conclude that the xvu-component of G that does not contain P consists of xvu and a vertex z adjacent to x, v and u with $|L(z)| = 3$. It follows that $|L(u)| > 3$, and since $|L'(u)| = 3$, we have $z \in X$ and $|L(u)| = 4$. The inspection of the choice of X shows that (X4) holds, i.e., $u = v_1$, $z = v_2$ and $x = v_3$. However, note that the condition of (X4b) holds; hence $u \in X$, contrary to the assumption that $u \in V(G')$. This completes the proof of Theorem 6. $\qquad\square$

2 Drawings with crossings

In this section, we introduce terminology concerning graphs drawn in the plane with crossings.

Let G be a graph. A *drawing* $\mathcal{G}$ of a graph G in the plane consists of a set $\mathcal{V} = \{p_v \mid v \in V(G)\}$ of distinct points in the plane and a set of simple polygonal curves $\mathcal{E} = \{c_e \mid e \in E(G)\}$ such that

- if $uv \in E(G)$, then p_u and p_v are the endpoints of c_{uv},

- no internal point of any $c_e \in \mathcal{E}$ belongs to $\mathcal{V}$, and

- any point in the plane that does not belong to $\mathcal{V}$ is contained in at most two of the curves in $\mathcal{E}$, and any two curves in $\mathcal{E}$ have at most one point in common.

A *crossing* of $\mathcal{G}$ is a point in the plane that belongs to two of the curves in $\mathcal{E}$, but not to $\mathcal{V}$. An edge e is *incident with the crossing* x if $x \in c_e$. An edge e is *crossed* if it is incident with some crossing, and *non-crossed* otherwise. For a

crossing x, we define G_x to be the graph consisting of the two edges incident with x. Two vertices of G are *crossing-adjacent* if they belong to G_x for some crossing x and are not adjacent in G_x. By a slight abuse of terminology, we will generally use e to refer to the curve p_e and v to refer to the point p_v in the drawing, and we will refer by G to both the graph G and its drawing $\mathcal{G}$.

Removal of $\bigcup \mathcal{E}$ splits the plane into several connected subsets, which we call *faces* of G. Slightly abusing the terminology again, we sometimes identify a face with its boundary and hence speak about the vertices, edges and crossings of the face. Let us remark that due to the existence of crossings, the boundary of a face may have somewhat more complicated structure than in the plane case. Let u and v be two consecutive vertices in the boundary of a face f. The part of the boundary of f between u and v is either formed by an edge (which is non-crossed), or by parts of two crossing edges ux and vy (and u is then crossing-adjacent to v). In the latter case, x and y may or may not be incident with f themselves (and even if say x is incident with f, there may be a part of the edge ux that is not contained in the boundary of f). Let us remark that if both parts of edges ux and vy between the crossing and x and y are contained in the boundary of f, then we can redraw G to eliminate the crossing by "flipping" the part of the drawing of G that contains v and x.

By a *chord* or *1-chord* of the face f, we mean an edge e of G such that both endpoints of e are contained in the boundary of f, and either e is crossed, or e is non-crossed and not contained in the boundary of f. A *k-chord* of f for $k \geq 2$ is a path $v_0 v_1 \ldots v_k$ such that v_0 and v_k are contained in the boundary of f and v_i is not contained in the boundary of f for $i = 1, \ldots, k-1$, and the edges of the path do not cross each other. Note that edges of a k-chord may or may not be crossed by other edges not belonging to the k-chord.

For a cycle K whose edges do not cross each other in G, let $\text{Int}_K(G)$ denote the subgraph of G consisting of the vertices and edges drawn fully inside the closed disc bounded by K (i.e., excluding the edges crossing with the edges of K), and let $\text{Ext}_K(G)$ denote the subgraph of G obtained by removing all vertices and edges intersecting the open disc bounded by K (so again, $\text{Ext}_K(G)$ does not contain the edges crossing with the edges of K).

Suppose that G is 2-connected, and consider a k-chord Q of the outer face F of G. Let c be a closed curve consisting of Q and of an arbitrary curve in the interior of F joining the endpoints of Q. Let G_1 be the subgraph of G consisting of the vertices and edges drawn fully inside the closed disc bounded by c, and let G_2 be the subgraph of G obtained by removing all

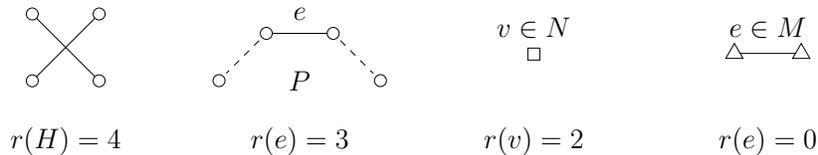$$r(H) = 4 \qquad r(e) = 3 \qquad r(v) = 2 \qquad r(e) = 0$$

Figure 2: Special subgraphs and their ranks.

vertices and edges intersecting the open disc bounded by c (so neither G_1 nor G_2 contains the edges of G that cross Q). We say that G_1 and G_2 are the *Q-components* of G (note that since G is 2-connected, the Q-components are uniquely determined, independently of the exact choice of the curve c).

3 Near-planar graphs

We now aim to show that graphs drawn in the plane with crossings far apart are 5-choosable. For the purposes of the induction, it will be useful to allow other kinds of irregularities (adjacent vertices with list of size three, as well as vertices with list of size four not incident with the outer face, which arise when some vertices incident with a crossing are colored and their colors are removed from the lists of their neighbors), subject to distance constraints. This results in a rather technical statement, and we devote this section to its exact formulation.

Let us fix a drawing of a graph G in the plane, possibly with crossings. Let P be a path of length at most three contained in the boundary of the outer face F of G (where in particular, no edge of P is crossed), let N be a subset of $V(G)$ and M a subset of $E(G)$, and let L be a list assignment for G (generally, the set N will contain vertices with list of size 4, while the edges of M will join vertices with lists of size three). We say that a 5-tuple (G, P, N, M, L) is a *target*.

For a target (G, P, N, M, L), we define some subgraphs H of G to be *special*, and assign a *rank $r(H)$* to each such subgraph (see Figure 2). Specifically, H is *special* if it falls into one of the following four cases:

- H consists of the two edges incident with a crossing. In this case, its rank is 4.
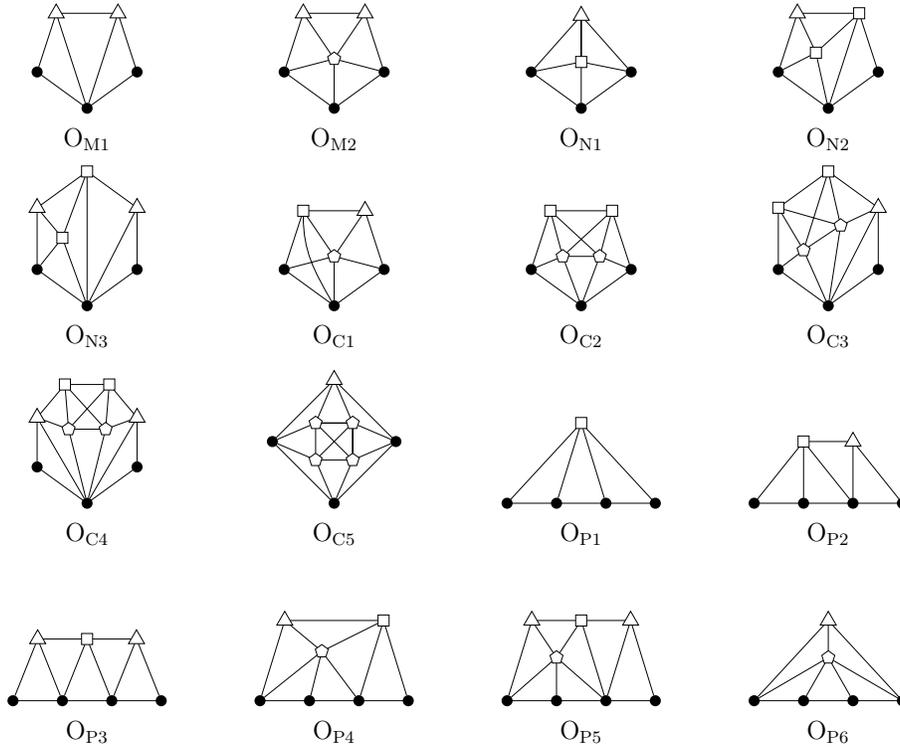
Figure 3: The obstructions used in Theorem 7.

- P has length three and H consists of the middle edge of P; the rank of H is 3.

- H is equal to a vertex of N, and $r(H) = 2$.

- H is equal to an edge of M, and $r(H) = 0$.

For two subgraphs $H_1, H_2 \subseteq G$, the *distance* $d(H_1, H_2)$ between H_1 and H_2 is the minimum of the distances between the vertices of H_1 and H_2, i.e., the minimum k such that there exists a path $v_0 v_1 \ldots v_k$ in G with $v_0 \in V(H_1)$ and $v_k \in V(H_2)$.

Given a target (G, P, N, M, L), a subgraph $O \subseteq G$ is an *obstruction* if O (with its drawing induced from the drawing of G) is isomorphic to one of the graphs drawn in Figure 3 and sizes of the lists of its vertices match those prescribed by the figure, where the full-circle vertices have list of size one, triangle vertices have list of size three, square vertices have list of size

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four and pentagonal vertices have list of size five. Note that each obstruction contains a special subgraph of G.

We say that the target (G, P, N, M, L) is *valid* if the following conditions are satisfied (with F denoting the outer face of G):

(S) $|L(v)| \geq 5$ for $v \in V(G) \setminus (V(F) \cup N)$, $|L(v)| \geq 3$ for $v \in V(F) \setminus V(P)$ and $|L(v)| = 1$ for $v \in V(P)$,

(N) $|L(v)| \geq 4$ for $v \in N \setminus V(F)$,

(M) if $|L(u)| = |L(v)| = 3$ and u and v are adjacent, then $uv \in M$,

(P) L gives a proper coloring to the subgraph induced by $V(P)$,

(T) if a vertex v has three neighbors w_1, w_2, w_3 in $V(P)$, then $L(v) \neq L(w_1) \cup L(w_2) \cup L(w_3)$,

(C) if x is a crossing and G_x contains a vertex with list of size three, then all other vertices of G_x have lists of size 1 or ≥ 5,

(D) $d(H_1, H_2) \geq r(H_1) + r(H_2) + 7$ for every pair $H_1 \neq H_2$ of special subgraphs of G, and

(O) if the target contains an obstruction O, then O is L-colorable.

We prove the following claim, which implies our main result, Theorem 3 (since each graph satisfying the assumptions of Theorem 3 can be transformed into a legal target in a natural way, with P consisting of a single vertex so that the target contains no obstructions).

Theorem 7. *If (G, P, N, M, L) is a valid target, then G is L-colorable.*

We sometimes refer to (D) as the *distance condition*. The purpose of the introduced rank function is the following. In our inductive arguments, we will occasionally construct a smaller graph G' and introduce a new special subgraph H' in a vicinity of a special subgraph H that would no longer exist in G'. If H' has smaller rank than H, then the distance condition for special subgraphs in G' would still hold, and the induction hypothesis can be applied.

Let us remark that if the distance condition holds, then G can contain at most one of the obstructions. For further reference we exhibit in Figure 4 all possible list assignments for which the obstructions are not colorable. In particular, observe that the following holds:

(2) *Let H be one of the obstructions and let Q be the path in H consisting of full-circle vertices. Suppose that Q has length two and that H is neither O_{M1} nor O_{C1}. Let q be the middle vertex of Q and let L be a list assignment such that each vertex v drawn by a k-gon has $|L(v)| = k$, while the vertices of Q have lists consisting of all possible colors. Then there exists a color b such that every L-coloring ψ of Q with $\psi(q) \neq b$ extends to an L-coloring of H.*

Let us now give a quick outline of the proof of Theorem 7; we work out all the details in the following section. Essentially, we follow the proof of Theorem 6. First, we show that the outer face of a hypothetical minimal counterexample G has no chords and then we also restrict its 2-chords. This is somewhat more complicated due to the presence of crossings and the condition (O). Next, we find the set X and its partial coloring φ defined in the same way as in the proof of Theorem 6, and use it to construct the graph G' with the list assignment L'. By the minimality of G, we conclude that G' violates one of the assumptions of the theorem. A straightforward case analysis shows that (O) holds, the definition of the rank function ensures that (D) holds, and the conditions (S), (P) and (T) follow in the same way as in the proof of Theorem 6; but (M), (N) and (C) can be violated in ways which do not enable us to obtain a contradiction directly. However, we observe that in such a case, there is a special subgraph S near to X. In this situation, we apply the symmetric argument on the other side of the path P, and obtain another set X' and a special subgraph S' close to it. By the distance condition, we have $S = S'$, and thus there exists a short path from X to X' passing through S. In this situation, we consider all the possible combinations of X, X' and their positions relatively to S, and obtain a contradiction similarly to the way we deal with 2-chords.

Let us note that the assumption (C) is a product of a somewhat delicate tradeoff. We believe the claim still essentially holds even without this assumption, and avoiding it would greatly reduce the number of possible bad cases and simplify the last part of the proof. However, the list of obstructions in (O) would be significantly larger, making the first part of the proof longer and more complicated. Moreover, if we omit (C) completely, then there exists an obstruction with a precolored path of length one (see Figure 5(a)), which would be a major problem (we could not easily get rid of chords of F). One could consider excluding Figure 5(a) by forbidding vertices with lists of sizes three or four joined by a crossed edge. This would still simplify the last part of the proof a lot. However, in addition to having more than 10
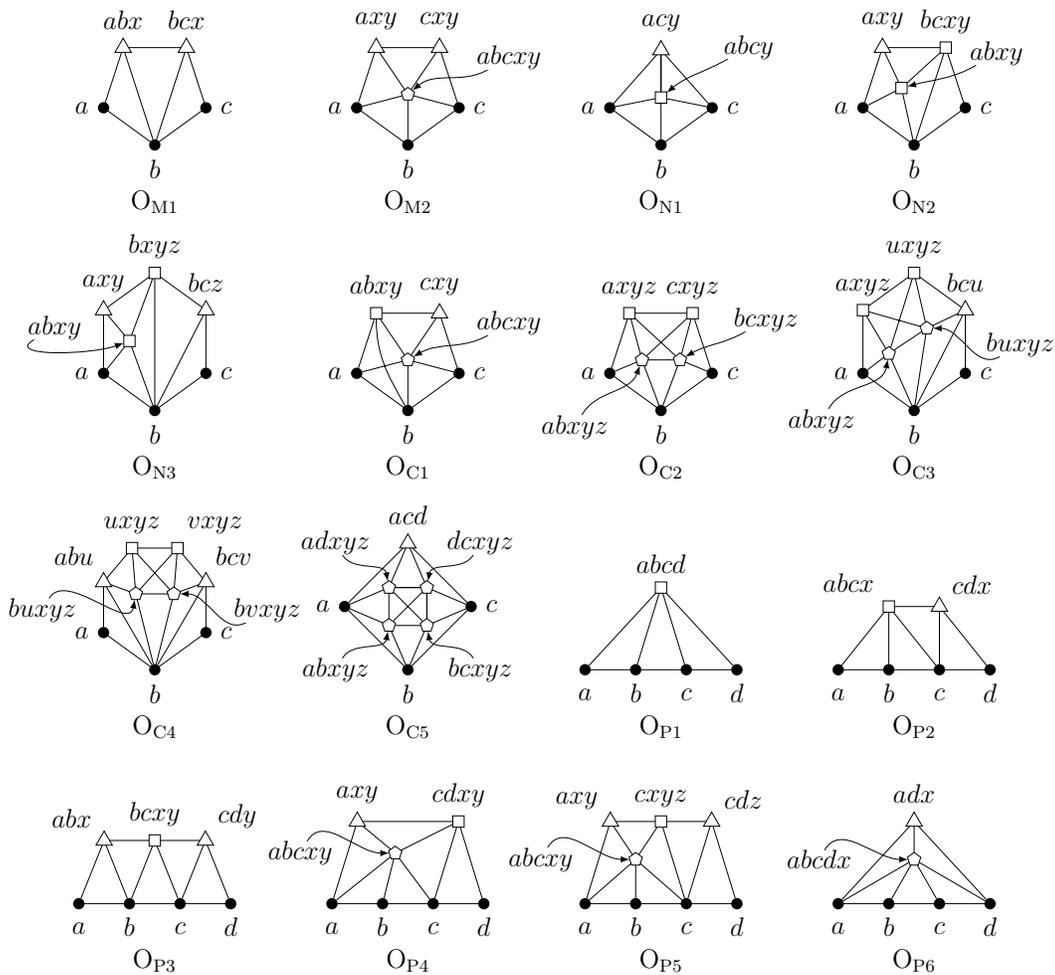
13

Figure 4: The lists for which the obstructions cannot be colored. Colors represented by different letters may be equal to each other if they do not occur in the same list for a particular obstruction.
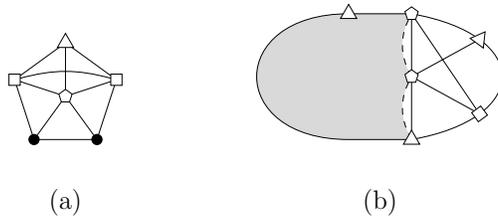
14

(a)   (b)

Figure 5: Why is condition (C) needed?

new obstructions, we do not see a way how to reduce the 2-chord depicted in Figure 5(b), which would need to be dealt with somehow.

Before we proceed with the proof of Theorem 7, let us show how it implies Theorem 4.

Proof of Theorem 4. Let G be a graph with crossing number at most two. We may assume that G is nonplanar. Consider a drawing of G in the plane with one or two crossings and let L be a list assignment such that each vertex has five admissible colors. Let xy and uv be two edges crossing each other at the crossing q. Suppose first that the edges xy and uv do not participate in another crossing. Now remove the two edges and add the edges xu, uy, yv, and vx (if they are not already present). This gives rise to a graph G' with at most one crossing, and we can redraw it so that the cycle $xuyv$ bounds the outer face. Let L' be a list assignment such that $L'(x) \subset L(x)$, $L'(y) \subset L(y)$, and $L'(u) \subset L(u)$ are pairwise distinct sets of size 1, $L'(v) = L(v) \setminus L'(u)$, and $L'(z) = L(z)$ for $z \in V(G) \setminus \{u, v, x, y\}$. Theorem 7 implies that G' has an L'-coloring φ, and the choice of L' ensures that φ is also an L-coloring of G.

If the edge uv participates in another crossing, then xy does not participate in another one. Suppose that the segment of uv from u to the crossing q does not contain the other crossing. Then we proceed similarly as above: we remove the edges xy and uv and add edges xu and uy. The resulting graph G' is planar and the path $P = xuy$ is part of a facial walk. Let L' be defined as before; by Theorem 7 with $N = \{v\}$, G' is L'-colorable, and thus G is L-colorable. □

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4 The proof of Theorem 7

We say that a target (G, P, N, M, L) is a *counterexample* if it is valid and G is not L-colorable. We proceed by contradiction, showing that no counterexample exists. A counterexample (G, P, N, M, L) is *minimal* if there exists no counterexample (G', P', N', M', L') satisfying

- $|V(G')| + |E(G')| < |V(G)| + |E(G)|$, or

- $|V(G')| + |E(G')| = |V(G)| + |E(G)|$ and $\sum_{v \in V(G')} |L'(v)| < \sum_{v \in V(G)} |L(v)|$.

We follow the outline of the proof of Theorem 6. Let us first start by establishing some basic properties of hypothetical minimal counterexamples.

Lemma 8. *Every minimal counterexample $B = (G, P, N, M, L)$ with the outer face F has the following properties.*

(a) *Every vertex $v \in V(G)$ satisfies $\deg(v) \geq |L(v)|$.*

(b) *G is 2-connected and $\ell(P) \geq 1$.*

(c) *Every non-crossed chord of F is incident with exactly one vertex of P, and this vertex is internal in P.*

(d) *If K is a triangle in G and no edge of K is crossed, then K is not separating. If K is a separating 4-cycle without crossed edges, then $\text{Int}_K(G) - V(K)$ is either a vertex in N or a complete graph on 4 vertices involving a crossing.*

(e) *Every vertex $v \in V(G)$ satisfies $|L(v)| \leq 5$.*

(f) *If $v \in V(G) \setminus V(P)$ is adjacent to a vertex $p \in V(P)$, then $L(p) \subseteq L(v)$.*

Proof. For (a), if $v \in V(G)$ has degree less than $|L(v)|$, then every L-coloring of $G - v$ extends to an L-coloring of G, and thus the target obtained from B by removing v is a counterexample contradicting the minimality of B.

For (f), it suffices to note that if $vp \in E(G) \setminus E(P)$ and $L(p) \cap L(v) = \emptyset$, then removing e would result in a smaller counterexample. Together with the property (P), this also implies that no chord of F joins two vertices of P.

Hence, for (c), it suffices to consider a non-crossed chord uv of G that is not incident with an internal vertex of P. Let G_1 and G_2 be the uv-components of G such that $P \subseteq G_1$. By the minimality of B, there exists an L-coloring φ of G_1. Let $L'(u) = \{\varphi(u)\}$, $L'(v) = \{\varphi(v)\}$ and $L'(x) = L(x)$

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for $x \in V(G_2) \backslash \{u, v\}$. Since G is not L-colorable, it follows that G_2 is not L'-colorable, and thus $(G_2, uv, N \cap V(G_2), M \cap E(G_2), L')$ is a counterexample contradicting the minimality of B.

The claim (b) and the first part of claim (d) (concerning triangles) is proved similarly to the previous paragraph. For the second part of (d), let $K = v_1 v_2 v_3 v_4$ be a separating 4-cycle without crossed edges. If all vertices of K have a common neighbor $x \in N \cap V(\mathrm{Int}_K(G))$, then by the first part of (d), $\mathrm{Int}_K(G) - V(K)$ consists only of x. Otherwise, we can choose the labels of the vertices of K so that v_4 is not adjacent to a vertex in $N \cap V(\mathrm{Int}_K(G))$. Furthermore, using (D), we can choose the labels so that v_4 is not incident with a crossed edge other than possibly the chord $v_4 v_2$. By the minimality of B, there exists an L-coloring φ of the subgraph of G consisting of $\mathrm{Ext}_K(G)$ and all chords of K. Consider the graph $G' = \mathrm{Int}_K(G) - v_4$ with the list assignment L' given by $L'(v_i) = \{\varphi(v_i)\}$ for $i = 1, 2, 3$, $L'(x) = L(x) \backslash \{\varphi(v_4)\}$ for all neighbors x of v_4 distinct from v_1, v_2, and v_3, and $L'(x) = L(x)$ for all other vertices x of G'. Since G is not L-colorable, it follows that G' is not L'-colorable, and by the minimality of B, the target $B' = (G', v_1 v_2 v_3, N \cap V(G'), \emptyset, L')$ is not valid. By the choice of v_4, we have $|L'(x)| \geq 4$ for all $x \in V(G') \backslash \{v_1, v_2, v_3\}$, and thus the target B' may only violate (O); and if it violates (O), then B' contains the obstruction O_{C2}. However, together with the first part of (d), this implies that $\mathrm{Int}_K(G) - V(K)$ is a complete graph on 4 vertices involving a crossing.

Property (e) is achieved by removing colors from lists of size 6 or more. The only problem that may arise is that we obtain an obstruction; however, inspection of bad lists for the obstructions exhibited in Figure 4 shows that we can always remove one of the colors so that (O) still holds. □

Next, we strengthen Lemma 8(d) for triangles.

Lemma 9. *Let $B = (G, P, N, M, L)$ be a minimal counterexample and let T be a triangle in G (possibly with a crossed edge). If T does not bound the outer face of G, then $V(\mathrm{Int}_T(G)) = V(T)$.*

Proof. Let $T = v_1 v_2 v_3$. The claim follows by Lemma 8(d) if no edge of T is crossed; hence, suppose that say $v_1 v_2$ is crossed by an edge uw, where the part of the edge uw next to w is drawn in the open disk bounded by T. Let $G_1 = \mathrm{Ext}_T(G) + uw$. Suppose for a contradiction that $V(\mathrm{Int}_T(G)) \neq V(T)$, and thus $G_1 \neq G$ (since $\deg(w) > 1$ if $w \neq v_3$ by Lemma 8(a)). By the minimality of B, there exists an L-coloring φ of G_1. Let $G_2 = \mathrm{Int}_T(G)$ and let

L_2 be the list assignment such that $L_2(v_i) = \{\varphi(v_i)\}$ for $i = 1, 2, 3$, $L_2(w) = L(w) \setminus \{\varphi(u)\}$ if $w \neq v_3$, and let $L_2(z) = L(z)$ for $z \in V(G_2) \setminus \{v_1, v_2, v_3, w\}$. Since G is not L-colorable, G_2 is not L_2-colorable. Let $N_2 = N \cap V(G_2)$ if $w = v_3$, and $N_2 = (N \cap V(G_2)) \cup \{w\}$ otherwise. Then $(G_2, v_1v_2v_3, N_2, \emptyset, L_2)$ is a counterexample contradicting the minimality of B. $\qquad\square$

We get the following easy corollary.

Lemma 10. *No minimal counterexample (G, P, N, M, L) contains an obstruction.*

Proof. Suppose for a contradiction that $H \subseteq G$ is an obstruction. Note that H contains a special subgraph, and by (D), if $e \in E(H)$ is not crossed in H, then e is not crossed in G, either. Furthermore, no vertex of the outer face of H belongs to N, and thus all of them are incident with the outer face of G. Lemmas 8(c) and 9 imply that $G = H$. However, then G is L-colorable by (O), which is a contradiction. $\qquad\square$

Analogously, we can strengthen Lemma 8(d) for 4-cycles.

Lemma 11. *Let $B = (G, P, N, M, L)$ be a minimal counterexample and let K be a 4-cycle in G (possibly with a crossed edge). If K does not bound the outer face of G and $V(\mathrm{Int}_K(G)) \neq V(K)$, then one of the following holds:*

- $\mathrm{Int}_K(G) - V(K)$ *is isomorphic to K_4 drawn with a crossing, or*

- $V(\mathrm{Int}_K(G)) = V(K) \cup \{z\}$ *for a vertex z adjacent to all vertices of K such that either $z \in N$ or z is incident with an edge crossing an edge of K.*

Proof. Let $K = v_1v_2v_3v_4$. If no edge of K is crossed, then the claim follows from Lemma 8(d); hence, suppose that say v_3v_4 is crossed by an edge uz, where the part of the edge uz next to z is drawn in the open disk bounded by K. If $\mathrm{Int}_K(G)$ contained a chord of K, then $V(\mathrm{Int}_K(G)) = V(K)$ by Lemma 9; hence, we can assume that K is an induced cycle in $\mathrm{Int}_K(G)$.

Let $G_1 = \mathrm{Ext}_K(G) + uz$. Suppose that $V(\mathrm{Int}_K(G)) \neq V(K)$, and thus $G_1 \neq G$ (since $\deg(z) > 1$ if $z \notin V(K)$ by Lemma 8(a)). By the minimality of B, there exists an L-coloring φ of G_1. Consider the graph $G' = \mathrm{Int}_K(G) - v_4$ with the list assignment L' given by $L'(v_i) = \{\varphi(v_i)\}$ for $i = 1, 2, 3$, $L'(z) = L(z) \setminus \{\varphi(u), \varphi(v_4)\}$ if $z \notin \{v_1, v_2\}$ and $zv_4 \in E(G)$, $L'(z) = L(z) \setminus \{\varphi(u)\}$ if $z \notin \{v_1, v_2\}$ and $zv_4 \notin E(G)$, $L'(x) = L(x) \setminus \{\varphi(v_4)\}$ for all neighbors x of v_4

distinct from v_1, v_2, v_3, and z, and $L'(x) = L(x)$ for all other vertices x of G'. Since G is not L-colorable, it follows that G' is not L'-colorable, and by the minimality of B, the target $B' = (G', v_1v_2v_3, N \cap V(G'), \emptyset, L')$ is not valid. Since the edge v_3v_4 is crossed, the distance condition for B implies that B' does not contain any obstruction, and thus it satisfies (O). Observe that B' satisfies all other conditions of validity except possibly for (T). If B' violates (T), then $z \notin V(K)$ and z is adjacent to all vertices of K, and by Lemma 9, we have $V(\text{Int}_K(G)) = V(K) \cup \{z\}$. $\qquad\square$

Next, let us consider separating 5-cycles, in a somewhat restricted situation.

Lemma 12. *Let $B = (G, P, N, M, L)$ be a minimal counterexample and let K be a 5-cycle in G such that no edge of K is crossed, K does not bound the outer face of G, and $V(\text{Int}_K(G)) \neq V(K)$. If G contains a special subgraph S such that $S \subseteq \text{Ext}_K(G)$ and $d(S, K) \leq 1$, then $V(\text{Int}_K(G)) = V(K) \cup \{z\}$ for a vertex z adjacent to all vertices of K.*

Proof. Observe that $K = v_1v_2v_3v_4v_5$ is an induced cycle in $\text{Int}_K(G)$, as otherwise Lemma 11 implies that $\text{Int}_K(G)$ contains a special subgraph at distance at most 1 from K, which together with S violates the assumption (D) for B. By the minimality of B, there exists an L-coloring φ of $\text{Ext}_K(G)$. Let $G' = \text{Int}_K(G) - \{v_4, v_5\}$ with the list assignment L' given by $L'(v_i) = \{\varphi(v_i)\}$ for $i = 1, 2, 3$ and $L'(x) = L(x) \setminus \{\varphi(v_i) : i \in \{1, 2\}, xv_i \in E(G)\}$ for $x \in V(G') \setminus \{v_1, v_2, v_3\}$. Since G is not L-colorable, it follows that G' is not L'-colorable, and by the minimality of B, the target $B' = (G', v_1v_2v_3, N \cap V(G'), \emptyset, L')$ is not valid. Note that by the distance condition for S and by Lemma 9, at most one vertex z of G' satisfies $|L'(z)| = 3$ and B' satisfies (O). Observe that B' satisfies all other conditions of validity except possibly for (T). If B' violates (T), then z is adjacent to all vertices of K, and by Lemma 9, we have $V(\text{Int}_K(G)) = V(K) \cup \{z\}$. $\qquad\square$

Our next goal is to show that the outer face of a minimal counterexample does not have chords. Let us start with excluding non-crossed chords.

Lemma 13. *If $B = (G, P, N, M, L)$ is a minimal counterexample and F is the outer face of G, then every chord of F is crossed.*

Proof. Let $k = \ell(P)$, and let $P = p_0p_1 \ldots p_k$. Suppose for a contradiction that F has a non-crossed chord uv. By Lemma 8(c), u is an internal vertex

of P, say $u = p_1$, while $v \notin V(P)$. Let G_1 and G_2 be the uv-components of G such that $p_0 \in V(G_1)$. Let $P_1 = p_0 p_1 v$ and $P_2 = v p_1 \ldots p_k$. For each color $c \in L(v) \setminus L(u)$, let L_c be the list assignment such that $L_c(v) = \{c\}$ and $L_c(z) = L(z)$ if $z \neq v$. Since G is not L-colorable, either G_1 or G_2 is not L_c-colorable. Furthermore, since both G_1 and G_2 are L-colorable (by the minimality of B), there exist distinct colors c_1 and c_2 such that G_1 is not L_{c_1}-colorable and G_2 is not L_{c_2}-colorable. For $i \in \{1, 2\}$, let $B_i = (G_i, P_i, N \cap V(G_i), M \cap E(G_i), L_{c_i})$. By the minimality of B, neither B_1 nor B_2 is a valid target. Clearly, both B_1 and B_2 satisfy (S), (N), (M), (C), and (D). Hence, each of them violates (P), (T), or (O).

We claim that we can choose the labeling of P and the chord $uv = p_1 v$ so that with the notation as in the previous paragraph, the following holds.

  (i) B_1 does not contain any obstruction,

  (ii) no edge at distance at most two from p_1 in G_1 is crossed, and

  (iii) p_0, p_1, and v do not have a common neighbor z with $|L(z)| = 3$.

Indeed, if B_1 contains an obstruction, then it contains a special subgraph whose distance to p_1 is at most two. In that case, (D) implies that $\ell(P) = 2$ and that B_2 contains no obstruction. Hence, we can exchange the labels of p_0 with p_2, G_1 with G_2, $\ldots$, and B_1 with B_2, so that (i) holds. Similarly, if an edge at distance at most two from p_1 in G_1 is crossed, then (D) implies that $\ell(P) = 2$, B_2 contains no obstruction, and no edge at distance at most two from p_1 in G_2 is crossed, and thus exchanging the labels in the same way ensures that both (i) and (ii) hold. Finally, if p_0, p_1, and v have a common neighbor $z \in V(G_1)$ with list of size three, then z is incident with F and by (ii) and by Lemma 8(c) and (d), we conclude that $V(G_1) = \{p_0, p_1, v, z\}$; in this case, we consider the chord $p_1 z$ instead of the chord $p_1 v$ (so that G_1 becomes equal to the triangle $p_0 p_1 v$), which ensures that (i), (ii), and (iii) hold.

By (i) and (iii), B_1 satisfies (O) and (T). Therefore, B_1 violates (P), and thus $p_0 v \in E(G)$. By (ii) and by Lemma 8(c) and (d), we conclude that G_1 is equal to the triangle $p_0 p_1 v$.

Let $S = L(v) \setminus \bigcup_{p \in V(P), vp \in E(G)} L(p)$. Recall that by Lemma 10, B does not contain the obstruction $\mathrm{O_{P1}}$. Together with the property (T) of B, this implies that $S \neq \emptyset$. Consider any $c \in S$, and let $B_c = (G_2, P_2, N \cap V(G_2), M \cap E(G_2), L_c)$. Since G_1 is L_c-colorable, we conclude that G_2 is not L_c-colorable,

and thus B_c is not a valid target. Note that B_c satisfies (S), (N), (M), (C), and (D), and that B_c satisfies (P) since $c \in S$.

If B_c contained an obstruction H, then Lemmas 8(c) and 9 would imply that $G_2 = H$. However, the inspection of the obstructions shows that either $G = H \cup G_1$ would be L-colorable, or B would contain an obstruction. Therefore, B_c satisfies (O).

It follows that B_c violates (T), i.e., there exists a vertex w adjacent to v and to vertices $p, p' \in V(P) \setminus \{p_0\}$ such that $L(w) = L(p) \cup L(p') \cup \{c\}$. Since this holds for all $c \in S$, it follows that $|S| = 1$. If $|L(v)| = 3$, then $vw \in M$, (D) for B implies that $\ell(P) = 2$, and B contains O_{M1}, which contradicts Lemma 10. Hence, suppose that $|L(v)| \geq 4$, and since $|S| = 1$, v has at least three neighbors in P. If $\ell(P) = 2$, then $|L(v)| = 4$ and the edges vp_2 and wp_1 cross, which contradicts (C) for G. Hence, $\ell(P) = 3$, and by (D) for B, neither v nor w is incident with a crossed edge. Therefore, v is adjacent to p_0, p_1, and p_2, $|L(v)| = 4$, and w is adjacent to p_2 and p_3. However, this implies that G contains O_{P2}, which contradicts Lemma 10. $\qquad\square$

Next, we exclude crossed chords that are not incident with an internal vertex of P.

Lemma 14. *If $B = (G, P, N, M, L)$ is a minimal counterexample and F is the outer face of G, then every chord of F is incident with an internal vertex of P.*

Proof. Suppose for a contradiction that F has a chord uv of F that is not incident with an internal vertex of P. By Lemma 13, there exists an edge e crossing the edge uv. Let G_1 and G_2 be the uv-components of G such that $P \subseteq G_1$. Let $e = x_1 x_2$, where $x_1 \in V(G_1)$ and $x_2 \in V(G_2)$. By the minimality of G, there exists an L-coloring φ of G_1. Since $\varphi(u) \neq \varphi(v)$, we can by symmetry assume that $\varphi(x_1) \neq \varphi(u)$.

Let G' be the graph obtained from $G_2 - uv$ by adding new vertices y_1 and y_2, edges of the path $P' = uy_1y_2v$ and the edge y_1x_2. Let L' be the list assignment for G' such that $L'(u) = \{\varphi(u)\}$, $L'(v) = \{\varphi(v)\}$, $L'(y_1) = \{\varphi(x_1)\}$, $L'(y_2) = \{c\}$ for a new color c that does not appear in any of the lists and $L'(z) = L(z)$ for any other vertex z. Let $B' = (G', P', N \cap V(G_2), M \cap E(G_2), L')$.

Note that the edge y_1y_2 is a special subgraph in B' which does not appear in B. However, B' satisfies (D), since the crossing of G incident with x_2 does not belong to G' and any path from a special subgraph in G' to y_1y_2 passes

through one of the vertices u, v, x_2 of the crossing in G. Furthermore, B' satisfies (O) since y_2 has degree two in G'. Similarly, B' satisfies all other conditions of validity except possibly for (T).

Since G is not L-colorable, G' is not L'-colorable, and thus B' is not a valid target. Hence, B' violates (T). This implies that x_2 has list of size three and it is adjacent to u and v. By Lemmas 8(c) and 9, we have $V(G_2) = \{u, v, x_2\}$. Note that by (C), we conclude that each of $|L(u)|, |L(v)|$, and $|L(x_1)|$ is either 1 or 5. Let a be a color in $L(x_2)$ distinct from the colors of its neighbors in P, which exists by (T). Let $G'' = G - x_2$ with the list assignment L'' such that $L''(z) = L(z) \setminus \{a\}$ for $z \in \{u, v, x_1\}$ and $L''(z) = L(z)$ otherwise. Let $B'' = (G'', P, N \cup \{x_1\}, M, L'')$, and observe that G'' is not L''-colorable and that B'' satisfies all the conditions of validity except possibly for (O).

By the minimality of G, it follows that B'' violates (O). Let H be an obstruction in B''. By Lemma 10, H is not an obstruction in B, and thus $V(H) \cap \{u, v, x_1\} \neq \emptyset$. Thus, the distance between the special subgraph of H and between $\{u, v, x_1\}$ is at most 2, and since u, v, and x_1 are incident with a crossing in G that does not appear in H, the condition (D) for B implies that x_1 with $|L''(x_1)| = 4$ is a special subgraph of H and that $\ell(P) = 2$. Consequently, B'' contains one of O_{N1}, O_{N2} or O_{N3}, in which the interior vertex with list of size 4 is x_1. However, inspection of these graphs shows that $|L''(u)| = 3$ or $|L''(v)| = 3$, and thus $|L(u)| \leq 4$ or $|L(v)| \leq 4$, which contradicts (C). $\qquad\square$

Finally, let us exclude all the remaining chords.

Lemma 15. *The outer face of a minimal counterexample has no chords.*

Proof. Let $B = (G, P, N, M, L)$ be a minimal counterexample and let F be the outer face of G. Suppose for a contradiction that F has a chord uv. By Lemmas 13 and 14, uv is crossed and incident with an internal vertex of P. By (D), we conclude that $\ell(P) = 2$; let $P = p_0 p_1 p_2$, where say $u = p_1$. Let e be the edge crossing uv and let G_1 and G_2 be the uv-components of $G - e$ such that $p_0 \in V(G_1)$ and $p_2 \in V(G_2)$. Let $P_1 = p_0 p_1 v$ and $P_2 = p_2 p_1 v$, and let $e = x_1 x_2$, where $x_i \in V(G_i)$ for $i \in \{1, 2\}$.

If G_i contains an edge f different from $p_{2i-2} p_1$, $p_{2i-2} v$, and $p_1 v$, then by the minimality of G there exists an L-coloring φ_{3-i} of $G - f \supseteq G_{3-i} + x_1 x_2$. If additionally $|L(x_i)| \in \{1, 5\}$, then define L_i to be the list assignment for G_i such that $L_i(v) = \{\varphi_{3-i}(v)\}$, $L_i(x_i) = L(x_i) \setminus \{\varphi_{3-i}(x_{3-i})\}$, and $L_i(z) = L(z)$ for any other vertex z. Let $B_i = (P_i, (N \cap V(G_i)) \cup \{x_i\}, M \cap E(G_i), L_i)$,

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and note that B_i satisfies all the conditions of validity except possibly for (P), (T), or (O). Observe that G_i is not L_i-colorable, and by the minimality of B, we conclude that B_i violates (P), (T) or (O). Since φ_{3-i} is a coloring of $G - f$, (P) is satisfied by B_i. Since B satisfies (D) and all chords of F are crossed by Lemma 13, it follows that B_i satisfies (T). Thus, B_i violates (O). The corresponding obstruction is O_{N1} since all others either have a special subgraph that would violate the distance condition in G, or have a non-crossed chord incident with p_1 which contradicts Lemma 13.

Hence, for $i \in \{1, 2\}$, either $E(G_i) \subseteq \{p_{2i-2}p_1, p_{2i-2}v, p_1v\}$, or $|L(x_i)| \in \{3, 4\}$, or B_i as defined in the previous paragraph contains O_{N1}. Together with Lemmas 9 and 13, we conclude that one of the following holds:

- $x_i \in V(P_i)$ and either $G_i = P_i$ or G_i is the triangle on $V(P_i)$, or

- $|L(x_i)| \in \{3, 4\}$, or

- G_i is equal to O_{N1} and x_i is its vertex with list of size four.

By Lemma 14, at most one of x_1 and x_2 has a list of size three or four. By symmetry, we can assume that $|L(x_1)| \in \{1, 5\}$.

If $|L(x_2)| \in \{1, 5\}$, then observe that all possible combinations of graphs G_1 and G_2 (each of which is a path, a triangle, or O_{N1}) are either L-colorable or equal to O_{C1}. Therefore, $|L(x_2)| \in \{3, 4\}$, and thus x_2 is incident with F. By Lemmas 13 and 14, we have $x_1 \notin V(F)$, and thus G_1 is O_{N1}.

Let w be the vertex of G_1 with list of size three, let $G' = G - \{w, p_0\} - p_1v$ and let L' be the list assignment such that $L'(x_1) = \{\varphi_1(x_1)\}$, $L'(v) = \{\varphi_1(v)\}$ and $L'(z) = L(z)$ otherwise. Let $B' = (G', p_2p_1x_1v, N \cap V(G'), M \cap E(G'), L')$ and note that G' is not L'-colorable and that the target B' satisfies all the conditions of validity except possibly for (T) and (O).

If v has degree at least 5 in G, then it has degree at least three in G'. Together with Lemmas 9 and 13, this implies that $x_2 \in V(F)$ is not adjacent to v, hence B' satisfies (T). If v has degree at most four, then $|L(v)| \leq 4$ by Lemma 8(a), and by (C), $|L(x_2)| = 4$, and again B' satisfies (T). By the minimality of B, we conclude that B' violates (O). Since x_1 has degree three in G' and it is adjacent to a vertex x_2 with list of size three or four, G' contains (and by Lemmas 9 and 13, is equal to) O_{P1} or O_{P2}. However, then p_1x_2 is a non-crossed chord of F, which contradicts Lemma 13. $\qquad\square$

Let us show an easy corollary.

Lemma 16. *If $B = (G, P, N, M, L)$ is a minimal counterexample, then no vertex of P is incident with a crossed edge.*

Proof. Suppose for a contradiction that vp is a crossed edge with $p \in V(P)$. By Lemma 15, we have $v \notin V(F)$. Furthermore, since P is incident with a crossing, (D) implies that $\ell(P) \leq 2$. Let L' be the list assignment such that $L'(v) = L(v) \setminus L(p)$ and L' matches L on the rest of the vertices of G. Note that $G - vp$ is not L'-colorable, and by the minimality of B, we conclude that $B' = (G - vp, P, N \cup \{v\}, M, L')$ contains O_{N1}, O_{N2} or O_{N3}, whose internal vertex with list of size 4 is v. Note that B' cannot contain O_{N1}, since v is not adjacent to all vertices of P in $G - vp$. Similarly, B' cannot contain O_{N3}, since the edge vp would be crossed twice. Finally, B' does not contain O_{N2}, since G does not contain O_{C1} by Lemma 10. This is a contradiction. $\qquad\square$

Next, we prove the following claim that is useful when considering the condition (T) for targets derived from minimal counterexamples.

Lemma 17. *If $B = (G, P, N, M, L)$ is a minimal counterexample, then every vertex of G has at most two neighbors in P.*

Proof. Let $k = \ell(P)$ and $P = p_0 p_1 \ldots p_k$. Suppose for a contradiction that a vertex $v \in V(G)$ has at least three neighbors $p_a, p_b, p_c \in V(P)$, where $0 \leq a < b < c \leq k$. By Lemma 15, v is not incident with the outer face F of G. Let K be the cycle $p_a p_{a+1} \ldots p_c v$, and note that K has a chord vp_b. By Lemma 16, none of the edges vp_a, vp_b and vp_c is crossed. By Lemma 8(d) applied to the two cycles in $K + vp_b$ containing the edge vp_b, the cycle K is not separating (Lemma 8(d) allows a vertex of N or a K_4 with a crossed edge in the interior of K; however, this would only be possible if $\ell(P) = 3$, yielding two special subgraphs at distance 1).

Suppose first that $c - a = \ell(P)$, and let G_2 be the $p_a v p_c$-component of G that does not contain P. Since $v \notin V(F)$, and $v \notin N$ if $\ell(P) = 3$, there exists a color $g \in L(v)$ that does not appear in the lists of vertices in P. Let L_2 be the list obtained from L by setting $L_2(v) = \{g\}$. Observe that G_2 is not L_2-colorable, and thus $B_2 = (G_2, p_a v p_c, N, M, L_2)$ is not a valid target. This is only possible if B_2 violates either (T) or (O). In the former case, G is either O_{N1} or O_{P6}, which contradicts Lemma 10. In the latter case, we have $\ell(P) = 2$ by the distance condition, and (2) together with Lemmas 9 and 13 imply that G is L-colorable unless G_2 is either O_{M1} or O_{C1}. If G_2 is O_{M1}, then G is O_{M2}, and if G_2 is O_{C1}, then G is L-colorable; in both cases, we obtain a contradiction.

Hence, $c - a < \ell(P)$, and thus $\ell(P) = 3$; by symmetry, we can assume that v is adjacent to say p_0, p_1 and p_2, and v is not adjacent to p_3. If $L(p_0) = L(p_2)$, then $G - vp_2$ gives a counterexample contradicting the minimality of B. Therefore, $L(p_0) \neq L(p_2)$. Since the edges vp_0, vp_1, and vp_2 are not crossed by Lemma 16, Lemma 8(d) implies that the degree of p_1 is three. Let $G' = G - p_1 + p_0 p_2$, with the list assignment L' such that $L'(v) = L(v) \setminus L(p_1)$ and $L'(z) = L(z)$ for $z \in V(G') \setminus \{v\}$. Let $B' = (G, p_0 p_2 p_3, N \cup \{v\}, M, L')$. Note that B' satisfies (D), since the rank of the special subgraph $p_1 p_2$ in B is greater than the rank of the special subgraph v in B', and any path Q between two special subgraphs S_1 and S_2 that uses the new edge $p_0 p_2$ gives rise to paths between S_1 or S_2 and the middle edge $p_1 p_2$ of P in G, thus implying $\ell(Q) \geq 14 + r(S_1) + r(S_2) + 2r(p_1 p_2) - 1 > 7 + r(S_1) + r(S_2)$. Since G' is not L'-colorable, the minimality of B implies that B' is not valid, and this is only possible if B' violates (O). Hence, B' contains O_{N1}, O_{N2}, or O_{N3}, with v as the vertex with list of size 4 not incident with the outer face. However, then B contains O_{P6}, O_{P4}, or O_{P5}, respectively, which contradicts Lemma 10. $\qquad\square$

Let us now derive a variation on Lemma 12.

Lemma 18. *Let $B = (G, P, N, M, L)$ be a minimal counterexample. Suppose that $Q = x_1 x_2 \dots x_{t-1} x_t$ is a path in G, where $t \leq 6$ and $x_1 x_2$ crosses $x_{t-1} x_t$. Let c be the closed curve consisting of the path $x_2 \dots x_{t-1}$ and parts of the edges $x_1 x_2$ and $x_{t-1} x_t$. If x_1 is not drawn in the open disk Λ bounded by c, then no vertex of G is contained in Λ.*

Proof. Observe first that the curve c is not crossed since all its edges are close to a crossing. Let G' be the subgraph of G consisting of the vertices and edges drawn fully in the closure of Λ. Suppose for a contradiction that $x_1 \notin \Lambda$ (and hence $x_t \notin \Lambda$) and that Λ contains a vertex v of G. By the minimality of B, there exists an L-coloring φ of $G - v$. Let L' be the list assignment for G' defined by $L'(x_i) = \{\varphi(x_i)\}$ for $i = 2, \dots, t - 1$ and by $L'(z) = L(z)$ for $z \in V(G) \cap \Lambda$. Note that G' is not L'-colorable. However, then $B' = (G', x_2 \dots x_{t-1}, N \cap V(G'), \emptyset, L')$ is a counterexample contradicting the minimality of B (note that B' satisfies (D) even if $t = 6$, since the middle edge of the path $x_2 x_3 x_4 x_5$ has smaller rank then the crossing, whose distance to $x_3 x_4$ in G is one). $\qquad\square$

Next, we consider an analogue of a chord formed by parts of two crossing edges.

Lemma 19. *Let $B = (G, P, N, M, L)$ be a minimal counterexample, and let F denote the outer face of G. Let uv and xy be crossing edges with $u, x \in V(F)$, and let c denote the curve formed by the parts of uv and xy between the crossing and u and x, respectively. If c is not a part of the boundary of F, then the part of G separated from P by c consists only of the edge ux.*

Proof. By Lemma 16, neither u nor x belongs to P, and by Lemma 15, we have $v, y \notin V(F)$. Suppose that c is not part of the boundary of F. Let G_2 be the subgraph of G consisting of the vertices and edges that are fully drawn inside the closed disc bounded by c and the part of the boundary of F between u and x that does not contain P. Note that there are two possible situations, depending on whether G_2 includes the vertices v and y or not. In either case, we can write $G = G_1 \cup G_2$ for a subgraph G_1 of G such that $G_1 \cap G_2$ consists only of vertices u and x. Let G_2' be the graph obtained from G_2 by adding a common neighbor w of u and x. By the minimality of B, there exists an L-coloring φ of G_1. Let L_2 be a list assignment such that $L_2(u) = \{\varphi(u)\}$, $L_2(x) = \{\varphi(x)\}$, $L_2(w) = \{a\}$ for some color a distinct from $\varphi(u)$ and $\varphi(v)$, and $L_2(z) = L(z)$ for $z \in V(G_2') \setminus \{u, x, w\}$. Let $B_2 = (G_2', uwx, N \cap V(G_2'), M \cap E(G_2'), L_2)$. Note that G_2' is not L_2-colorable, and by the minimality of B, it follows that B_2 is not a valid target. This is only possible if B_2 violates (P), and thus $ux \in E(G)$.

If $v, y \notin V(G_2)$, then by Lemmas 15 and 18, it follows that G_2 consists only of the edge ux. If $v, y \in V(G_2)$, then we can redraw the edge ux along c so that ux becomes a chord of F, which contradicts Lemma 15. $\qquad\square$

Now, we shall consider the 2-chords of F.

Lemma 20. *Let $B = (G, P, N, M, L)$ be a minimal counterexample, and let F denote the outer face of G. Let uvw be a 2-chord of F such that vw is not crossed and neither u nor w is an internal vertex of P. If G_1 and G_2 are the uvw-components of G with $P \subset G_1$, then one of the following holds (cf. Figure 6):*

- $V(G_2) = \{u, v, w\}$, *and either uv is not crossed and $uw \in E(G)$, or uv is crossed by an edge incident with w; in the latter case, uw may or may not be an edge.*

- $V(G_2) = \{u, v, w, z\}$ *for a vertex z with list of size three, and either uv is not crossed and $uz, vz, wz \in E(G)$, or uv is crossed by an edge incident with z, $zw \in E(G)$ and at least one of uz and vz is an edge.*
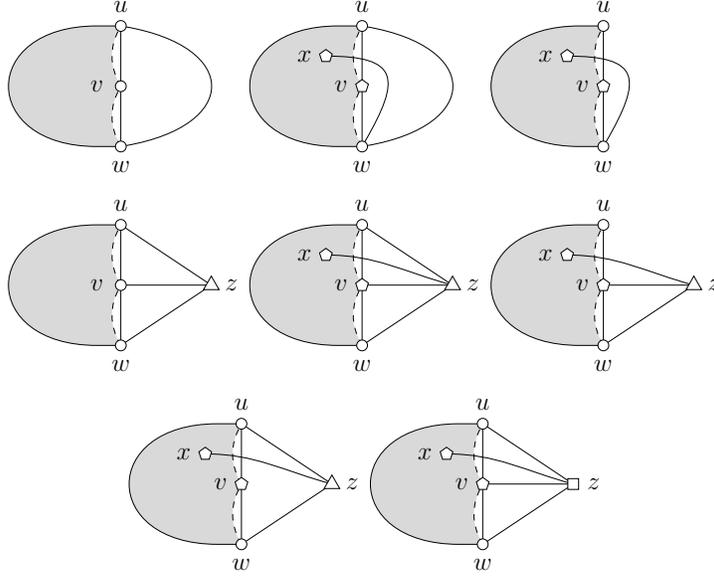
Figure 6: Possible cases for G_2 for a 2-chord uvw.

- $V(G_2) = \{u, v, w, z\}$ *for a vertex z with list of size four adjacent to u,*
  v, w and incident with an edge crossing uv.

Proof. For a contradiction, let us consider a 2-chord uvw that does not satisfy the conclusion of the lemma such that $(\star)$ uv is non-crossed if possible, and subject to that, G_2 is maximal. Let us distinguish the cases depending on whether uv is crossed or not.

First, suppose that uv is not crossed. By the minimality of B, there exists an L-coloring φ of G_1. Let $L_2(z) = \{\varphi(z)\}$ for $z \in \{u, v, w\}$ and $L_2(z) = L(z)$ for $z \in V(G_2) \setminus \{u, v, w\}$, and let $B_2 = (G_2, uvw, N \cap V(G_2), M \cap E(G_2), L_2)$. Note that G_2 is not L_2-colorable, and thus the target B_2 is not valid. This is only possible if it violates (P), (T), or (O). If B_2 violates (P) or (T), then by Lemmas 9 and 15 the conclusion of Lemma 20 holds for uvw. Therefore, we conclude that B_2 violates (O).

Since the obstruction in B_2 violating (O) contains a special subgraph with a vertex distinct from v and $v \notin V(F)$, the condition (D) for B implies that $v \notin N$, and hence $|L(v)| = 5$. By Lemmas 9 and 13 we also conclude that G_2 is equal to the obstruction. Let S be the set of L-colorings of uvw that do not extend to an L-coloring of G_2. The inspection of the obstructions with

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$\ell(P) = 2$ in Figure 4 shows that one of the following holds:

(R1) there exists a set A of at most two colors and S contains only colorings ψ such that $\psi(v) \in A$, and furthermore, if $|A| = 2$ then $|L(u)| \neq 3$ and $|L(w)| \neq 3$; or,

(R2) S contains only colorings ψ such that $\psi(u) = \psi(w)$, and neither u nor w has list of size three.

Indeed, by (2), all obstructions except for O_{M1} and O_{C1} satisfy (R1) with $|A| = 1$. If G_2 is O_{M1} or O_{C1}, then neither u nor w has list of size three, by (M) together with the distance condition and by (C). The inspection of the colorings shows that if G_2 is O_{C1}, then (R1) holds with $|A| = 2$, and if G_2 is O_{M1}, then either (R1) holds with $|A| = 2$, or (R2) holds (the latter is the case when the two lists of size 3 are equal, i.e., $a = c$ in Figure 4).

If (R1) holds, then let $G' = G_1$, with the list assignment L' such that $L'(v) = L(v) \setminus A$ and $L'(z) = L(z)$ for $z \neq v$. Note that if $|A| = 2$, then v has no neighbor in G_1 with list of size three by (R1) and by ($\star$)—the edge from v to such a neighbor cannot be crossed by the condition (D) for B and the existence of the obstruction in B_2. If (R2) holds, then let $G' = G_1 + uw$ with the list assignment $L' = L$. Let $B' = (G', P, N \cap V(G'), M \cap E(G)', L')$. In either case, since G is not L-colorable, we conclude that G' is not L'-colorable. The target B' satisfies (D)—in the latter case, any path Q between special subgraphs H_1 and H_2 using the added edge uw gives rise to paths from H_1 and H_2 to the special subgraph H of G_2, and since $d(u, H) \leq 2$ and $d(w, H) \leq 2$, we have $\ell(Q) \geq 14 + r(H_1) + r(H_2) + 2r(H) - 3 > r(H_1) + r(H_2) + 7$. Furthermore, B' satisfies (T) by Lemma 17, and if B' violated (C) or (O), then v or uw would have to belong to a crossing or to an obstruction in B', and the distance between its special subgraph and the special subgraph of B_2 would be at most 4, which contradicts the condition (D) for B. Note that B' cannot violate (P), as otherwise $u, w \in V(P)$ and G_2 is O_{M1}, and by Lemma 8 and 17, v would have degree four and list of size five. Therefore, B' is a counterexample contradicting the minimality of B.

Hence, we conclude that Lemma 20 holds for all 2-chords without crossed edges.

Suppose now that uv **is crossed** by an edge xy, where $x \in V(G_1)$ and $y \in V(G_2)$. If $y = w$, then the conclusion of Lemma 20 holds for uvw by Lemmas 19 and 18, hence assume that $y \neq w$. Furthermore, $x \neq w$ by Lemma 19, and $uw \notin E(G)$ by Lemma 9. Let G'_1 be the graph obtained

from G_1 by adding the edges ux and vx (if they are not present already). Note that this can be done without introducing any new crossings. Let $B_1' = (G_1', P, N \cap V(G_1), M \cap E(G_1), L)$. Since u, v and x are incident with a crossing in G, the target B_1' satisfies (D) and (M). Furthermore, B_1' does not contain any obstruction, as its special subgraph would be at distance at most 2 from the crossing. By Lemma 16, u, v, and x cannot belong to P, hence B_1' satisfies (P) and (T). We conclude that B_1' is a valid target, and by the minimality of B, there exists an L-coloring φ of G_1'.

Let G_2' be the graph obtained from $G_2 - uv$ by adding the vertex x and edges ux, vx, yx. Consider the list assignment L_2' for G_2' such that $L_2'(z) = \{\varphi(z)\}$ for $z \in \{u, v, w, x\}$ and $L_2'(z) = L(z)$ otherwise. Let $B_2' = (G_2', uxvw, N \cap V(G_2'), M \cap E(G_2'), L_2')$. Note that G_2' is not L_2'-colorable and that B_2' satisfies (D).

Since $y \neq w$ and since $uw \notin E(G_2')$, the target B_2' satisfies (P). If B_2' violates (T), then by Lemma 9 we have that $|L(y)| = 3$ and y is adjacent to at least two of u, v and w. In particular, $y \in V(F)$. Observe that if $vy \in E(G)$, then $wy \in E(G)$, since Lemma 20 holds for the non-crossed 2-chord yvw. Thus $wy \in E(G)$ in any case. By Lemma 18 applied to the path $xywvu$ in G, and by Lemma 19, we have $V(G_2) = \{u, v, w, y\}$ and the conclusion of Lemma 20 holds for uvw. Hence, we can assume that B_2' satisfies (T). By the minimality of B, the target B_2' is not valid, and we conclude that it violates (O). By Lemmas 8 and 9, G_2' is equal to one of O_{P1}, ..., O_{P6}, but not to O_{P3} since x has degree 3 in G_2'.

If G_2' is O_{P1}, then the conclusion of Lemma 20 holds for uvw. Otherwise, let us define S as the set of colorings ψ of the path $uxvw$ that do not extend to an L-coloring of G_2' and satisfy $\psi(u) \neq \psi(v)$. The inspection of the obstructions and their problematic list assignments displayed in Figure 4 shows that either (R1) or one of the following holds:

(R3) G_2' is O_{P2} and there exists a color c such that S contains only colorings ψ such that either $\psi(u) = c$ and $\psi(x) = \psi(w)$, or $\psi(x) = c$ and $\psi(u) = \psi(w)$. Moreover, $|L(u)| \neq 3$ and $|L(w)| \neq 3$.

(R4) G_2' is O_{P4} and there exists a color c such that S contains only colorings ψ satisfying either $\psi(v) = c$ or $\psi(x) = c$. Moreover, $|L(u)| \neq 3$.

Let us remark that for O_{P2} we have (R1) if the colors a, b, c, d in Figure 4 are different; we have (R3) if $b = d$ or $a = d$. To argue for O_{P4}, O_{P5}, O_{P6} we observe that $\psi(x)$ and $\psi(v)$ should be taken from the difference of the lists of

the two neighbors of u (so these are colors b, c in Figure 4). This yields (R1) with the only exception in the case of O_{P4}, where we cannot argue about $|L(w)| \neq 3$, so we need (R4) in this (and only this) case.

The conclusions that the specified vertices do not have lists of size three follow in all applicable cases by noting that otherwise either (C) or the distance condition would be violated. For example, for (R3) we argue as follows. Since x has degree 3 in G_2', the vertex z of O_{P2} with list of size 3 is not the vertex y, and v, w are both adjacent to z. Since $|L(z)| = 3$ and the edge wz is close to a crossing in G, we conclude that $wz \notin M$ and hence $|L(w)| \neq 3$. Since $|L(y)| = 4$, (C) implies that $|L(u)| \neq 3$.

If (R1) holds, then let $G_1' = G_1$, let $M_1 = M \cap E(G_1)$, let $N_1 = N \cap V(G_1)$, and let L_1 be the list assignment obtained from L by removing A from the list of v. If (R3) holds, then we let $G_1' = G_1 + uw$ with the list assignment L_1 obtained from L by removing c from the list of u (note that $|L(u)| \neq 1$ by Lemma 16), let $N_1 = N \cap V(G_1)$, and let $M_1 = (M \cap E(G_1)) \cup \{uz\}$ if u has a neighbor z with list of size three and $M_1 = M \cap E(G_1)$ otherwise.

If (R4) holds and $|L(x)| = 5$, then let $G_1' = G_1$ with the list assignment L_1 obtained by removing c from the lists of x and v, let $N_1 = (N \cap V(G_1)) \cup \{x\}$ and let $M_1 = M \cap E(G_1)$. In all the cases, $B_1' = (G_1', P, N_1, M_1, L_1)$ is a valid target. Indeed, all the conditions except for (P), (T), and (O) are easy to argue. (P) holds since $u \notin V(P)$ by Lemma 16. Similarly, (T) follows by Lemmas 15 and 17. Finally, (O) holds since by the distance condition for B, we could only create O_{M1} or O_{M2} if (R3) holds, and $O_{N1}, O_{N2},$ or O_{N3} if (R4) holds, and each of them is excluded by Lemma 15 or 17, or by the condition $|L(u)| \neq 3$. However, any L_1-coloring of G_1' would extend to an L-coloring of G, and thus B_1' is a counterexample contradicting the minimality of B.

Finally, consider the case that (R4) holds and $|L(x)| \in \{3, 4\}$. By Lemma 19, all neighbors of u distinct from x belong to G_2. By Lemma 16, we have $u \notin V(P)$, and since $\deg(u) \geq |L(u)| \geq 4$, we conclude that u is adjacent to x and $|L(u)| = 4$. Since G_2' is O_{P4}, every L-coloring of x, v and w extends to an L-coloring of the graph consisting of G_2, x, and all the edges between x and $V(G_2)$. We conclude that G_1 is not L-colorable, and thus $(G_1, P, N \cap V(G_1), M \cap E(G_1), L)$ contradicts the minimality of B. $\square$

Similarly, one can prove the following.

Lemma 21. *Let $B = (G, P, N, M, L)$ be a minimal counterexample, and let F denote the outer face of G. Let $u, w \in V(F)$ be distinct vertices which*

are not crossing-adjacent and neither of which is an internal vertex of P. Suppose that $v \notin V(F)$ is a vertex adjacent to w and crossing-adjacent to u. Let c be the closed curve consisting of vw, parts of the crossed edges incident with u and v, and a part of the boundary of F between u and w that does not contain P, and let G_2 be the subgraph of G consisting of vertices and edges fully drawn in the closed disc bounded by c. Then G_2 does not contain the crossing and satisfies one of the following:

(a) *$V(G_2) = \{u, v, w\}$ and $uw \in E(G)$, or*

(b) *$V(G_2) = \{u, v, w, z\}$, $|L(z)| = 3$ and z is adjacent to u, v and w.*

Proof. By Lemma 20, it suffices to consider the case that $uv \notin E(G)$. Let c_1 be the closed curve consisting of vw, parts of the crossed edges incident with u and v, and a part of the boundary of F between u and w that contains P. Let G_1 be the subgraph of G drawn in the closed disk bounded by c_1. Let G'_1 be the graph obtained from G_1 as follows: If $uw \in E(G)$, then we add the edge uw. If u, v and w have a common neighbor z with list of size three, then we add z and incident edges. If $V(G'_1) = V(G)$, then (a) or (b) holds.

Otherwise, there exists an L-coloring φ of G'_1 by the minimality of B. Let $G'_2 = G_2 + uv$ with the list assignment L' such that $L'(v) = \{\varphi(v)\}$, $L'(w) = \{\varphi(w)\}$, $L'(u) = \{a\}$ for a new color a, $L'(x) = (L(x) \setminus \{\varphi(u)\}) \cup \{a\}$ for each neighbor x of u distinct from v and w and $L'(x) = L(x)$ for all other vertices x of G'_2. Note that G'_2 is not L'-colorable, and by the minimality of G, it follows that $B' = (G'_2, uvw, N \cap V(G'_2), M \cap E(G'_2), L')$ is not a valid target. Note that B' satisfies (P) and (T) by the construction of G'_1 and the choice of φ. The only other condition that can be violated by B' is (O). By the distance condition, the only obstruction that can appear in B' is O_{C1}. However, letting t be the neighbor of u in G_2 with list of size four, the 2-chord wvt contradicts Lemma 20. $\square$

Let us now introduce a way of defining list assignments that will be used throughout the rest of the paper. Let (G, P, N, M, L) be a valid target. Let φ be any proper partial L-coloring of G such that $\varphi(v) \notin L(p)$ for every pair of adjacent vertices $v \in \mathrm{dom}(\varphi)$ and $p \in V(P)$. For each vertex $z \in V(G) \setminus V(P)$, let

$$R_z = \bigcup_{p \in V(P) \setminus \mathrm{dom}(\varphi), zp \in E(G)} L(p).$$

For $z \in V(P)$, let $R_z = \emptyset$. We define L_φ to be the list assignment such that

$$L_\varphi(z) = \Big(L(z) \setminus \{\varphi(x) : x \in \mathrm{dom}(\varphi), xz \in E(G)\} \Big) \cup R_z$$

for each $z \in V(G)$. Let us also define $G_\varphi = G - \mathrm{dom}(\varphi)$. That is, from the list of each vertex z we remove the colors of its neighbors according to φ, except for those colors that are also forbidden by the neighbors of z in $P \cap G_\varphi$.

Consider any L_φ-coloring ψ of G_φ. We claim that the combination of φ with ψ is a proper L-coloring of G. Indeed, for any $z \in V(G_\varphi)$, we clearly have $\psi(z) \notin R_z$, and thus $\psi(z) \in L_\varphi(z)$ is different from the colors of the neighbors of z in $\mathrm{dom}(\varphi)$. Since G is not L-colorable, we conclude that G_φ is not L_φ-colorable.

Suppose now that (G, P, N, M, L) is a valid target and that G contains a subgraph H isomorphic to one of the graphs drawn in Figure 3 such that the subgraph of H corresponding to full-circle vertices is equal to P, triangle vertices have lists of size *at least* three, square vertices have lists of size *at least* four and pentagonal vertices have lists of size five. Then we say that H is a *near-obstruction*.

Lemma 22. *Let $B = (G, P, N, M, L)$ be a minimal counterexample, and let F denote the outer face of G. If H is a near-obstruction in B, then H is isomorphic to one of O_{M1}, O_{N2}, O_{N3} or O_{P3}. Furthermore, $|(V(H) \cap V(F)) \setminus V(P)| \leq 1$, and if $(V(H) \cap V(F)) \setminus V(P) \neq \emptyset$, then H is O_{N2} or O_{N3}.*

Proof. By Lemma 17, H is isomorphic to one of O_{M1}, O_{N2}, O_{N3}, O_{C2}, O_{C3}, O_{C4}, O_{C5} or O_{P3}. Let $k = \ell(P)$ and $P = p_0 p_1 \ldots p_k$. Note that $G \neq H$, since otherwise G is L-colorable by (O).

Let us exclude several of the possible near-obstructions.

- If H is O_{C5}, then let w be the vertex of the outer face of H not belonging to P. By Lemma 20, G is obtained from H either by adding the edge $p_0 p_2$, or a vertex z of degree three adjacent to p_0, w, and p_2. However, the distance condition implies that $w \notin N$, so that $|L(w)| = 5$. In both cases, this implies that G is L-colorable, which is a contradiction.

- If H is O_{C2}, then let $p_0 w_1 w_2 p_2$ be the path in the outer face of H. If $w_1, w_2 \in V(F)$, then $V(G) = V(H)$ and G is L-colorable by (O). Hence, by symmetry we can assume that $w_2 \notin V(F)$, and thus $|L(w_2)| = 5$. If

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$w_1 \in V(F)$, then since w_2 has degree at least 5, by Lemma 20 we have that G is obtained from H by adding a vertex z is adjacent to w_1, w_2 and p_2. However, then G is L-colorable. Therefore, $w_1 \notin V(F)$.

Let φ be an L-coloring of H, let G_2 be the $p_0w_1w_2p_2$-component of G that does not contain P, with the list assignment L_2 obtained from L by setting $L_2(x) = \{\varphi(x)\}$ for $x \in \{p_0, w_1, w_2, p_2\}$, and let $B_2 = (G_2, p_0w_1w_2p_2, N \cap V(G_2), M \cap E(G_2), L)$. Since φ does not extend to an L-coloring of G_2, it follows that B_2 is not a valid target, which is only possible if it violates (P), (T) or (O). Since both w_1 and w_2 have degree at least 5 in G, Lemma 9 implies that $p_0w_2 \notin E(G)$ and $w_1p_2 \notin E(G)$, and thus B_2 satisfies (P).

Suppose that B_2 violates (T). Then a vertex z with $|L(z)| = 3$ is adjacent to three vertices among p_0, w_1, w_2 and p_2. If it is adjacent to all four of them, then B contains O_{C5} as a near-obstruction, which has already been excluded. Otherwise, since w_1 and w_2 have degree at least 5 in G, Lemma 11 implies that z is not adjacent to p_0, w_1, p_2. By symmetry, we may assume that z is adjacent to p_0, w_1, w_2. Then Lemma 20 applied to the 2-chord zw_2p_2 shows that there is a vertex z' adjacent to z with $|L(z')| = 3$, and thus the edge zz' contradicts either (M) or (D) in B. Hence, B_2 satisfies (T).

Finally, if B_2 violates (O), then the obstruction is equal to one of O_{P1}, O_{P2}, O_{P3}, O_{P4}, O_{P5} or O_{P6}. However, then it is easy to see (by comparing bad lists for the obstructions) that G is L-colorable, which is a contradiction.

- If H is O_{C3}, then let w_1 be the vertex of H drawn by the triangle and w_2 the vertex of P that is not adjacent to it in G. If H is O_{C4}, then let w_1 and w_2 be the vertices of H drawn by triangles. By symmetry, we can assume that w_1 is the neighbor of p_2. Let $w_1x_1x_2w_2$ be the path in H formed by neighbors of p_1. Note that $|L(w_i)| \in \{1, 5\}$ by Lemma 15.

  Choose an L-coloring φ of the subgraph of G induced by $V(P) \cup \{w_1, w_2\}$ such that $\varphi(w_1) \neq \varphi(w_2)$ and either $|L_\varphi(x_1)| \geq 4$ or $L_\varphi(x_1) \neq L_\varphi(x_2)$. Note that this is possible since $|L(w_1)| = 5$. Let $G' = G - \{p_1, x_1, x_2\} + w_1w_2$ with the list assignment L' such that $L'(z) = \{\varphi(z)\}$ for $z \in \{w_1, w_2\}$ and $L'(z) = L(z)$ otherwise. Observe that every L'-coloring of G' extends to an L-coloring of G, and thus G' is not L'-colorable. Let $P' = w_2w_1p_2$ if H is O_{C3} and $P' = p_0w_2w_1p_2$ if H is O_{C4},

and let $B' = (G', P', N \cap V(G'), M \cap E(G'), L')$. By the minimality of B, we conclude that B' is not a valid target.

Note that the choice of φ ensures that B' satisfies (P). Observe that B' may only violate (T) or (O). In the former case, by symmetry we can assume that there exists a vertex $z \in V(G)$ such that $|L(z)| = 3$ and z is adjacent to p_2, w_1 and either w_2 or p_0. It follows that G contains a separating 4-cycle formed by non-crossed edges, and by Lemma 8(d) the interior of this 4-cycle contains K_4. By Lemmas 15 and 20, there are no other vertices in G. Now, it is easy to see that the resulting graph G is L-colorable. This is a contradiction.

Therefore, B' violates (O). By the condition (D) for G, we conclude that $\ell(P') = 3$ (and thus H is O_{C4}) and B' contains one of O_{P1}–O_{P6}. Note that the edge $w_1 w_2$ is contained in a triangle in G'; let z be the common neighbor of w_1 and w_2. By Lemma 8, the 4-cycle $w_1 z w_2 p_1$ surrounds K_4 in G. However, then G is obtained from one of the obstructions O_{P1}–O_{P6} with the precolored path $p_0 w_2 w_1 p_2$ by adding the vertex p_1 joined to the vertices p_0, w_1, w_2, p_2, deleting the edge $w_1 w_2$, and adding K_4 inside the 4-cycle $w_1 z w_2 p_1$, and all such graphs are easily seen to be L-colorable. This is a contradiction.

We conclude that H is one of O_{M1}, O_{N2}, O_{N3}, and O_{P3}. If H is O_{M1} or O_{P3}, then none of the vertices in $V(H) \setminus V(P)$ belongs to F since this would contradict Lemma 15. In the other cases, at most one of the vertices of $V(H) \setminus V(P)$ can belong to F by the same reason. $\qquad\square$

Next, we analyze the part of the boundary of the outer face of a minimal counterexample next to the precolored path.

Lemma 23. *Let $B = (G, P, N, M, L)$ be a minimal counterexample, and let F denote the outer face of G. Let $k = \ell(P)$ and let $p_k \ldots p_1 p_0 v_1 v_2 \ldots v_s$ be the vertices contained in the boundary of F in the cyclic order around it. Then $k \geq 2$, $s \geq 3$ and $v_1 v_2, v_{s-1} v_s \in E(G)$. Furthermore, if $|L(v_1)| > 3$ and $|L(v_2)| > 3$, then v_1 and v_2 are crossing-adjacent and $|L(v_1)| = |L(v_2)| = 4$.*

Proof. Let us define $v_{s+1} = p_k$, $v_{s+2} = p_{k-1}$, $\ldots$. Note that for $i = 1, \ldots, s$, we either have $v_i v_{i+1} \in E(G)$, or v_i and v_{i+1} are crossing-adjacent. Lemma 16 implies that $p_0 v_1, p_k v_s \in E(G)$.

If $k < 2$, then let $P' = v_1 p_0 \ldots p_k$ and let L' be the list assignment obtained from L by setting $L'(v_1) = \{a\}$ for a color $a \in L(v_1)$ chosen so that

$B' = (G, P', N, M, L')$ satisfies (P) and (O). Then it is easy to see that B' is a valid target, and by the minimality of B, we conclude that G is L'-colorable. This is a contradiction, since this also gives an L-coloring of G. Therefore, we have $k \geq 2$.

If $s = 0$, then let φ be the L-coloring of p_0; observe that $(G_\varphi, p_1 \ldots p_k, N \cap V(G_\varphi), \emptyset, L_\varphi)$ is a counterexample contradicting the minimality of B. Hence, $s \geq 1$.

Suppose for a contradiction that $s = 1$. Let φ be a partial coloring that assigns a color in $L(v_1) \setminus (L(p_0) \cup L(p_k))$ to v_1. If v_1 is adjacent to a vertex x by a crossed edge, then let $N' = (N \cap V(G_\varphi)) \cup \{x\}$, otherwise let $N' = N \cap V(G_\varphi)$. By the minimality of B, we conclude that the target $B' = (G_\varphi, P, N', \emptyset, L_\varphi)$ violates (O). Let H be an obstruction in B'; by Lemma 22, H is one of O_{M1}, O_{N2}, O_{N3} or O_{P3}. Note that H contains a vertex z such that $|L_\varphi(z)| = 3$, and by the choice of φ, we conclude that $z \in N$ and $zv_1 \in E(G)$. However, since H also contains a special subgraph, we obtain a contradiction with the condition (D) for B. Therefore, $s \geq 2$.

Suppose for a contradiction that $v_1 v_2 \notin E(G)$, and thus v_1 and v_2 are crossing-adjacent. Note that $\ell(P) = 2$ by the distance condition. Let φ be a partial coloring that assigns a color from $L(v_1) \setminus L(p_0)$ to v_1 and the color from $L(p_0)$ to p_0. Let y be the vertex adjacent to v_1 by the crossed edge, and let $B' = (G_\varphi, p_1 p_2, (N \cap V(G_\varphi)) \cup \{y\}, M \cap E(G_\varphi), L_\varphi)$. By the minimality of B, we conclude that B' is not a valid target. By Lemma 15, we have $|L(y)| = 5$, and if $|L_\varphi(y)| < 4$, then y is adjacent to p_0; however, by Lemma 9, v_2 would be adjacent to p_0, contrary to Lemma 15. Therefore, we have $|L_\varphi(y)| \geq 4$. By Lemmas 15 and 17, we conclude that only the condition (M) can be violated by B'. In that case, p_0 and v_1 have a common neighbor $u \neq y$ adjacent to a vertex w with $|L(w)| = 3$. This contradicts Lemma 20. Therefore, $v_1 v_2 \in E(G)$, and by symmetry, $v_{s-1} v_s \in E(G)$.

Suppose for a contradiction that $s = 2$. By symmetry, assume that if v_2 is incident with a crossed edge, then v_1 is incident with a crossed edge as well. If $v_1 v_2 \in M$, then let φ be an L-coloring of v_1 and v_2 such that $\varphi(v_1) \notin L(p_0)$ and $\varphi(v_2) \notin L(p_k)$. Otherwise, let φ be a coloring of v_1 by a color in $L(v_1) \setminus L(p_0)$ such that if $|L(v_2)| = 3$, then $\varphi(v_1) \notin L(v_2) \setminus L(p_k)$. Note that this is possible by Lemma 8(f). Let us remark that when $|L(v_2) \setminus \{\varphi(v_1)\}| = 2$, then $L(p_k) = \{\varphi(v_1)\}$ and $L_\varphi(v_2) = L(v_2)$ by the definition of L_φ, and thus we always have $|L_\varphi(v_2)| \geq 3$. If v_1 is incident with a crossed edge $v_1 x$, then let $N' = N \cup \{x\}$; if v_1 is adjacent to a vertex $y \in N$, then let $N' = N \setminus \{y\}$; otherwise, let $N' = N \cap V(G_\varphi)$. If v_1 and v_2 have a common neighbor z

belonging to N and $|L_\varphi(v_2)| = 3$, then let $M' = M \cup \{v_2z\}$; otherwise let $M' = M \cap E(G_\varphi)$. Let $B' = (G_\varphi, P, N', M', L_\varphi)$; by the minimality of B, the target B' is not valid.

The choice of φ, M' and N' ensures that B' satisfies (S), (N), (M), (P), and (D). It satisfies (T) by Lemma 17. If B' violated (C), then v_2 would have to be incident with a crossing, and by the choice of v_1 and the distance condition, the vertex v_1 would be incident with the same crossing, which then would not appear in G_φ. Therefore, B' satisfies (C). Hence, B' violates (O) and G contains a near-obstruction H. By Lemma 22, H is O_{M1}, O_{N2}, O_{N3} or O_{P3}. Observe that $v_1v_2 \notin M$, since otherwise the distance between v_1v_2 and the special subgraph of H (which is also special in G) is at most 3. Every vertex with list of size three according to L_φ either belongs to N or is equal to v_2. If $v_2 \notin V(H)$, then H contains only one vertex with list of size three, hence H is O_{N2}. However, then N contains two adjacent vertices, which is a contradiction. Similarly, we exclude the case that $v_2 \in V(H)$ and H is O_{N3} or O_{P3}. Therefore $v_2 \in V(H)$ and H is O_{M1} or O_{N2}. The former is excluded by Lemma 15. If H is O_{N2}, then we have $V(G) = V(H) \cup \{v_1\}$ by Lemma 9. If v_1 is incident with a crossed edge, then G contains O_{C2}. On the other hand, if v_1 is not incident with a crossed edge, then $|L(v_1)| = 3$, $|L(v_2)| = 4$, $|N| = 1$ and G is L-colorable. This is a contradiction, and thus $s \geq 3$.

Suppose for a contradiction that v_1 and v_2 are not crossing-adjacent, $|L(v_1)| > 3$, and $|L(v_2)| > 3$. We remove a color from the list of v_1; let B' denote the resulting target. If some edge v_1x crosses an edge e, then $|L(x)| = 5$ by Lemma 15, and both vertices incident with e have list of size five by Lemmas 16 and 19, hence B' satisfies (C). By Lemma 22, no obstruction arises (since all vertices with lists of size three or four in the new list assignment are contained in $V(F) \cup N$, implying that the corresponding near-obstruction would have at least two vertices in $V(F) \setminus V(P)$). We conclude that B' contradicts the minimality of B; hence, if $|L(v_1)| > 3$ and $|L(v_2)| > 3$, then v_1 and v_2 are crossing-adjacent. A similar argument shows that in this case $|L(v_1)| = |L(v_2)| = 4$. $\qquad\square$

Consider a minimal counterexample $B = (G, P, N, M, L)$ with outer face F. Let $k = \ell(P)$ and let $p_k \ldots p_1 p_0 v_1 v_2 \ldots v_s$ be the vertices contained in the boundary of F in the cyclic order around it, with $v_{s+1} = p_k$, $v_{s+2} = p_{k-1}$, $\ldots$. Suppose that $v_1v_2, v_2v_3 \notin M$. If $|L(v_1)| = 3$ or $|L(v_2)| = 3$, then let the set $X \subseteq V(F) \setminus V(P)$ and its partial L-coloring φ be defined as in (X1)–

(X4) in the proof of Theorem 6. Let us remark that the choice of X and φ is performed independently on whether v_2v_3 or v_3v_4 is an edge, or whether they are crossing-adjacent (note that when v_3 is crossing-adjacent to v_2 or v_4, then the case (X4) cannot apply by (C)). If $|L(v_1)| > 3$ and $|L(v_2)| > 3$, then v_1 and v_2 are crossing-adjacent by Lemma 23, and we define X and φ as follows.

(X5) If $|L(v_1)| = |L(v_2)| = 4$ and $|L(v_3)| \neq 3$, then $X = \{v_1\}$ and $\varphi(v_1) \in L(v_1) \setminus L(p_0)$ is chosen arbitrarily.

(X6) If $|L(v_1)| = |L(v_2)| = 4$ and $|L(v_3)| = 3$, then $X = \{v_2\}$ and $\varphi(v_2) \in L(v_2) \setminus L(v_3)$ is chosen arbitrarily.

Let m denote the largest index such that $v_m \in X$. Let us note that $m = 1$ in (X1) and (X5), $m = 3$ in (X4), and $m = 2$ otherwise. Also, $X = \text{dom}(\varphi)$ in all cases except for (X4b), when $X = \{v_1, v_2, v_3\}$ and $\text{dom}(\varphi) = \{v_1, v_3\}$. We say that the triple (X, φ, m) is the *probe for B at the p_0 side of P*. The *probe for B at the p_k side of P* is defined symmetrically, exchanging the role of p_0 with p_k, v_1 with v_s, ...

Lemma 24. *Consider a minimal counterexample $B = (G, P, N, M, L)$ with outer face F. Let $k = \ell(P)$ and let $p_k \ldots p_1 p_0 v_1 v_2 \ldots v_s$ be the vertices contained in the boundary of F in the cyclic order around it. If $v_1v_2, v_2v_3 \notin M$, then let (X, θ, m) be the probe for B at the p_0 side of P. One of the following cases holds, see Figure 7 for the illustration of the possibilities.*

(A1) *$v_1v_2 \in M$ or $v_2v_3 \in M$.*

(A2) *Either v_1 and v_2, or two distinct vertices in $\text{dom}(\theta)$ have a common neighbor in N.*

(A3) *There exists a crossing q and two crossing-adjacent vertices $w_1, w_2 \in V(G_q)$ such that $V(G_q) \cap X = \emptyset$, w_1 has a neighbor in $\text{dom}(\theta)$ and w_2 has two neighbors in $\text{dom}(\theta)$.*

(A4) *$v_m v_{m+1} \in E(G)$, there exists a crossing q such that $V(G_q) \cap X = \emptyset$ and $v_{m+1}, v_{m+2} \in V(G_q)$, and either $|L(v_{m+1})| = |L(v_{m+2})| = 4$ or $|L(v_{m+1})| = 5$ and $|L(v_{m+2})| = 3$.*

(A5) *$v_m v_{m+1} \in E(G)$, $|L(v_{m+1})| \in \{3, 4\}$ and there exists a crossing q such that $V(G_q) \cap X = \emptyset$, $v_{m+1} \in V(G_q)$ and a neighbor $w \notin V(F)$ of v_m is crossing-adjacent to v_{m+1}.*
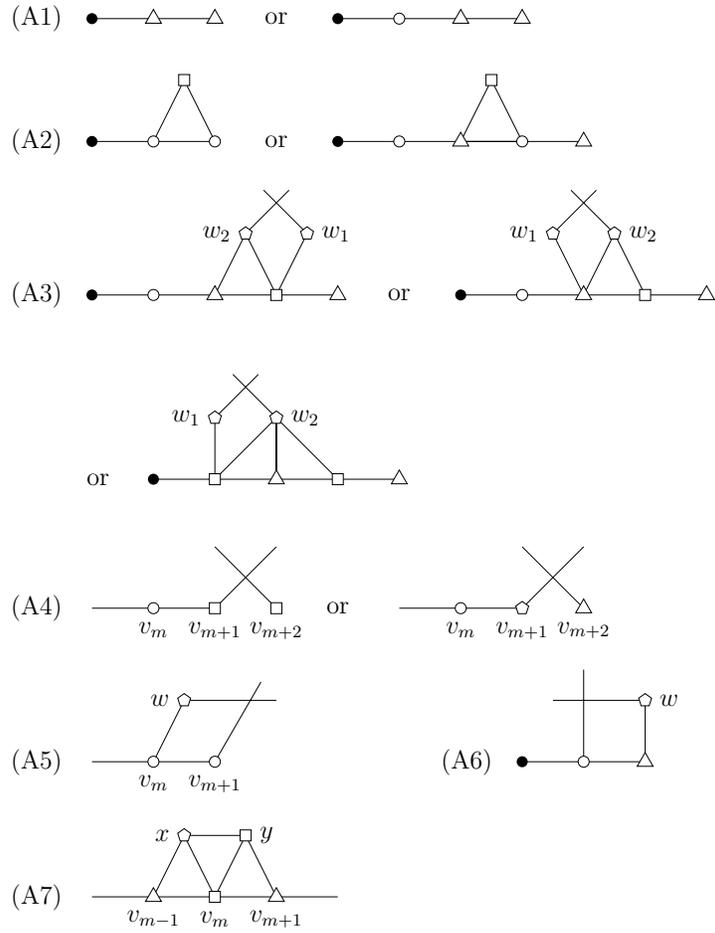
Figure 7: Possible outcomes of Lemma 24.

(A6) $v_1 \notin X$ and there exists a crossing q such that $V(G_q) \cap X = \emptyset$, $v_1 \in V(G_q)$ and a neighbor $w \notin V(F)$ of v_2 is crossing-adjacent to v_1.

(A7) $|X| \geq 2$ and there exists a path $v_{m-1}xyv_{m+1}$, where x and y are neighbors of v_m and $y \in N$.

Proof. We can assume that (A1) does not hold, and thus the probe (X, θ, m) is defined.

Let us consider the graph $G_0' = G - \mathrm{dom}(\theta)$ with the list assignment L_θ, and let G' be the graph obtained from G_0' by repeatedly removing vertices whose list is larger than their degree. If $\mathrm{dom}(\theta) \neq X$, then we have case (X4b) and $X \setminus \mathrm{dom}(\theta) = \{v_2\}$. In this case, if v_2 were incident with a crossing, then Lemma 9 implies that an edge incident with v_2 crosses an edge incident with v_1 or v_3, and since $|L(v_1)| = |L(v_3)| = 4$ and $|L(v_2)| = 3$, this would contradict (C). Hence, v_2 is not incident with a crossing and its degree in G_0' is 1, and since $|L_\theta(v_2)| \geq 2$, the vertex v_2 is not present in G'. This shows that $G' \subseteq G - X$. Observe also that G' is not L_θ-colorable.

Note that every vertex $v \in V(G')$ not incident with the outer face of G' satisfies $|L_\theta(v)| \geq 4$ (and $|L_\theta(v)| = 4$ only if either $v \in N$ or v is incident with a crossed edge with the other end in $\mathrm{dom}(\theta)$). Let N' be the set of vertices $v \in V(G')$ not incident with the outer face of G' such that $|L_\theta(v)| = 4$.

If a vertex $v \in V(G') \setminus V(P)$ satisfies $|L_\theta(v)| \leq 2$, then $v \notin V(F)$ by the choice of X and θ and by Lemma 15. Since $|\mathrm{dom}(\theta)| \leq 2$, it follows that $v \in N$ and v has two neighbors in $\mathrm{dom}(\theta)$, and thus (A2) holds. Hence, we can assume that $|L_\theta(v)| \geq 3$ for all $v \in V(G') \setminus V(P)$.

Let M' consist of all edges of G' that join vertices with list of size three, and let $B' = (G', P, N', M', L_\theta)$. Note that B' satisfies (S), (N), and (M) by the choice of N' and M'. Since $L_\theta(p) = L(p)$ for all $p \in V(P)$, the target B' satisfies (P). By Lemma 17, B' satisfies (T).

Now, let us consider property (C). Let q be a crossing in G' and suppose that (C) is violated at q, i.e., there exist distinct $u, v \in V(G_q)$ such that $|L_\theta(u)| = 3$ and $|L_\theta(v)| \in \{3, 4\}$. Note that each of u and v is either incident with F or adjacent to a vertex in $\mathrm{dom}(\theta)$. Let us distinguish several cases.

- If both u and v belong to F, then by Lemmas 15 and 19 we have that u and v are crossing-adjacent, $\{u, v\} = \{v_{m+1}, v_{m+2}\}$ and $L(v_{m+2}) = L_\theta(v_{m+2})$. It follows that $|L(v_{m+1})| \neq 3$ and that (A4) holds.

- If $|\{u, v\} \cap V(F)| = 1$ and u and v are not crossing-adjacent, then since $V(G_q) \cap X = \emptyset$, Lemma 20 implies that (A4) holds.

- If $u \in V(F)$, $v \notin V(F)$, and u and v are crossing-adjacent, then we apply Lemma 21. The outcome (a) of Lemma 21 gives (A5) or (A6). The outcome (b) gives a vertex $w \in X$ that is adjacent to v, and a vertex z with $|L(z)| = 3$ that is adjacent to u, v and w. Therefore, $|L(u)| \neq 3$, so u has a neighbor in X, necessarily equal to z. Note that $|X| \geq 2$ and $|L(z)| = 3$ can only be satisfied in the subcase (X4a) of the definition of X with $z = v_2$, $w = v_3$ and $u = v_1$, and thus (A6) holds.

- If $u \notin V(F)$, $v \in V(F)$, and u and v are crossing-adjacent, then u has two neighbors in $\mathrm{dom}(\theta)$. By Lemma 21, one of the neighbors of u in X is also adjacent to v and has list of size three, and by the choice of X, we conclude that (A6) holds.

- Finally, if $u, v \notin V(F)$, then $|\mathrm{dom}(\theta)| = 2$, u is adjacent to both vertices in $\mathrm{dom}(\theta)$ and v is adjacent to at least one of them. Hence, u and v are crossing-adjacent by Lemma 9 and the fact that $V(G_q) \cap X = \emptyset$, and (A3) holds.

Therefore, we can assume that B' satisfies (C). Let us now consider the newly created special subgraphs in B'.

- If $v \in N' \setminus N$, then v is adjacent to a vertex of $\mathrm{dom}(\theta)$ by an edge containing a crossing q, and only one vertex of G_q belongs to X. Therefore, $|N' \setminus N| \leq 1$.

- If $xy \in M' \setminus M$, then $|L_\theta(x)| = |L_\theta(y)| = 3$, and at least one of x and y has a list of size at least 4 in the list assignment L.

  Suppose that $x, y \notin N$. If $x, y \notin V(F)$, then both x and y have two neighbors in $\mathrm{dom}(\theta)$. By Lemma 20, this implies that x and y are crossing-adjacent in G via the edges joining x, y with $\mathrm{dom}(\theta)$. If $x, y \in V(F)$, then by Lemma 15 we can assume that $x = v_{m+1}$ and $y = v_{m+2}$; but then $|L_\theta(x)| \neq 3$ or $|L_\theta(y)| \neq 3$ by the choice of X, which is a contradiction. Finally, suppose that say $x \in V(F)$ and $y \notin V(F)$; then y has two neighbors in $\mathrm{dom}(\theta)$ and, in particular, we have cases (X2) or (X4). By Lemma 20, we have $x \in \{v_1, v_{m+1}\}$. If $x = v_1$, then y would be a common neighbor of v_1, v_2 and v_3, contradicting the choice of X (assumptions of (X4b) are satisfied, hence we would have $v_1 \in X$). If $x = v_{m+1}$, then since $|L(v_m)| \geq 4$, Lemma 20 implies

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that $|L(v_m)| = 4$ and v_m is incident with an edge crossing either the edge $v_{m-1}y$ or the edge $v_{m+1}y$. However, by the choice of X we have $|L(v_{m-1})| = |L(v_{m+1})| = 3$, contradicting (C).

Therefore, either $\{x, y\} \cap N \neq \emptyset$, or there exists a crossing q such that $\{x, y\} = V(G_q) \setminus X$.

It follows that $d(S_1, S_2) \geq 7 + r(S_1) + r(S_2)$ whenever S_1 is a special subgraph of G that is also special in G' and S_2 is any special subgraph of G'. Suppose now that S_1 and S_2 are both distinct newly created special subgraphs in G'. Note that $|N' \setminus N| \leq 1$ and if $N' \setminus N \neq \emptyset$, then $M' \setminus M = \emptyset$. It follows that $S_1, S_2 \in M' \setminus M$. As proved in the previous paragraph, each edge in $M' \setminus M$ is incident with a special subgraph in G that is adjacent to X. By the distance condition, we conclude that there exists a path xyz in G' such that $|L_\theta(x)| = |L_\theta(y)| = |L_\theta(z)| = 3$ and $y \in N$. Note that at most one of x and z can have two neighbors in $\operatorname{dom}(\theta)$, as otherwise G would contain a crossing at distance at most one from y; thus we may assume that $x \in V(F)$. Since y has a neighbor in $\operatorname{dom}(\theta)$, Lemma 20 implies that $x \in \{v_1, v_{m+1}, v_{m+2}\}$. If $x = v_{m+2}$, then we would have $|L(v_{m+1})| = |L(x)| = 3$ and $v_{m+1}x \in M$ would be at distance one from $y \in N$, which is a contradiction; therefore, $x \neq v_{m+2}$. If $x = v_1$, then (A2) holds. Finally, suppose that $x = v_{m+1}$. In this case, $z \notin V(F)$ has two neighbors in $\operatorname{dom}(\theta)$, and X was chosen according to (X2) or (X4). However, then $|L(v_m)| \geq 4$, hence $\deg(v_m) \geq 4$ and v_m is adjacent to y by Lemma 8, and (A7) holds. Hence, we can assume that B' satisfies (D).

By the minimality of B, we conclude that B' violates condition (O), and thus B contains a near-obstruction H. By Lemma 22, H is one of O_{M1}, O_{N2}, O_{N3} or O_{P3}.

- If H is O_{M1}, then let xy be the edge of H that belongs to M', where x is adjacent to p_2. Note that $x, y \notin V(F)$ by Lemma 22 and $xy \notin M$. If $x \notin N$, then x has two neighbors v_i and v_j in $\operatorname{dom}(\theta)$, where $i < j$. By Lemma 20 applied to p_2xv_i, we have $j = i + 1$ and by the choice of X, $|L(v_j)| = 4$; hence $\deg(v_j) \geq 4$, and by Lemma 8, v_j is incident with a crossing and thus $y \notin N$. Consequently, y is also adjacent to v_i and v_j. However, note that $|L(v_i)| = 3$, contradicting (C) for G. Therefore, $x \in N$ is adjacent to v_j, and y is adjacent to both v_i and v_j. By Lemma 20 applied to p_0yv_j, we have $i = 1$, $j = 2$ and $|L(v_1)| = 3$, and by Lemma 20 applied to p_2xv_2, we have that $s = 3$ and $|L(v_3)| = 3$. However, then G is L-colorable.

- Next, suppose that H is O_{N2} and let x and y be the vertices in the outer face of H such that $|L_\theta(x)| = 3$ and $|L_\theta(y)| = 4$. By Lemma 15, $y \notin V(F)$. If $x \in V(F)$, then by Lemma 20 we have $s \le 2$, which is a contradiction, hence $x \notin V(F)$. Thus x has two neighbors in $\mathrm{dom}(\theta)$ and y has one, and by Lemma 20 we conclude that $s = 3$ and $|L(v_1)| = |L(v_3)| = 3$. It follows that $X = \{v_1, v_2\}$, x is adjacent to v_1 and v_2, and y is adjacent to v_2. There are two cases, either v_2 is incident with a crossed edge or $|N| = 1$; in both of them, G is L-colorable.

- If H is O_{N3}, then let xyz be the path in the outer face of H such that $|L_\theta(x)| = |L_\theta(z)| = 3$, $|L_\theta(y)| = 4$ and z is adjacent to p_1. By Lemma 15, $z \notin V(F)$, thus z has two neighbors $w_1, w_2 \in \mathrm{dom}(\theta)$, and by Lemma 20, we can assume that the neighbors of w_1 are w_2, z and an endvertex of P, and that $|L(w_1)| = 3$. Since $y \notin V(F)$ and $|L_\theta(y)| = 4$, y is adjacent to w_2. Since x cannot have more than one neighbor in $\mathrm{dom}(\theta)$, we have $x \in V(F)$. If $xw_2 \notin E(G)$, then $|L(x)| = |L_\theta(x)| = 3$, and thus x has no neighbor with list of size three (since H is an obstruction in B', x is either adjacent to a vertex in N, or incident with a crossing, and thus it is not incident with an edge of M). Lemma 20 applied to the 2-chord xyw_2 implies that the edge xy is crossed by an edge incident with w_2. However, then $\deg(w_2) = 4$ and (C) implies that $|L(w_2)| = 5$, which is a contradiction. We conclude that $xw_2 \in E(G)$. By the choice of X, $|L(x)| = 3$. Again, we distinguish two cases depending on whether w_2 is incident with a crossed edge (in this case $|L(w_2)| = 5$ by (C)) or $|N| = 1$. In both cases, G is L-colorable.

- If H is O_{P3}, then two of the vertices of H have two neighbors in $\mathrm{dom}(\theta)$, hence G contains a crossing at distance at most one from P, which contradicts the distance condition for B.

These contradictions imply that at least one of (A1)–(A7) holds. □

Each case among (A1)–(A7) in Lemma 24 contains a special subgraph. Thus, a minimal counterexample $B = (G, P, N, M, L)$ with $P = p_0 \ldots p_k$ contains a special subgraph S whose distance from p_0 is at most $2 + r(S)$. Consequently, $\ell(P) = k = 2$. Next, we consider the probe (X', θ', b) at the p_k side of P. This probe satisfies one of the conditions symmetric to (A1)–(A7) (satisfying one of (A1)–(A7) with v_i replaced by v_{s+1-i}, and X by X'),

which we will refer to as (A1')–(A7'). In particular, there exists a special subgraph S' whose distance from p_2 is at most $2 + r(S')$. It follows that $d(S, S') \leq 6 + r(S) + r(S')$, and by (D), we conclude that $S = S'$.

To avoid repeating the definitions each time, let us fix the following notation (†) for the following lemmas.

- $B = (G, P, N, M, L)$ is a minimal counterexample, with $P = p_0 p_1 p_2$.

- $p_2 p_1 p_0 v_1 v_2 \ldots v_s$ are the vertices contained in the boundary of the outer face F of G in the cyclic order.

- If (A1) does not hold, then (X, θ, m) is a probe for B at the p_0 side of P.

- If (A1') does not hold, then (X', θ', b) is a probe for B at the p_2 side of P.

- S is the special subgraph in B at distance at most 4 from P.

Lemma 25. *With the notation* (†), *S is a crossing; hence, B satisfies one of* (A3)–(A6) *and one of* (A3')–(A6').

Proof. Suppose for a contradiction that B does not contain such a crossing. Hence, it satisfies (A1), (A2), or (A7).

Suppose first that B satisfies (A1), and thus S is an edge of M and B also satisfies (A1'). It follows that $s \leq 4$. Since $s \geq 3$, we can by symmetry assume that $S = v_2 v_3$. If v_2, v_3 and v_i have no common neighbor for $i \in \{1, 4\}$ ($i = 1$ if $s = 3$), then let φ be an arbitrary L-coloring of S (such that $\varphi(v_3) \notin L(p_2)$ if $s = 3$). Let $B' = (G_\varphi, P, N, \emptyset, L_\varphi)$. Observe that B' cannot contain an obstruction since its special subgraph would be a special subgraph in B, too close to the special subgraph S. Now it is easy to check using previously proved properties of B that B' satisfies all conditions of Theorem 7. (The same reasoning will be applied in the sequel without repeating it.) Therefore, B' with the list coloring L_φ is a counterexample to Theorem 7, contradicting the minimality of B. Thus, by symmetry, we may assume that v_1, v_2 and v_3 have a common neighbor w. In that case, w is not adjacent to v_4 by Lemma 20. Let φ be an L-coloring of v_1 and v_3 such that $\varphi(v_1) \notin L(p_0)$, $\varphi(v_3) \notin L(p_2)$ and $|L_\varphi(v_2)| \geq 2$. Observe that $(G - \{v_1, v_2, v_3\}, P, N, \emptyset, L_\varphi)$ is a counterexample contradicting the minimality of B, since any L_φ-coloring of $G - \{v_1, v_2, v_3\}$ can be extended to v_2 by using a color in $L_\varphi(v_2)$, and can henceforth be extended to G. Therefore, B does not satisfy (A1).

Let us now consider the case that B satisfies (A2) or (A7), and thus $S \in N$. Let i and j be the smallest and the largest integer, respectively, such that S is adjacent to v_i and v_j. By Lemma 20 we have $j \in \{i+1, i+2\}$. We consider the two possible values of j separately:

- Suppose first that $j = i + 1$. If $|X| \geq 2$, then $|L(v_m)| \geq 4$ and $|L(v_{m-1})| = 3$, hence (A7) and (A7') cannot both be true. If both (A2) and (A2') hold, then since $s \geq 3$, we can assume that $v_2, v_3 \in X$ have a common neighbor in N. By the choice of X, we have $|L(v_4)| = 3$, hence $s = 4$ and $v_2, v_3 \in X'$. However, then $|L(v_1)| \geq 4$ by the choice of X and $|L(v_1)| = 3$ by the choice of X', which is a contradiction.

  Hence, we can assume that B satisfies (A7) and (A2'); then we either have $s = m + 1$, or we have $s = m + 2$ and $X' = \{v_m, v_{m+1}\}$. If there exists an L-coloring φ of v_{m-1} and v_{m+1} such that their colors are distinct from the colors of their neighbors in P and $|L_\varphi(v_m)| \geq 3$, then $B' = (G - \{v_{m-1}, v_m, v_{m+1}\}, P, N \setminus \{S\}, \emptyset, L_\varphi)$ contradicts the minimality of B; observe that B' satisfies (O), since no special subgraph of G is at distance at most two from S. A new special subgraph (an edge joining two vertices with lists of size three) would appear in B' only if S were adjacent to v_{m+2}, which is not the case since $j = i + 1$.

  We conclude that no such coloring exists. Observe that this is only possible if both v_{m-1} and v_{m+1} have a neighbor in P (i.e., $m = 2$ and $s = 3$), $|L(v_1)| = |L(v_3)| = 3$, and $L(v_2)$ is the disjoint union of $L(v_1) \setminus L(p_0)$ and $L(v_3) \setminus L(p_2)$. Let w' be the common neighbor of S and v_1. Suppose that there exists a color $c \in L(w')$ different from the colors of the neighbors of w' in P such that either $c \notin L(v_2)$, or v_1 has degree three and $c \notin L(v_1) \setminus L(p_0)$. In this case, we let φ be the partial coloring such that $\varphi(w') = c$ and let $G' = G - \{w', v_2\}$ if $c \notin L(v_2)$ and $G' = G - \{w', v_1, v_2\}$ if $c \in L(v_2)$. Observe that G' is not L_φ-colorable. We conclude that $(G', P, N, M \cup \{Sv_3\}, L_\varphi)$ is a counterexample contradicting the minimality of B (the condition (O) holds by Lemma 22, the distance condition and Lemma 15). Therefore, no such color c exists.

  Since $|L(w')| > |L(v_2)|$, it follows that w' has a neighbor in P. By Lemma 20, w' is not adjacent to p_2, hence it is adjacent to p_0 or p_1. However, then Lemmas 11 and 15 imply that v_1 has degree three, and

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since $|L(v_1) \setminus L(p_0)| = 2$ and w' has at most two neighbors in P, the color c exists. This is a contradiction.

- It remains to consider the case when $j = i+2$. In this case S is adjacent to v_i and v_{i+2}, and by Lemma 20 we conclude that v_{i+1} is a vertex of degree 3 with neighbors v_i, v_{i+2}, and S. Thus, $|L(v_{i+1})| = 3$. Suppose first that B satisfies both (A7) and (A7'). If there exists a coloring φ of S by a color different from the colors of its neighbors in P such that $\varphi(S) \notin L(v_i) \cap L(v_{i+1}) \cap L(v_{i+2})$, then $(G - \{S, v_i, v_{i+1}, v_{i+2}\}, P, N \setminus \{S\}, \emptyset, L_\varphi)$ is a counterexample contradicting the minimality of B. Otherwise, since S is not adjacent to p_0 or p_2 by Lemma 20, we conclude that S is adjacent to p_1 and $L(S) \setminus L(p_1) = L(v_{i+1}) \subseteq L(v_i) \cap L(v_{i+2})$. However, in this case we let $\varphi = \theta$ (where θ is the coloring from the probe for B at the p_0 side of P), and note that $\varphi(v_i) \notin L(v_{i+1}) = L(S) \setminus L(p_1)$. Hence, the construction of L_φ ensures that $|L_\varphi(S)| = 4$, and we conclude that $(G - X, P, N \setminus \{S\}, \emptyset, L_\varphi)$ is a counterexample contradicting the minimality of B.

Therefore, B does not satisfy both (A7) and (A7'), and by symmetry, we can assume that B satisfies (A2'). Let us first consider the case that B satisfies (A2). Note that $v_{i+2} \notin X$, as otherwise $|L(v_{i+3})| = 3$ by the choice of X, and thus $v_{i+1} \notin X'$, contradictory to the assumption that B satisfies (A2'). Symmetrically, $v_i \notin X'$. Since $|L(v_{i+1})| = 3$, we cannot have $\{v_i, v_{i+1}\} \subseteq X$, and since (A2) holds, we have $i = 1$ and symmetrically, $s = 3$. Observe that we cannot color S by a color $\varphi(S) \notin L(v_{i+1})$, as otherwise $(G - \{S, v_{i+1}\}, P, N \setminus \{S\}, \emptyset, L_\varphi)$ would contradict the minimality of B. Therefore, S has a neighbor in P, and by Lemma 20, this neighbor is p_1. By Lemma 11, the 4-cycle $p_1 p_0 v_1 S$ is not separating, and by Lemma 15, v_1 has degree three. This contradicts Lemma 8(a), since $|L(v_1)| > 3$.

Therefore, B satisfies (A7). Note that v_{i+1} cannot be the element of X' with the smallest index, and thus $i + 2 = s$. As before, we exclude the case that S can be colored by a color not belonging to $L(v_i) \cap L(v_{i+1})$, hence S has a neighbor in P. By Lemma 20, S is adjacent to p_1. However, by Lemma 11, the 4-cycle $p_1 S v_{i+2} p_2$ is not separating, and by Lemma 15, v_{i+2} is not adjacent to p_1. Thus, v_{i+2} has degree three and list of size at least four, which is a contradiction.

Therefore, B does not satisfy (A2) or (A7), either. $\qquad\square$

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Note that if B satisfies (A4) or (A4'), then $|V(G_q) \cap V(F)| = 2$ by Lemma 15. If B satisfies (A6) or (A6'), then by Lemmas 15, 16 and 19 we have $|V(G_q) \cap V(F)| = 1$. If B satisfies (A3) or (A3'), then by Lemmas 20, 15 and 16, we have $|V(G_q) \cap V(F)| \le 1$, and if B satisfies (A5) or (A5') then $1 \le |V(G_q) \cap V(F)| \le 2$.

Lemma 26. *With the notation* (†), *B satisfies neither* (A3) *nor* (A3').

Proof. Suppose for a contradiction that B satisfies (A3). Let w_1 and w_2 be as in the description of (A3). Note that w_2 is adjacent to v_{m-1} and v_m (even if $v_{m-1} \notin \mathrm{dom}(\theta)$, in the case (X4b)) and that $|L(v_{m-1})| = |L(v_{m+1})| = 3$.

Let q denote the crossing contained in S. Let us first consider the case that $|V(G_q) \cap V(F)| = \emptyset$. In this case B satisfies (A3'), i.e., there exists $w_2' \in V(G_q)$ adjacent to v_b and v_{b+1}, and another vertex w_1' of G_q crossing-adjacent to w_2' with a neighbor in $\mathrm{dom}(\theta')$. Since $|L(v_b)| \ne 3$, we have $b \notin \{m-1, m+1\}$. Consequently, $|X \cap X'| \le 1$, and $w_2' \ne w_2$ by Lemma 20.

We now distinguish two cases regarding whether w_2 is adjacent or crossing-adjacent to w_2' in G_q.

- Suppose that $w_2 w_2'$ is a crossed edge. Then $b \ne m$ by Lemma 9 and the assumption that G_q is disjoint with F; thus $b \ge m + 2$. Let G_1 and G_2 be the $v_m w_2 w_2' v_b$-components of G, such that $P \subset G_1$. Since w_1 and w_2 are crossing-adjacent, we have $w_1 \ne w_2'$, and symmetrically $w_1' \ne w_2$. By Lemma 20, if $w_1 = w_1'$, then w_1 belongs to G_2. Hence, by symmetry, we can assume that w_1 belongs to G_2, and thus w_1 is adjacent to v_m.

  If w_1 is adjacent to v_b, then $b = m + 2$ by Lemma 20. Let $T = \{v_m, v_{m+1}, v_{m+2}, w_1\}$. By using Lemma 18 it is easy to see that $|L(t)| = \deg(t)$ for each $t \in T \setminus \{w_1\}$ and that $\deg(w_1) \le 6$. By the minimality of B, there exists an L-coloring φ of $G - T$. Consider the subgraph G' of G induced by T with the list assignment L_φ. We have $|L_\varphi(v_{m+1})| = 3$ and $|L_\varphi(z)| \ge 2$ for $z \in T \setminus \{v_{m+1}\}$. If $L_\varphi(w_1) \ne L_\varphi(v_m)$, then color w_1 by a color in $L_\varphi(w_1) \setminus L_\varphi(v_m)$ and extend this coloring to the rest of G'. Similarly, G' is L_φ-colorable if $L_\varphi(w_1) \ne L_\varphi(v_{m+2})$. If $L_\varphi(v_m) = L_\varphi(w_1) = L_\varphi(v_{m+2})$, then we color v_{m+1} by a color in $L_\varphi(v_{m+1}) \setminus L_\varphi(w_1)$ and again we can extend this to an L_φ-coloring of G'. It follows that G is L-colorable, which is a contradiction.

  Therefore, w_1 is not adjacent to v_b, and in particular $w_1 \ne w_1'$ and $w_1' \in V(G_1)$. Let φ be an L-coloring of G_1, which exists by the minimality
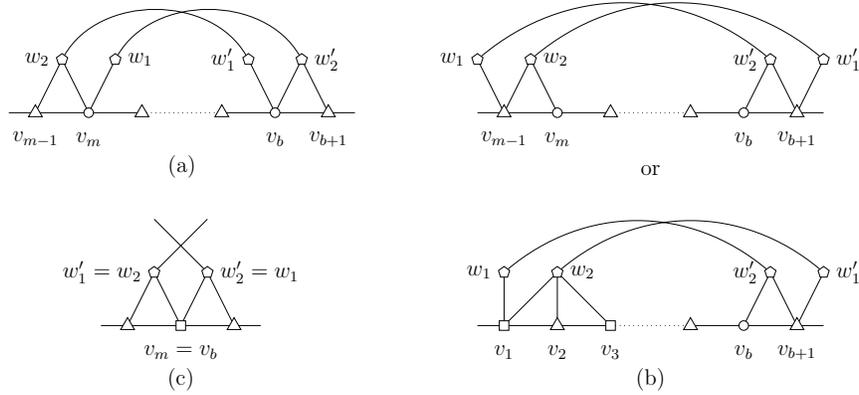
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Figure 8: Subcases when w_2 and w_2' are crossing-adjacent.

of B. Since w_1 is not adjacent to v_b, note that w_1 has at most three neighbors in G_1 different from w_2'. Hence, we can additionally choose a color $\varphi(w_1)$ for w_1 different from the colors of its neighbors in G_1 so that $\varphi(w_1) \neq \varphi(w_2')$. Let $G_2' = G_2 - w_2 + w_1 w_2'$ and $P' = v_m w_1 w_2' v_b$, let L' be the list assignment for G_2' obtained from L by setting $L'(z) = \{\varphi(z)\}$ for $z \in V(P')$, and let $B_2' = (G_2', P', N \cap V(G_2'), M \cap E(G_2'), L')$. Note that the added edge $w_1 w_2'$ of G_2' can be drawn without crossings following the crossed edges of G that are no longer in G_2'. Note that φ does not extend to an L-coloring of G_2', and thus the target B_2' is not valid. Note that B_2' may only violate (T) or (O). In the former case, the vertex violating (T) must be v_{m+1} and we would have $b = m+2$. Consequently, v_b would have degree three by Lemma 18, which is a contradiction since $|L(v_b)| \geq 4$. In the latter case, since $|L(v_{m+1})| = |L(v_{b-1})| = 3$ and v_b has degree at least three in G_2', we have that G_2' is equal to O_{P5} or O_{P6}. In both cases, any L-coloring of $G_1 - \{v_m, v_b\}$ would extend to an L-coloring of G, a contradiction.

- Suppose now that w_2 is crossing-adjacent to w_2'. Let G_1 and G_2 be the subgraphs of G intersecting in $\{v_b, w_2', w_2, v_m\}$, where $P \subset G_1$ and $G_1 \cup G_2$ is equal to $G - E(G_q)$. We have two subcases: either $b > m$ or $b = m$.

    - If $b > m$, then Lemma 20 implies that w_2' has no neighbor in X,

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and thus $w_1 \neq w_2'$. Symmetrically, $w_1' \neq w_2$. Since w_1 is crossing-adjacent to w_2, and w_1' is crossing-adjacent to w_2', we conclude that the edges of G_q are $w_1 w_2'$ and $w_1' w_2$.

If $w_1, w_1' \notin V(G_1)$ (see Figure 8(a)), then $w_1 v_m, w_1' v_b \in E(G)$. Let φ be an L-coloring of $G_1 + \{w_1, w_1', w_1 w_2', w_1' w_2, w_1 w_1'\}$ which exists by the minimality of G, and note that φ does not extend to an L-coloring of $G_2' = G_2 + w_1 w_1'$. Let $P' = v_m w_1 w_1' v_b$ and let L' be the list assignment for G_2' obtained from L by setting $L'(z) = \{\varphi(z)\}$ for $z \in V(P')$. Observe that it is possible to choose φ so that $B_2' = (G_2', P', N \cap V(G_2'), M \cap E(G_2'), L')$ is a counterexample contradicting the minimality of B (once the coloring of $G_1 - \{v_m, v_b\}$ is fixed, we still have two possible choices for the colors of v_m and v_b and three possible choices for the colors of w_1 and w_1', which suffices to ensure that B_2' satisfies (T) and (O)). This is a contradiction. The case that $w_1, w_1' \in V(G_1)$ (see Figure 8(b)) is excluded similarly.

− If $b = m$, then let $w_2 z$ and $w_2' z'$ be the edges of G_q (note that we have $w_1 = w_2'$ and $w_1' = w_2$). Suppose that $z, z' \in V(G_2)$. Note that $V(G_2) \neq \{z, z', w_2, w_2', v_m\}$, since otherwise z would have degree at most four and $|L(z)| = 5$. Therefore, the subgraph of G induced by $V(G_1) \cup \{z, z'\}$ has an L-coloring ψ by the minimality of G. Let L' be the list assignment for $G_2' = G_2 - \{z, z'\}$ obtained from L by removing the colors of z and z' according to ψ from the lists of their neighbors and by setting $L'(w_2) = \{\psi(w_2)\}$, $L'(v_m) = \{\psi(v_m)\}$ and $L'(w_2') = \{\psi(w_2')\}$. Note that $(G_2', w_2 v_m w_2', N \cap V(G_2), \emptyset, L')$ satisfies (O) and (C) by the distance condition and (P) by the choice of ψ, and since G is not L-colorable, we conclude that B_2' violates (T). Therefore, G_2 contains a vertex adjacent to w_2, w_2', v_m, z and z'. This contradicts Lemma 9.

Therefore, we have $z, z' \in V(G_1)$, see Figure 8(c). By Lemma 18, $\deg(v_m) = 4$. Let $S_1 = L(v_2)$ if $m = 3$ and $S_1 = L(v_1) \setminus L(p_0)$ if $m = 2$. Note that $S_1 \subset L(v_m)$, as otherwise we consider the partial coloring φ with $\varphi(v_{m-1}) \in S_1 \setminus L(v_m)$ and conclude that $(G_\varphi, P, N, \emptyset, L_\varphi)$ contradicts the minimality of G.

Suppose that there exists a color $c \in L(w_2) \setminus L(v_m)$, or that $\deg(v_{m-1}) = 3$ and there exists a color $c \in L(w_2) \setminus S_1$, such that this color c is distinct from the colors of the neighbors of
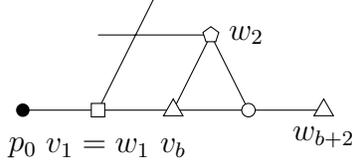
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Figure 9: Subcase combining (A3) and (A5').

w_2 in P. Let φ be the partial coloring with $\varphi(w_2) = c$, and let $G' = G - \{w_2, v_m\}$ if $\deg(v_{m-1}) > 3$ and $G' = G - \{w_2, v_m, v_{m-1}\}$ if $\deg(v_{m-1}) = 3$. Note that G' is not L_φ-colorable, and that $L_\varphi(v_{m-1}) = L(v_{m-1})$ if v_{m-1} belongs to $V(G')$ since $\varphi(w_2) \notin S_1$. By the minimality of B, the target $B' = (G', P, N \cup \{z\}, \emptyset, L_\varphi)$ is not valid. This is only possible if B' violates (O), and by Lemma 22 and the distance condition, G' contains O_{N2} or O_{N3}. However, then z is adjacent to two vertices of P and to z' and w_2', and at least one of z' and w_2' has a list of size three according to L', which is a contradiction since $|L(z')| = |L(w_2')| = 5$.

We conclude that there exists no such color c. Since $|L(v_m)| = 4$ and $|L(w_2)| = 5$, we conclude that w_2 has a neighbor in P. By Lemma 20, w_2 is not adjacent to p_2, and if it were adjacent to p_0, then we would have $m = 2$, $\deg(v_1) = 3$ and there would exist a color $c \in L(w_2) \setminus (S_1 \cup L(p_0) \cup L(p_1))$. Therefore, w_2 is adjacent to p_1. By symmetry, w_2' is adjacent to p_1 as well. However, the edges $w_2 p_1$ and $w_2' p_1$ are not crossed by Lemma 16, and thus the crossing is contained inside the 4-cycle $v_m w_2 p_1 w_2'$, contrary to Lemma 11.

We conclude that $V(G_q) \cap V(F) \neq \emptyset$. By Lemma 15, $w_2 \notin V(F)$. Let w be the vertex joined to w_2 by a crossed edge, and let $w_1 w'$ be the other crossing edge. Since $V(G_q) \cap X = \emptyset$, by Lemmas 16 and 20 we have $w \notin V(F)$. Since $|L(v_m)| \geq 4$, v_m has degree at least four; hence, we cannot have $w_1 = v_{m+1}$, and by Lemmas 15 and 16, we have $w_1 \notin V(F) \setminus \{v_1\}$. If $w_1 \notin V(F)$ and $x \in X$ is a neighbor of w_1, then the 2-chord $x w_1 w'$ separates P from either w_2 or w, and neither w_2 nor w belongs to F, contrary to Lemma 20. We conclude that $w_1 = v_1$ and $V(G_q) \cap V(F) = \{v_1\}$, hence $v_1 \notin X$ and X was chosen according to (X4a).

Since $|V(G_q) \cap V(F)| = 1$, B must satisfy (A3'), (A5') or (A6'). If

B satisfied (A3'), the conclusions of the preceding paragraph would apply symmetrically and we would have $v_1 = v_s$, which is a contradiction. Similarly, X' cannot satisfy (A6'). The remaining possibility is that (A5') holds for X'. Then $v_1 = v_{b-1}$ and $v_b = v_2$. The situation is shown in Figure 9. Since X was chosen according to (X4a), we have $|L(v_b)| = |L(v_{b+2})| = 3$; in particular, $s \geq 4$ and $b \leq s - 2$. This is only possible if X' has been chosen according to (X4), but then $|L(v_b)| > 3$. This is a contradiction, showing that B does not satisfy (A3). Symmetrically, B does not satisfy (A3'). $\qquad\square$

Next, we exclude the case that the crossing near P is incident with two vertices of F.

Lemma 27. *With the notation* (†), *let q denote the crossing contained in S. Then $|V(G_q) \cap V(F)| = 1$.*

Proof. Since B does not satisfy (A3), if $|V(G_q) \cap V(F)| \neq 1$ then $|V(G_q) \cap V(F)| = 2$ and B satisfies (A4) or (A5). Symmetrically, B satisfies (A4') or (A5'). By Lemmas 15 and 19, $V(G_q) \cap V(F) = \{v_{m+1}, v_{m+2}\}$ and v_{m+1} is crossing-adjacent to v_{m+2}. Let $v_{m+1}w$ and $v_{m+2}w'$ be the crossed edges. By symmetry, we can assume that $|L(v_{m+1})| \geq |L(v_{m+2})|$. By (C), either $|L(v_{m+1})| \geq |L(v_{m+2})| \geq 4$ or $|L(v_{m+1})| = 5$ and $|L(v_{m+2})| = 3$. Therefore, X was chosen according to the rules (X1) or (X3) and $|L(v_m)| = 3$.

If $L(v_{m+2}) \neq L(v_{m+1})$, then let c be a color in $L(v_{m+1}) \setminus L(v_{m+2})$. If v_{m+1} is not adjacent to v_{m+2}, then let c be an arbitrary color in $L(v_{m+1})$. In both cases, let us define a partial L-coloring φ as follows. Let $\varphi(v_{m+1}) = c$. If $m = 1$, then choose $\varphi(v_1) \in L(v_1) \setminus L(p_0)$ distinct from c. If $m = 2$ and v_1, v_2, and v_3 have a common neighbor, then choose $\varphi(v_1) \in L(v_1) \setminus L(p_0)$ so that $|L_\varphi(v_2)| \geq 2$. Otherwise, choose $\varphi(v_m) \in L(v_m)$ distinct from c. Note that $G_\varphi - v_m$ is not L_φ-colorable, and by the minimality of B, we conclude that $B' = (G_\varphi - v_m, P, N \cup \{w\}, \emptyset, L_\varphi)$ is not a valid target. This is only possible if B' violates (O). By Lemma 22, B' contains O_{N2} or O_{N3}. It follows that w is adjacent to p_1 and to p_0 or p_2. However, if w is adjacent to p_0, then by Lemma 20, v_{m+2} is incident with a chord of F, contradicting Lemma 15. If w is adjacent to p_2, then $v_{m+2} = v_s$ by Lemma 20. This contradicts the assumption that B satisfies (A4') or (A5'). We conclude that $L(v_{m+1}) = L(v_{m+2})$ (and in particular, $|L(v_{m+1})| = |L(v_{m+2})| = 4$ by (A4) and (A5)), and $v_{m+1}v_{m+2} \in E(G)$. By the choice of X', we have $|L(v_{m+3})| = 3$.

Suppose now that $w'v_m \in E(G)$. Note that v_{m+1} has degree at least four, so it is adjacent to w'. Let $S_1 = L(v_m)$ if $m \neq 1$ and $S_1 = L(v_m) \setminus L(p_0)$ if $m = 1$. Note that $S_1 \subseteq L(v_{m+1})$, as otherwise we can choose an L-coloring φ of v_m such that $\varphi(v_m) \in S_1 \setminus L(v_{m+1})$, and $(G - \{v_m, v_{m+1}\}, P, N, \emptyset, L_\varphi)$ is a counterexample contradicting the minimality of B. Since $L(v_{m+1}) = L(v_{m+2})$, we conclude that $S_1 \subseteq L(v_{m+2})$. Let G' be the graph obtained from $G - v_{m+1}$ by identifying v_m with v_{m+2} to a new vertex z, and let L' be the list assignment obtained from L by setting $L'(z) = L(v_m)$. Let $B' = (G', P, N, \{zv_{m+3}\}, L')$. Observe that every coloring of G' gives rise to an L-coloring of G, and thus G' is not L'-colorable. Note that B' satisfies (O), since B' contains neither O_{M1} nor O_{M2}, and the exclusion of other obstructions is obvious. It follows that B' is a counterexample contradicting the minimality of B. Therefore, $w'v_m \notin E(G)$, and by symmetry, $wv_{m+3} \notin E(G)$.

Let $S_2 = L(v_{m+3})$ if $m + 3 \neq s$ and $S_2 = L(v_{m+3}) \setminus L(p_2)$ if $m + 3 = s$. Suppose now that there exists an L-coloring φ of v_{m+1} and v_{m+2} by distinct colors such that $\varphi(v_{m+1}) \notin S_1$ and $\varphi(v_{m+2}) \notin S_2$. Let $M' = \{ww'\}$ if $ww' \in E(G)$, and $M' = \emptyset$ otherwise. By the minimality of B, we conclude that $B' = (G_\varphi, P, N, M', L_\varphi)$ is not a valid target. This is only possible if B' violates (O). By Lemma 22, B' contains O_{M1} (the other cases are easily excluded: O_{N2} and O_{N3} since no internal vertex gets a reduced list and O_{P3} since $\ell(P) = 2$). But then w' is adjacent to p_0, and the 2-chord $p_0w'v_{m+2}$ contradicts Lemma 20. Therefore, no such coloring φ exists. It follows that $|S_1| = |S_2| = 3$ and $S_1 \subseteq L(v_{m+1})$. Since $L(v_{m+1}) = L(v_{m+2})$, we also have that $S_1 = S_2$. Since $|S_1| = |S_2| = 3$, claim Lemma 8(f) implies that $m = 2$ and $s = 6$. Similarly, we conclude that $L(v_1) = L(p_0) \cup L(v_2)$ and $L(v_6) = L(p_2) \cup L(v_5)$, as otherwise we can color and remove v_1 or v_6.

Let us now consider the case that v_2, v_3 and w' have no common neighbor. If v_1, v_2 and v_3 have no common neighbor, then let φ be an L-coloring of v_2, v_3 and v_4 such that $\varphi(v_4) \notin L(v_5)$. Otherwise, let φ be an L-coloring of v_1, v_3 and v_4 such that $\varphi(v_4) \notin L(v_5)$ and $\varphi(v_1) = \varphi(v_3)$. In the former case, let $G' = G_\varphi$, in the latter case let $G' = G_\varphi - v_2$. Let $M' = \{ww'\}$ if $ww' \in E(G)$, and $M' = \emptyset$ otherwise. Note that $B' = (G', P, N, M', L_\varphi)$ satisfies (O) by Lemma 22, since w' cannot be adjacent to p_0. We conclude that B' is a counterexample contradicting the minimality of B. Therefore, v_2, v_3 and w' have a common neighbor x', and by symmetry, v_4, v_5 and w have a common neighbor x (see Figure 10).

By Lemma 20, we have $x \neq x'$ and x is adjacent neither to p_0 nor to p_2. Furthermore, if $xp_1 \in E(G)$, then consider the cycle $K = p_1p_2v_6v_5x$. Since
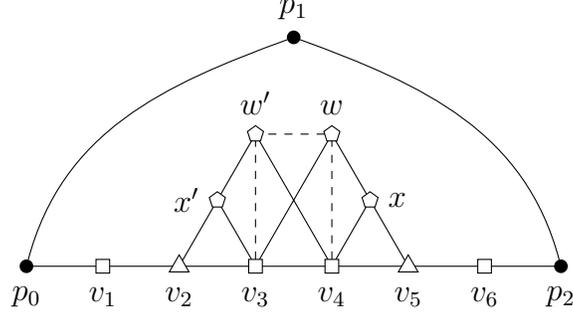
Figure 10: A subcase in the proof when $|V(G_q) \cap V(F)| = 2$. The dotted edges may or may not be present.

v_6 has degree at least four, we conclude by Lemma 12 that K has two chords incident with v_6. However, that contradicts Lemma 15. Therefore, x (and symmetrically x') has no neighbor in P. By Lemma 20, neither w nor w' is adjacent to p_0 or p_2. Lemmas 9 and 11 imply that $x'w, xw', xx' \notin E(G)$. Since both w and w' have degree at least 5, we conclude that each of them is adjacent either to p_1 or to a vertex not shown in Figure 10. Suppose that $w'p_1 \notin E(G)$. Then let φ be an L-coloring of x and w' such that $\varphi(x), \varphi(w') \notin L(v_4)$ (note that these colors do not belong to the lists of v_2, v_3 and v_5, as well as to $L(v_1) \setminus L(p_0)$ and $L(v_6) \setminus L(p_2)$). Let $G' = G - \{x, w', v_3, v_4\}$ if $\deg(w) > 5$ and $G' = G - \{x, w', v_3, v_4, w\}$ if $\deg(w) = 5$. Note that G' is not L_φ-colorable since any L_φ-coloring of G' extends to G. Furthermore, the only possible vertices with list of size three in G' are v_2, v_5, w and a common neighbor u of x and w' distinct from w and v_4, if such a vertex exists. By Lemma 11, if u exists, then $\deg(w) = 5$ and $w \notin V(G')$. Furthermore, by Lemma 11, u and w are not adjacent to v_2 and v_5. Therefore, $(G', P, N, \emptyset, L_\varphi)$ is a counterexample contradicting the minimality of B.

We conclude that $w'p_1 \in E(G)$. Let G_1 and G_2 be the $p_1w'v_4$-components of G, where G_1 contains p_0. Consider an L-coloring φ of G_2. Note that v_3 has only two neighbors in $G_2 - w'$, and thus φ can be extended to v_3 in such a way that $\varphi(v_3) \neq \varphi(w')$. Let $G' = G_1 - v_4 + w'v_3$, $P' = p_0p_1w'v_3$ and let L' be the list assignment obtained from L by setting $L'(z) = \{\varphi(z)\}$ for $z \in V(P')$. By the minimality of B, we conclude that $B' = (G', P', N, \emptyset, L')$ is not a valid target. This is only possible if B' violates (O). Observe that only v_1 and v_2 and vertices in N have list of size at most four in B' and

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that x' is a common neighbor of v_3 and w'. Therefore, x' is a vertex in the corresponding obstruction K, and v_2 is a vertex in K with list of size 3. It follows that K is equal to O_{P4} or O_{P6}. However, then $v_1 p_1 \in E(G)$ or $v_2 p_0 \in E(G)$, contradicting Lemma 15. $\qquad\square$

We are now ready to finish the proof of the main theorem.

Proof of Theorem 7. We proceed by contradiction; if Theorem 7 is false, then there exists a counterexample, and we can choose B as a minimal counterexample. With the notation (†), note that S is a crossing by Lemma 25. Let q be the crossing contained in S. By Lemma 27, we have $|V(G_q) \cap V(F)| = 1$, and thus B satisfies (A5) or (A6), and (A5') or (A6'). Since $s \geq 3$, we can by symmetry assume that B satisfies (A5').

Suppose first that B satisfies (A6), and thus $b = 2$. Let z be the neighbor of v_1 along the crossed edge. Since $v_1 \notin X$, the inspection of possible cases for X and X' shows that we have $|L(v_2)| = 3$, $X' = \{v_2\}$, and $s = 3$. If v_1, v_2 and v_3 have no common neighbor, then consider any L-coloring φ of v_1 and v_2 such that $\varphi(v_1) \notin L(p_0)$, and observe that $(G_\varphi, P, N \cup \{z\}, \emptyset, L_\varphi)$ is a counterexample contradicting the minimality of B (since v_1, v_2, v_3 do not have a common neighbor, we do not get adjacent vertices with lists of size 3, and (O) is satisfied since z is not adjacent to p_0 and p_2 by Lemmas 20 and 16).

Hence, we can assume that v_1, v_2 and v_3 have a common neighbor w, and thus $\deg(v_2) = 3$. Similarly, we conclude that $L(v_1) = L(p_0) \cup L(v_2)$ (if not, we color v_1 with a color in $L(v_1) \setminus (L(p_0) \cup L(v_2))$ and then consider $G' = G - \{v_1, v_2\}$) and that $L(v_3) = L(p_2) \cup L(v_2)$ (if not, we can color v_3 by a color in $L(v_3) \setminus (L(p_2) \cup L(v_2))$ and then consider $G' = G - \{v_2, v_3\}$). By Lemmas 11, 16 and 20, w has no neighbor in P. Let u be the vertex adjacent to w by the crossed edge, let φ be an L-coloring of w such that $\varphi(w) \notin L(v_2)$ and let $G' = G - \{v_2, w\}$. Note that $B' = (G', P, N \cup \{u\}, \emptyset, L_\varphi)$ satisfies (O), since it contains no vertex with list of size three. Thus, B' is a counterexample to Theorem 7 contradicting the minimality of G, which implies that B does not satisfy (A6).

Therefore, B satisfies both (A5) and (A5'), and we have $b = m + 2$. Moreover, Lemma 11 implies that the neighbors w and w' of v_m and v_b in $V(G_q) \setminus \{v_{m+1}\}$ are distinct. Let y be the vertex joined to v_{m+1} by a crossed edge. If $|L(v_{m+1})| \neq 3$, then both X and X' are chosen by cases (X1) or (X3) and $|L(v_m)| = |L(v_{m+2})| = 3$. The condition (A5) implies

$|L(v_{m+1})| = 4$. However, in that case we have $|L(v_{m+2})| \neq 3$ both in (X1) and (X3), which is a contradiction. Therefore, $|L(v_{m+1})| = 3$. Consequently, X and X' were chosen by (X2) or (X4) and we have $|L(v_m)|, |L(v_{m+2})| \geq 4$ and $|L(v_{m-1})| = |L(v_{m+3})| = 3$. Since $\deg(v_m) \geq 4$, Lemmas 18 and 20 imply that w has no neighbor in F other than p_1, v_m and v_{m+1}, and by symmetry, the only possible neighbors of w' in F are p_1, v_{m+1} and v_{m+2}.

Let $S_1 = L(v_{m-1})$ if $m = 3$ and $S_1 = L(v_{m-1}) \setminus L(p_0)$ if $m = 2$. Let $S_2 = L(v_{b+1})$ if $b = s - 2$ and $S_2 = L(v_{b+1}) \setminus L(p_2)$ if $b = s - 1$. By symmetry, we can assume that if $m = 2$, then $b = s - 1$. Let S be the set of colors $c \in L(v_{m+1})$ such that either

(a) $L(v_{m+2}) = S_2 \cup \{c\}$, or

(b) $|L(v_m)| = 4$, $c \notin S_1$ and $S_1 \cup \{c\} \subseteq L(v_m)$.

If $m = 2$, then we have $b = s - 1$, $|S_1| = |S_2| = 2$, there are at most two colors with the property (b) and no colors with the property (a). If $m = 3$, then $|S_1| = 3$ and $|S_2| \leq 3$, there is at most one color with the property (b) and at most one color with the property (a). It follows that $|S| \leq 2$. Let φ be an L-coloring of v_{m-1}, v_{m+1} and v_{m+2} chosen so that $\varphi(v_{m+2}) \notin S_2$, $\varphi(v_{m+1}) \notin S$, $\varphi(v_{m-1}) \in S_1$ and $|L(v_m) \setminus \{\varphi(v_{m-1}), \varphi(v_{m+1})\}| \geq 3$. Note that the choices for $\varphi(v_{m+2})$ and $\varphi(v_{m-1})$ are possible, since $\varphi(v_{m+1})$ does not satisfy (a) and (b), respectively.

Consider $G' = G - \{v_{m-1}, v_{m+1}, v_{m+2}\}$ with the list assignment L_φ. By Lemma 20, v_{m-1} has no common neighbor with v_{m+1} other than v_m, and v_{m-1} has no common neighbor with v_{m+2}, and the only common neighbor of v_{m+1} and v_{m+2} is w'. Therefore, the only vertices with list of size three are v_1 if $m = 3$, v_m, v_{m+3} and w'. Since w' is not adjacent to v_{m+3}, we conclude that $B' = (G', P, N \cup \{y\}, \emptyset, L_\varphi)$ satisfies (M). Furthermore, y is adjacent neither to p_0 nor to p_2 by Lemma 20, hence B' satisfies (O) by Lemma 22. Therefore, B' is a counterexample contradicting the minimality of B.

This shows that there exists no minimal counterexample, and thus Theorem 7 holds. $\qquad\square$

5   Acknowledgments

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