3-coloring triangle-free planar graphs with a precolored 9-cycle

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Abstract

Given a triangle-free planar graph G and a cycle C of length 9 in G, we characterize all situations where a 3-coloring of C does not extend to a proper 3-coloring of G. This extends previous results for the length of C up to 8.

1 Introduction

Let $[n] = \{1, 2, \ldots, n\}$. Graphs in this paper are finite and may have loops or parallel edges. Given a graph G, let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. We will also use $|G|$ for the size of $E(G)$. A *proper k-coloring* of a graph G is a function $\varphi : V(G) \to [k]$ such that $\varphi(u) \neq \varphi(v)$ for each edge $uv \in E(G)$. A graph G is *k-colorable* if there exists a proper k-coloring of G, and the minimum k where G is k-colorable is the *chromatic number* of G.

Garey and Johnson [17] proved that deciding if a graph is k-colorable is NP-complete even when $k = 3$. Moreover, deciding if a graph is 3-colorable is still NP-complete when restricted to planar graphs [11]. Therefore, even though planar graphs are 4-colorable by the celebrated Four Color Theorem [5, 6, 21], finding sufficient conditions for a planar graph to be 3-colorable has been an active area of research. A landmark result in this area is Grötzsch's Theorem [19], which is the following:

Theorem 1 ([19])**.** *Every triangle-free planar graph is* 3*-colorable.*

We direct readers to a nice survey by Borodin [8] for more results and conjectures regarding 3-coloring of planar graphs.

A graph G is *k-critical* if it is not $(k-1)$-colorable but every proper subgraph of G is $(k-1)$-colorable. Critical graphs are important since they are (in a certain sense) the

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minimal obstacles in reducing the chromatic number of a graph. Numerous coloring algorithms are based on detecting critical subgraphs. Despite its importance, there is no known characterization of k-critical graphs when $k \geq 4$. On the other hand, there has been some success regarding 4-critical planar graphs. Extending Theorem 1, the Grünbaum–Aksenov Theorem [1, 7, 20] states that a planar graph with at most three triangles is 3-colorable, and we know that there are infinitely many 4-critical planar graphs with four triangles. Borodin, Dvořák, Kostochka, Lidický, and Yancey [9] were able to characterize all 4-critical planar graphs with four triangles.

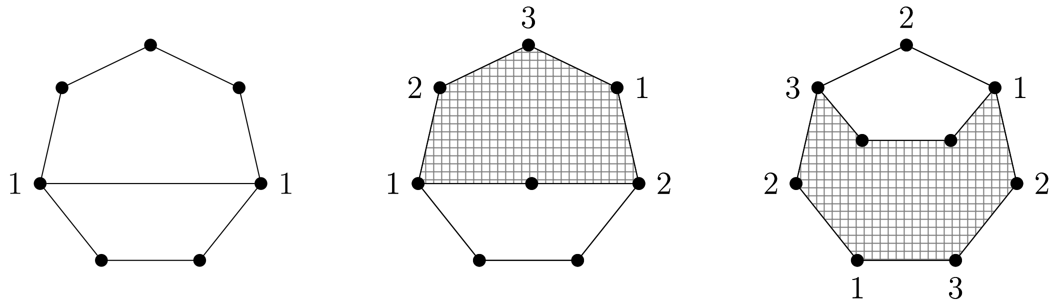
Given a graph G and a proper subgraph C of G, we say G is *C-critical for k-coloring* if for every proper subgraph H of G where $C \subseteq H$, there exists a proper k-coloring of C that extends to a proper k-coloring of H, but does not extend to a proper k-coloring of G. Roughly speaking, a C-critical graph for k-coloring is a minimal obstacle when trying to extend a proper k-coloring of C to a proper k-coloring of the entire graph. Note that $(k+1)$-critical graphs are exactly the C-critical graphs for k-coloring with C being the empty graph.

In the proof of Theorem 1, Grötzsch actually proved that any proper coloring of a 4-cycle or a 5-cycle extends to a proper 3-coloring of a triangle-free planar graph. This implies that there are no triangle-free planar graphs that are C-critical for 3-coloring when C is a face of length 4 or 5. This sparked the interest of characterizing triangle-free planar graphs that are C-critical for 3-coloring when C is a face of longer length. Since we deal with 3-coloring triangle-free planar graphs in this paper, from now on, we will write "C-critical" instead of "C-critical for 3-coloring" for the sake of simplicity.

The investigation was first done on planar graphs with girth 5. Walls [24] and Thomassen [22] independently characterized C-critical planar graphs with girth 5 when C is a face of length at most 11. The case when C is a 12-face was initiated in [22], but a complete characterization was given by Dvořák and Kawarabayashi in [13]. Moreover, a recursive approach to identify all C-critical planar graphs with girth 5 when C is a face of any given length is given in [13]. Dvořák and Lidický [12] implemented the algorithm from [12] and used a computer to generate all C-critical graphs with girth 5 when C is a face of length at most 16. The graphs were generated then used to reveal some structure of 4-critical graphs on surfaces without short contractible cycles. It would be computationally feasible to generate graphs with girth 5 even with C larger than 16.

The situation for planar graphs with girth 4, which are triangle-free planar graphs, is more complicated since the list of C-critical graphs is not finite when C has size at least 6. We already mentioned that there are no C-critical triangle-free planar graphs when C is a face of length 4 or 5. An alternative proof of the case when C is a 5-face was given by Aksenov [1]. Gimbel and Thomassen [18] not only showed that there exists a C-critical triangle-free planar graph when C is a 6-face, but also characterized all of them.

Theorem 2 (Gimbel and Thomassen [18])**.** *Let G be a plane triangle-free graph with outer face bounded by a cycle $C = c_1c_2\dots$ of length at most 6. The graph G is C-critical if and only if C is a 6-cycle, all internal faces of G have length exactly four and G contains no separating 4-cycles. Furthermore, if φ is a 3-coloring of $G[V(C)]$ that does not extend to a*



(a) (b) (c)

Figure 1: Critical graphs with a precolored 7-face.

3-*coloring of* G*, then* $\varphi(c_1) = \varphi(c_4)$*,* $\varphi(c_2) = \varphi(c_5)$ *and* $\varphi(c_3) = \varphi(c_6)$*.*

Aksenov, Borodin, and Glebov [3] independently proved the case when C is a 6-face using the discharging method, and also characterized all C-critical triangle-free planar graphs when C is a 7-face in [4]. The case where C is a 7-face was used in [9].

Theorem 3 (Aksenov, Borodin, and Glebov [3])**.** *Let* G *be a plane triangle-free graph with outer face bounded by a cycle* $C = c_1 \dots c_7$ *of length* 7*. The graph* G *is* C*-critical and* ψ *is a* 3*-coloring of* C *that does not extend to a* 3*-coloring of* G *if and only if* G *contains no separating cycles of length at most five and one of the following propositions is satisfied up to relabelling of vertices (see Figure 1 for an illustration).*

(a) The graph G *consists of* C *and the edge* c_1c_5*, and* $\psi(c_1) = \psi(c_5)$*.*

(b) The graph G *contains a vertex* v *adjacent to* c_1 *and* c_4*, the cycle* $c_1c_2c_3c_4v$ *bounds a 5-face and every face drawn inside the* 6*-cycle* $vc_4c_5c_6c_7c_1$ *has length four; furthermore,* $\psi(c_4) = \psi(c_7)$ *and* $\psi(c_5) = \psi(c_1)$*.*

(c) The graph G *contains a path* c_1uvc_3 *with* $u, v \notin V(C)$*, the cycle* $c_1c_2c_3vu$ *bounds a 5-face and every face drawn inside the* 8*-cycle* $uvc_3c_4c_5c_6c_7c_1$ *has length four; furthermore,* $\psi(c_3) = \psi(c_6)$*,* $\psi(c_2) = \psi(c_4) = \psi(c_7)$ *and* $\psi(c_1) = \psi(c_5)$*.*

Dvořák and Lidický [16] used a correspondence of nowhere-zero flows and colorings to give simpler proofs of the case when C is either a 6-face or a 7-face, and also characterized C-critical triangle-free planar graphs when C is an 8-face. For a plane graph G, let $S(G)$ denote the set of multisets of possible lengths of internal faces of G with length at least 5.

Theorem 4 (Dvořák and Lidický [16])**.** *Let* G *be a connected plane triangle-free graph with outer face bounded by a cycle* C *of length* 8*. The graph* G *is* C*-critical if and only if* G *contains no separating cycles of length at most five, the interior of every non-facial 6-cycle contains only faces of length four and one of the following propositions is satisfied (see Figure 2 for an illustration).*

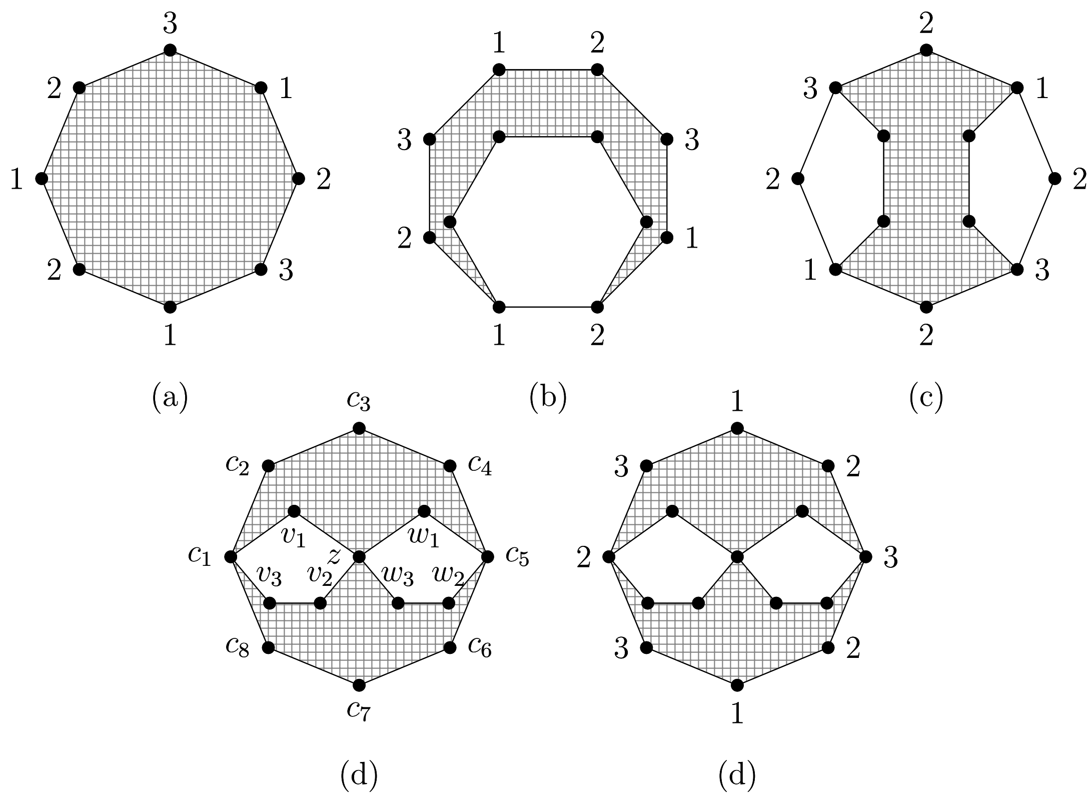


Figure 2: Graph described by Theorem 4 and examples of 3-colorings of C that do not extend.

(a) $S(G) = \emptyset$, *or*

(b) $S(G) = \{6\}$ *and the* 6*-face of* G *intersects* C *in a path of length at least one, or*

(c) $S(G) = \{5, 5\}$ *and each of the* 5*-faces of* G *intersects* C *in a path of length at least two, or*

(d) $S(G) = \{5, 5\}$ *and the vertices of* C *and the* 5*-faces* f_1 *and* f_2 *of* G *can be labelled in clockwise order along their boundaries so that* $C = c_1c_2 \dots c_8$, $f_1 = c_1v_1zv_2v_3$ *and* $f_2 = zw_1c_5w_2w_3$ *(where* w_1 *can be equal to* v_1, v_1 *can be equal to* c_2, *etc.)*

Theorem 4 has for example the following corollary. The corollary was not explicitly stated in [16] so we state it here.

Corollary 1 (Dvořák and Lidický [16])**.** *Let* G *be a plane triangle-free graph and* v *be a vertex of degree 4 in* G*. Then there exists a proper* 3*-coloring of* G *where all neighbors of* v *are colored the same.*

The corollary can be proven by splitting v into 4 vertices of degree two that are in one 8-face F and precoloring F by two colors; see Fig 3.

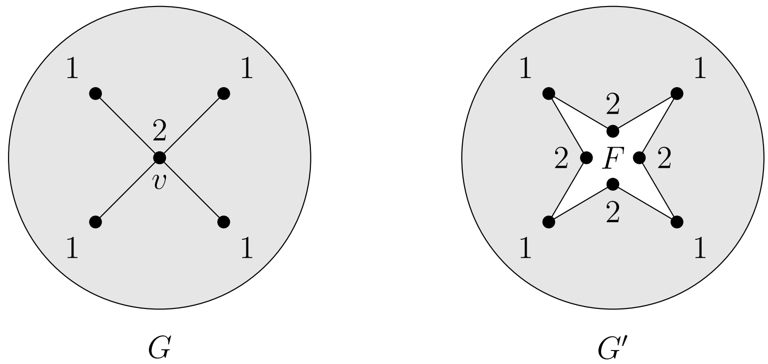


Figure 3: The coloring of a graph G where all neighbor of a 4-vertex v have the same color can be obtained by extending a precoloring of an 8-face F in G', where G' is obtained from G by splitting v into 4 vertices of degree two.

In this paper, we push the project further and characterize all C-critical triangle-free planar graphs when C is a face of length 9.

Theorem 5. *Let G be a connected plane triangle-free graph with outer face bounded by a cycle C of length 9. The graph G is C-critical for 3-coloring if and only if G contains no separating cycles of length at most five, the interior of every non-facial 6-cycle contains only faces of length four and one of the following propositions is satisfied (see Figure 4 for an illustration):*

(a) $S(G) = \{5\}$ *and the* 5*-face of* G *intersects* C *in a path of length at least two;*

(b) $S(G) = \{7\}$*;*

(c) $S(G) = \{5, 6\}$ *and the* 5*-face,* 6*-face, of* G *intersects* C *in a path of length at least two, and four, respectively;*

(d) $S(G) = \{5, 6\}$ *and* G *is depicted as* (d1) *or* (d2) *in Fig. 4;*

(e) $S(G) = \{5, 5, 5\}$ *and* G *is depicted as* (Bij) *in Fig. 4 for all* i, j.

Part of the proof of Theorem 5 is enumerating all integer solutions to several small set of linear constraints. It would be possible to do it by hand but we have decided to use computer programs to enumerate the solutions. Both computer programs and enumerations of the solutions are available online at `http://orion.math.iastate.edu/lidicky/pub/9cyc/`.

2 Preliminaries

Our proof of Theorem 5 uses the same method as Dvořák and Lidický [16]. The main idea is to use the correspondence between coloring of a plane graph G and flows in the dual of G.

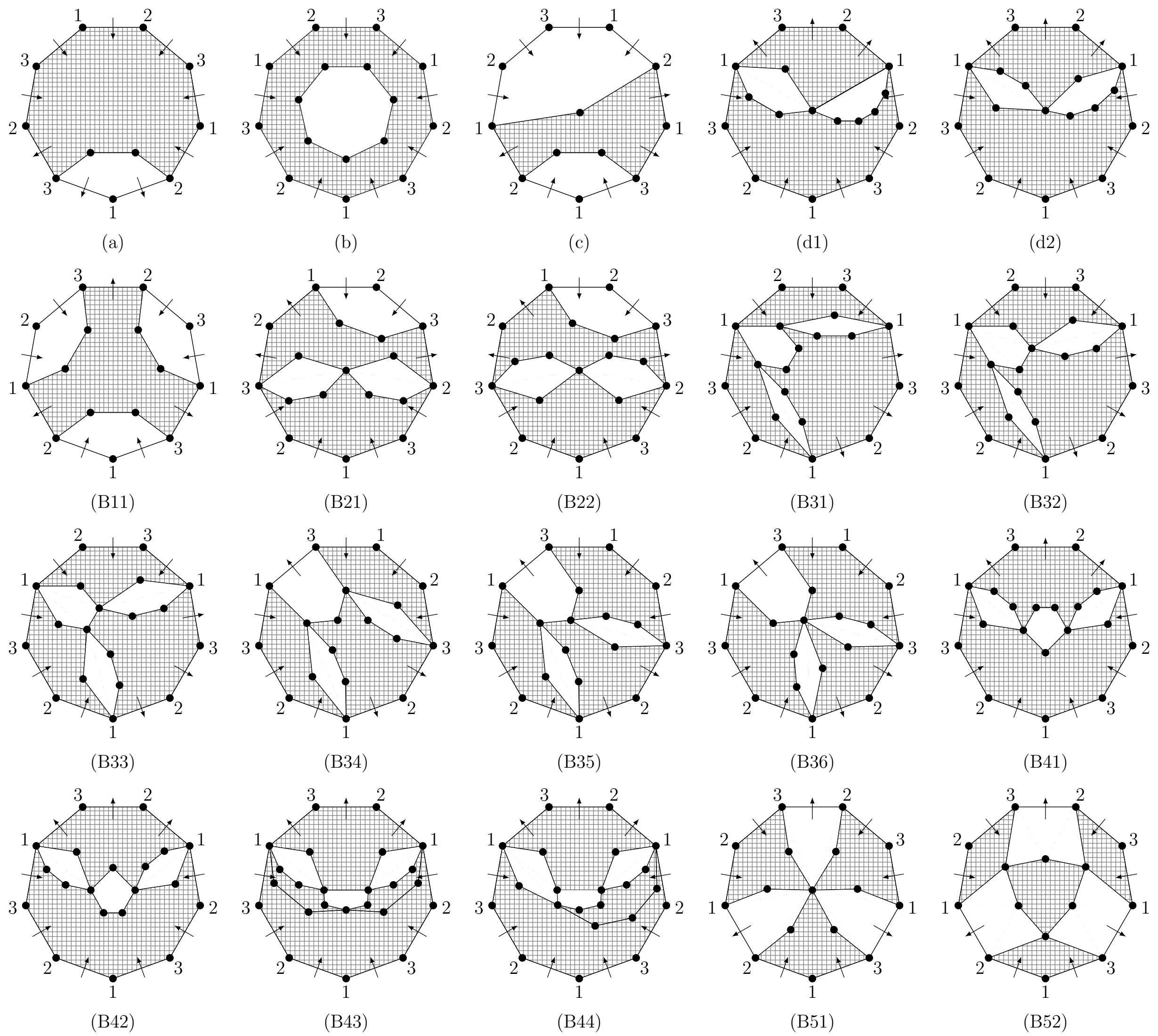


Figure 4: All C-critical triangle-free plane graphs where C is an outer 9-face. Note that each figure actually represents infinitely many graphs, including ones that can be obtained by identifying some of the depicted vertices. The arrows correspond to source edges and sink edges that are defined in Preliminaries.

In this paper, we give only a brief description of the correspondence and the lemmas useful in our case. A more detailed and general description can be found in [16].

Let $G^\star$ denote the dual of a plane graph G. Let φ be a proper 3-coloring of the vertices of G by colors $\{1, 2, 3\}$. For every edge uv of G, we orient the corresponding edge e in $G^\star$ in the following way. Let e have endpoints f, h in $G^\star$, where f,v,h is in the clockwise order from vertex u in the drawing of G. The edge e will be oriented from f to h if $(\varphi(u), \varphi(v)) \in \{(1, 2), (2, 3), (3, 1)\}$, and from h to f otherwise. See Figure 5(a) for an example of the orientation.

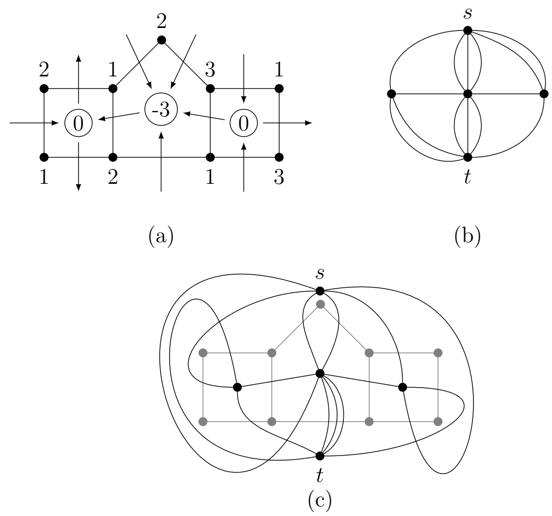


Figure 5: (a) a 3-coloring of a graph G and the corresponding orientation of the edges in $G^\star$; (b) $G^{q,\psi}$ corresponding to (a); (c) G and $G^{q,\psi}$ drawn in one figure.

Since φ is a proper coloring, every edge of $G^\star$ has an orientation. Tutte [23] showed that this orientation of $G^\star$ defines a nowhere-zero $\mathbb{Z}_3$-flow, which means that the in-degree and the out-degree of every vertex in $G^\star$ differ by a multiple of three. Conversely, every nowhere-zero $\mathbb{Z}_3$-flow in $G^\star$ defines a proper 3-coloring of G up to the rotation of colors.

Let h be the vertex in $G^\star$ corresponding to the outer face of G. Edges oriented away from h are called *source edges* and the edges oriented towards h are called *sink edges*. The orientations of edges incident to h depend only on the coloring of C, where C is the cycle bounding the outer face of G.

For a vertex f of $G^\star$, let $\delta(f)$ denote the difference of the out-degree and in-degree of f. Possible values of $\delta(f)$ depend on the size of the face corresponding to f, denoted by $|f|$. Clearly $|\delta(f)| \leq |f|$ and $\delta(f)$ has the same parity as $|f|$. Hence if $|f| = 4$, then $\delta(f) = 0$. Similarly, if $|f| \in \{5, 7\}$, then $\delta(f) \in \{-3, 3\}$ and if $|f| = 6$ then $\delta(f) \in \{-6, 0, 6\}$.

Next we convert the problem of extending a proper 3-coloring of C to the existence of a $\mathbb{Z}$-flow in an auxiliary graph obtained from $G^\star$. We call a function q assigning an integer to every internal face f of G a *layout* if $q(f) \leq |f|$, $q(f)$ is divisible by 3, and $q(f)$ has the same parity as $|f|$. Notice that $q(f)$ satisfies the same conditions as $\delta(f)$. Therefore it is sufficient to specify the q-values for faces of size at least 5, since $q(f) = 0$ if f is a 4-face.

Let ψ be a proper 3-coloring of C. The coloring ψ gives an orientation of the edges corresponding to the edges of C in $G^\star$. Denote by n^s the number of source edges and by n^t the number of sink edges. A layout q is *ψ-balanced* if $n^s + m = n^t$, where m is the sum of the q-values over all internal faces of G. A graph $G^{q,\psi}$ is obtained from $G^\star$ by removing the vertex h corresponding to the outer face of G and by adding two new vertices s and t. For every edge hf in $G^\star$ we add one edge sf if hf is a source edge and we add one edge tf if it is a sink edge. Moreover, for every internal face f with $q(f) > 0$, we add $q(f)$ parallel sf edges and for every internal face f with $q(f) < 0$, we add $-q(f)$ parallel tf edges. See Fig. 5 for an illustration. Note that q is ψ-balanced if and only if s and t have the same degree.

For a ψ-balanced layout q of G, let $c(q, \psi)$ denote the degree of the source vertex s (and also the sink vertex t) of $G^{q,\psi}$. For an edge cut K in $G^{q,\psi}$ separating s from t, the component of $G^{q,\psi} \setminus K$ containing s, or t, is called a *source component*, or a *sink component*, respectively.

For a set of faces F, let $\ell(F)$ denote the smallest length of a cycle in a critical graph that may contain all faces of F. Denote a face of size i by f_i. It is known [15] that $\ell(\{f_i\}) = i$ and $\ell(\{f_5, f_6\}) = 9$.

The next lemma describes interiors of cycles in critical graphs.

Lemma 6 ([14])**.** *Let G be a plane graph with outer face K. Let C be a cycle in G that does not bound a face, and let H be the subgraph of G drawn in the closed disk bounded by C. If G is K-critical for k-coloring, then H is C-critical for k-coloring.*

Lemma 7 is the key lemma that gives the correspondence between 3-colorings of C and flows. It implies that if a 3-coloring of C extends to the entire graph, then there is a $\mathbb{Z}$-flow from s to t of value $c(q, \psi)$.

Lemma 7 ([16])**.** *Let G be a connected plane triangle-free graph with the outer face C bounded by a cycle and let ψ be a 3-coloring of C. The coloring ψ extends to a 3-coloring of G if and only if there exists a ψ-balanced layout q such that the terminals of $G^{q,\psi}$ are not separated by an edge cut smaller than $c(q, \psi)$.*

The cuts showing that a 3-coloring of C does not extend are described by the following lemma.

Lemma 8 ([16])**.** *Let G be a connected plane triangle-free graph with the outer face C bounded by a cycle and let ψ be a 3-coloring of C that does not extend to a 3-coloring of G. If q is a ψ-balanced layout in G, then there exists a subgraph $K_0 \subseteq G$ such that either*

i) K_0 is a path with both ends in C and no internal vertex in C, and if P is a path in C joining the end vertices of K_0, n_s is the number of source edges of P, n_t is the number of the sink edges of P and m is the sum of the values of q over all faces of G drawn in the open disk bounded by the cycle $P + K_0$, then $|n_s + m - n_t| > |K_0|$. In particular, $|P| + |m| > |K_0|$.
Or,

ii) K_0 is a cycle with at most one vertex in C, and if m is the sum of the values of q over all faces of G drawn in the open disk bounded by K_0, then $|m| > |K_0|$.

3 Proof of Theorem 5

Let $\mathcal{S}_k$ be the set of possible multisets of sizes of faces of length at least five in a graph of girth at least 4 where the length of the precolored face is k. The result of Dvořák, Kráľ, and Thomas [15] implies among others that $\mathcal{S}_6 = \{\emptyset\}$, $\mathcal{S}_7 = \{\{5\}\}$, $\mathcal{S}_8 = \{\emptyset, \{6\}, \{5, 5\}\}$, and $\mathcal{S}_9 = \{\{7\}, \{5\}, \{6, 5\}, \{5, 5, 5\}\}$.

From now on, G is always a C-critical triangle-free plane graph and C is always the outer face of length 9. By the previous paragraph, we have four cases to consider when C has length 9. The case of one 7-face was already resolved by Dvořák and Lidický [16], and it is described in Theorem 5(b). We restate the result from [16] in the next section as Theorem 10. We resolve the remaining three cases in Lemmas 11, 17, and 18 in the following three subsections. In order to simplify the cases, we first solve the case when C has a chord.

If G is C-critical and C has a chord, then Lemma 6 implies that G can be obtained by identifying two edges of the outer faces of two different smaller critical graphs. Lemma 9 shows that the converse is also true.

Lemma 9. *Let G_i be a C_i-critical graph where $|C_i| = k_i$ for $i \in \{1, 2\}$. Let G be obtained from G_1 and G_2 identifying $e_1 \in E(C_1)$ and $e_2 \in E(C_2)$. Then G is C-critical, where $|C| = k_1 + k_2 - 2$.*

Proof. Let $e \in E(G) \setminus E(C)$.

Suppose first that $e \in E(G_i) - e_i$ for some $i \in \{1, 2\}$. Since G_i is C_i-critical, there exists a 3-coloring φ of C_i that extends to a proper 3-coloring of $G_i - e$ but does not extend to a proper 3-coloring of G_i. Since G_{3-i} is triangle-free, there exists a proper 3-coloring ϱ of G_{3-i} by Grötzsch's Theorem [19]. By permuting colors we can assume that φ and ϱ agree on e_i and e_{3-i}. This gives a proper 3-coloring of C showing that G is C-critical with respect to e.

The other case is that e is the result of the identification of e_1 and e_2. Let u, v be the vertices of e. Since $G - e$ is a triangle-free planar graph, there exists a proper 3-coloring φ of $G - e$ such that $\varphi(u) = \varphi(v)$. It is a result of Aksenov et al. [2] simplified by Borodin et al. [10]. Let ϱ be a restriction of φ to C. Clearly, ϱ can be extended to a proper 3-coloring of $G - e$ but not to a proper 3-coloring of G. □

Therefore, we can enumerate C-critical graphs G where C has a chord and has length 9 by identifying edges from two smaller critical graphs with outer faces of length either 4 and 7 or 5 and 6. Note that critical graph with outer face of length 4 or 5 is a 4-cycle or 5-cycle respectively. The resulting graphs are depicted in Fig. 4 (a) and (b), where some of the vertices must be identified.

In the following we assume that C has no chords. In the rest of the paper, ψ will always be a 3-coloring of C. Also, for a subset Z of the edges of C, we will use n^s_Z and n^t_Z to denote the number of source edges and sink edges of Z, respectively.

3.1 One 7-face

The case of one 7-face is solved by a more general result from [16]. The result works for a graphs with outer face of length k and one internal face of length $k-2$. Let $r(k)=0$ if $k \equiv 0 \pmod 3$, $r(k)=2$ if $k \equiv 1 \pmod 3$ and $r(k)=1$ if $k \equiv 2 \pmod 3$.

Theorem 10 ([16]). *Let G be a connected triangle-free plane graph with outer face bounded by a cycle C of length $k \geq 7$. Suppose that f is an internal face of G of length $k-2$ and that all other internal faces of G have length 4. The graph G is C-critical if and only if*

(a) $f \cap C$ is a path of length at least $r(k)$ (possibly empty if $r(k)=0$),

(b) G contains no separating 4-cycles, and

(c) for every $(\leq k-1)$-cycle $K \neq f$ in G, the interior of K does not contain f.

Furthermore, in a graph satisfying these conditions, a precoloring ψ of C extends to a 3-coloring of G if and only if $E(C) \setminus E(f)$ contains both a source edge and a sink edge with respect to ψ.

In our case, we apply Theorem 10 with $k=9$. Since $r(9)=0$, the 7-face does not have to share any edges with the outer face. The description is in Theorem 5(b) and depicted in Fig. 4(b).

3.2 One 5-face and one 6-face

Lemma 11. *If G contains one 5-face f_5 and one 6-face f_6, and all other faces are 4-faces, then G is described by Theorem 5(c),(d) and depicted in Fig. 4(c),(d1), and (d2).*

Proof. Let G be a C-critical graph containing one 5-face f_5 and one 6-face f_6.

Let $e \in E(G) \setminus E(C)$. We want to find a 3-coloring ψ of C that does not extend to a proper 3-coloring of G but extends to a proper 3-coloring of $G-e$. Note that $G-e$ has either one 5-face and one 8-face, or one 6-face and one 7-face, or one 9-face, or two 6-faces and one 5-face. We know that the smallest k such that $\mathcal{S}_k$ contains any of $\{5,8\}$, $\{6,7\}$, $\{9\}$, or $\{5,6,6\}$ is at least 11. Hence every precoloring of C extends to $G-e$. In particular, ψ extends to $G-e$. Therefore, we only need to characterize ψ that does not extend to G.

Let ψ be a proper 3-coloring of C that does not extend to a proper 3-coloring of G. By symmetry, we assume that C has more source edges than sink edges. Hence C has either 9 or 6 source edges. Let q be a ψ-balanced layout of G. By Lemma 7, there exists an edge-cut K in $G^{q,\psi}$ separating s from t such that $|K|$ is smaller than $c(q,\psi)$. Let $K_0 \subset G$ be obtained by Lemma 8 and let $k_0 = |K_0|$.

First suppose that K_0 is a cycle. Let m denote the sum of the q-values of the faces in the interior of K_0. By Lemma 8, $|m| > k_0$. If both f_5, f_6 are in the interior of K_0, then $|m| \leq 9$, contradicting the fact that $|m| > k_0$ since $k_0 \geq \ell(\{f_5, f_6\}) = 9$. If f_5 is in the interior of K_0, but f_6 is not, then $|m| = 3$, while $\ell(\{f_5\}) = 5$, a contradiction again. Similarly, we obtain a

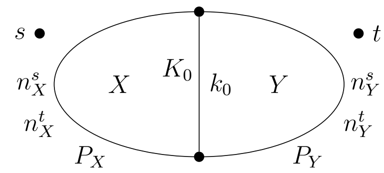


Figure 6: Structure of a cut in G.

contradiction when f_6 is in the interior of K_0 but f_5 is not, since $\ell(\{f_6\}) = 6$ and $|m| \leq 6$. Therefore K_0 is always a path joining two distinct vertices of C.

The graph G bounded by C is divided by K_0 into two closed disks X and Y intersecting at K_0, where faces in X correspond to the vertices in the component containing s in $G^{q,\psi} - K$. For $Z \in \{X, Y\}$, denote by P_Z the subpath of C such that Z is bounded by $P_Z + K_0$. Recall that n_Z^s and n_Z^t denote the number of source edges and sink edges in P_Z, respectively. The described structure is shown in Fig. 6.

Claim 12. *There are 6 source edges in C.*

Proof. Suppose for a contradiction that C contains 9 source edges. Hence there is just one ψ-balanced layout q with $q(f_5) = -3$, $q(f_6) = -6$, and $c(q,\psi) = 9$. Note that $n_X^s + n_Y^s = 9$ and $n_X^t + n_Y^t = 0$. If both f_5, f_6 belong to X then $|K| = k_0 + n_Y^s + 9 < 9$, a contradiction. If both f_5, f_6 belong to Y then $|K| = k_0 + n_Y^s < 9$, while the length of the boundary cycle of Y is $k_0 + n_Y^s \geq \ell(\{f_5, f_6\}) = 9$, which is a contradiction again. Now suppose that exactly one of f_5, f_6 belongs to X and let f_X denote such a face and f_Y the other one. Then $|K| = k_0 + n_Y^s + |q(f_X)| < 9$ and $k_0 + n_Y^s \geq |f_Y|$. If $f_X = f_5$ then $k_0 + n_Y^s + 3 < 9$ and $k_0 + n_Y^s \geq 6$, which is a contradiction. If $f_X = f_6$ then $k_0 + n_Y^s + 6 < 9$ and $k_0 + n_Y^s \geq 5$, a contradiction. $\square$

Claim 13. *If q is a ψ-balanced layout with $q(f_5) = -3$ and $q(f_6) = 0$, then f_5 belongs to Y and f_6 belongs to X.*

Proof. Assume that $q(f_5) = -3$ and $q(f_6) = 0$. Hence the six source edges are the only edges incident to s, thus $c(q,\psi) = 6$. Note that $n_X^s + n_Y^s = 6$ and $n_X^t + n_Y^t = 3$. First suppose that both f_5, f_6 belong to X. Then $n_X^s + n_X^t + k_0 \geq \ell(\{f_5, f_6\}) = 9$, and the size of the cut K is $3 + k_0 + n_X^t + n_Y^s < c(q,\psi) = 6$. By subtracting the two previous inequalities we get $n_X^s - n_Y^s > 6$, contradicting the fact that $n_X^s + n_Y^s = 6$. Now suppose that both f_5, f_6 belong to Y. Then $n_Y^s + n_Y^t + k_0 \geq \ell(\{f_5, f_6\}) = 9$ and $|K| = k_0 + n_X^t + n_Y^s < 6$. By subtracting them we get $n_Y^t - n_X^t > 3$, a contradiction with $n_Y^t + n_X^t = 3$. Finally we suppose that f_5 belongs to X and f_6 belongs to Y. Then $n_Y^s + n_Y^t + k_0 \geq \ell(\{f_6\}) = 6$ and $|K| = 3 + k_0 + n_X^t + n_Y^s < 6$. But then $n_Y^t - n_X^t > 3$, a contradiction again. Therefore f_5 is in Y and f_6 is in X. $\square$

Claim 14. *If q is a ψ-balanced layout with $q(f_5) = 3$ and $q(f_6) = -6$, then f_5 belongs to X and f_6 belongs to Y.*

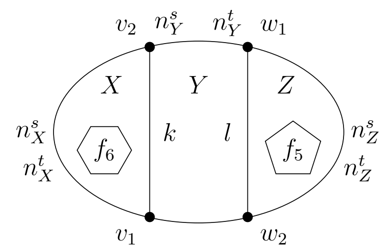


Figure 7: A structure for two non-crossing cuts.

Proof. Assume that $q(f_5) = 3$ and $q(f_6) = -6$. Since there are six source edges on C and three edges from s to f_5 in $G^{q,\psi}$, $c(q,\psi) = 9$. Note that $n_X^s + n_Y^s = 6$ and $n_X^t + n_Y^t = 3$. First suppose that both f_5 and f_6 belong to X. Then $k_0 + n_X^s + n_X^t \geq \ell(\{f_5, f_6\}) = 9$ and $|K| = 6 + k_0 + n_Y^s + n_X^t < c(q,\psi) = 9$. But then we obtain $n_X^s - n_Y^s > 6$, contradicting the fact that $n_X^s + n_Y^s = 6$. Now suppose that both f_5 and f_6 belong to Y. Then $k_0 + n_Y^s + n_Y^t \geq \ell(\{f_5, f_6\}) = 9$ and the size of K is $3 + k_0 + n_Y^s + n_X^t < 9$. But then we get $n_Y^t - n_X^t > 3$, contradicting $n_X^t + n_Y^t = 3$. Finally we suppose that f_6 belongs to X and f_5 belongs to Y. Then $|K| = 9 + k_0 + n_Y^s + n_X^t < 9$, a contradiction. □

Since C has 6 source edges, we have two different ψ-balanced layouts. Let q_1 and q_2 be the layouts where $q_1(f_5) = -3$, $q_1(f_6) = 0$, and $q_2(f_5) = 3$, $q_2(f_6) = -6$, respectively. Let K and L be the subgraphs of G obtained by Lemma 8 applied to q_1 and q_2, respectively, and let $k = |K|$ and $l = |L|$. Note that we already showed that each of K and L is a path joining pairs of distinct vertices of C. Denote these vertices by v_1, v_2 for K and by w_1, w_2 for L. The prescribed structure is depicted in Fig. 7 and Fig. 8.

If we can choose the labels of the endpoints of K and L so that the clockwise order along C is v_1, v_2, w_1, w_2, then we call K and L *non-crossing*, and we call K and L *crossing* otherwise. Notice that K and L are always non-crossing if they have a vertex of C in common.

We treat the cases of K and L being crossing and non-crossing separately.

Claim 15. *If K and L are non-crossing, then G is depicted in Fig. 4(c).*

Proof. Assume that K and L are non-crossing. See Figure 7. Note that K, L are not necessarily disjoint. The cuts K and L partition G into three parts. Denote by X the region of G containing f_6, by Z the region of G containing f_5, and by Y the rest of G. For an edge cut K' of $G^{q_1,\psi}$ corresponding to K, f_6 belongs to the source subdisk of G while f_5 belongs to the sink subdisk of G by Claim 13. Analogously, for an edge cut L' of $G^{q_2,\psi}$ corresponding to L, f_5 belongs to the source subdisk of G while f_6 belongs to the sink subdisk of G by Claim 14. For the edge cut K', $|K'| = k + n_X^t + n_Y^s + n_Z^s < c(q_1,\psi) = 6$. For the edge cut L', $|L'| = l + n_X^s + n_Y^s + n_Z^t < c(q_2,\psi) = 9$. By the assumptions that C has no chord, $k \geq 2$ and $l \geq 2$. Since X contains f_6, $k + n_X^s + n_X^t \geq \ell(\{f_6\}) = 6$ and even, and since Z contains f_5, $l + n_Z^s + n_Z^t \geq \ell(\{f_5\}) = 5$ and odd. Clearly $n_X^s + n_Y^s + n_Z^s = 6$ and $n_X^t + n_Y^t + n_Z^t = 3$.

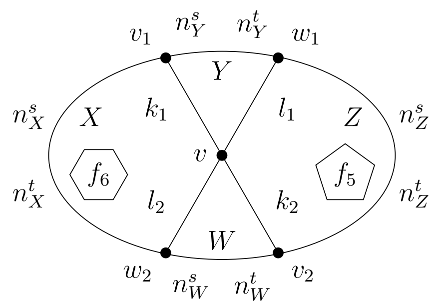


Figure 8: A structure for two crossed cuts.

Here is the summary of the constraints:

$$
\begin{aligned}
|K'| = k + n_X^t + n_Y^s + n_Z^s &< c(q_1, \psi) = 6 \\
|L'| = l + n_X^s + n_Y^s + n_Z^t &< c(q_2, \psi) = 9 \\
k + n_X^s + n_X^t &\geq \ell(\{f_6\}) = 6 \text{ and even} \\
l + n_Z^s + n_Z^t &\geq \ell(\{f_5\}) = 5 \text{ and odd} \\
n_X^s + n_Y^s + n_Z^s &= 6 \\
n_X^t + n_Y^t + n_Z^t &= 3 \\
\min\{k, l\} &\geq 2
\end{aligned}
$$

All integer solutions to these constraints are in the following table:

n_X^s	n_X^t	n_Y^s	n_Y^t	n_Z^s	n_Z^t	k	l
4	0	0	2	2	1	2	2
4	0	0	3	2	0	2	3

From these two solutions we obtain the graphs depicted in Fig. 4(c). □

Claim 16. *If K and L are crossing, then G is depicted in Fig. 4(d1) or (d2).*

Proof. Assume that K and L cross, hence G is divided by K and L into four regions. Let X be the region of G containing f_6, Z be the region containing f_5, and let W,Y be the two remaining regions. Since K and L cross, they have a common internal vertex v. Note that $K \cap L$ is a path and v can be any vertex on the path. Denote by k_1 the length of the subpath of K between X and Y up to v, and denote by k_2 the length of the rest of K. Denote by l_1 the length of the subpath of L between Y and Z up to v, and denote by l_2 the length of the rest of L. The prescribed structure is depicted in Fig. 8.

Note that $\min\{k_1, k_2, l_1, l_2\} \geq 1$ since v is an internal vertex. For an edge cut K' of $G^{q_1,\psi}$ corresponding to K, f_6 belongs to the source component while f_5 belongs to the sink component by Claim 13. Analogously, for an edge cut L' of $G^{q_2,\psi}$ corresponding to L, f_5 belongs to the source component while f_6 belongs to the sink component by Claim 14.

We obtain the following set of constraints that must be satisfied in this subcase.

$$|K'| = k_1 + k_2 + n_X^t + n_Y^s + n_Z^s + n_W^t < c(q_1, \psi) = 6 \quad (1)$$
$$|L'| = l_1 + l_2 + n_X^s + n_Y^s + n_Z^t + n_W^t < c(q_2, \psi) = 9 \quad (2)$$
$$k_1 + l_2 + n_X^s + n_X^t \geq \ell(\{f_6\}) = 6 \text{ and even} \quad (3)$$
$$l_1 + k_2 + n_Z^s + n_Z^t \geq \ell(\{f_5\}) = 5 \text{ and odd} \quad (4)$$
$$l_2 + k_2 + n_X^s + n_X^t + n_Y^s + n_Y^t + n_Z^s + n_Z^t \geq \ell(\{f_5, f_6\}) = 9 \text{ and odd} \quad (5)$$
$$\min\{k_1, l_1\} + n_Y^s + n_Y^t > \max\{k_1, l_1\} \quad (6)$$
$$\min\{k_2, l_2\} + n_W^s + n_W^t > \max\{k_2, l_2\} \quad (7)$$
$$n_X^s + n_Y^s + n_Z^s = 6 \quad (8)$$
$$n_X^t + n_Y^t + n_Z^t = 3 \quad (9)$$

Inequalities (1) and (2) come from the size of the cut being smaller than $c(q_1, \psi)$ and $c(q_2, \psi)$, respectively. Inequalities (3)–(5) come from the fact that interior of cycles are also critical graphs. Finally, if any of the inequalities (6)–(7) are violated then the cuts K and L can be taken as non-crossing.

We solve the system of constraints by computer programs. The programs are available at `http://orion.math.iastate.edu/lidicky/pub/9cyc/`.

From these solutions we get graphs depicted in Fig. 4(d1) and (d2). □

This finishes the proof of Lemma 11. □

3.3 One 5-face

Lemma 17. *If G contains one 5-face f_5 and all other faces are 4-faces, then G is described by Theorem 5(a) and depicted in Fig. 4(a).*

Proof. Let G be a C-critical graph containing one 5-face f_5. Let $e \in E(G) \setminus E(C)$. We want to find a 3-coloring ψ of C that does not extend to a proper 3-coloring of G but extends to a proper 3-coloring of $G - e$. Note that if $e \notin E(f_5)$ then $G - e$ has a 5-face, and if $e \in E(f_5)$ then 6-face and $G - e$ has a 7-face. This gives us two cases to consider.

Case 1: $G - e$ contains a 5-face and a 6-face.

Let ψ be a 3-coloring of C containing 9 source edges (i.e. the colors around C are $1, 2, 3, 1, 2, 3, 1, 2, 3$). Then ψ extends to a 3-coloring of $G - e$ by Claim 12. However, ψ does not extend to a 3-coloring of G since it is not possible to create a ψ-balanced layout for G.

Case 2: $G - e$ contains a 7-face f_7.

By Theorem 10, if ψ is a 3-coloring of C containing 9 source edges, then ψ does not extend to a proper 3-coloring of $G-e$ and if ψ is a 3-coloring of C containing 6 source edges and 3 sink edges, then ψ extends to a proper 3-coloring of $G-e$ if $E(C)\setminus E(f_7)$ contains both a sink and a source edge with respect to ψ. Since ψ must extend to $G-e$, we know that ψ contains 6 source edges and 3 sink edges. Now we need to construct such a proper 3-coloring ψ that does not extend to a proper 3-coloring of G and check the condition for f_7 and ψ.

Let q be a ψ-balanced layout of G. The only possibility is $q(f_5)=-3$ and $c(q,\psi)=6$. By Lemma 7, there exists an edge-cut K in $G^{q,\psi}$ separating s from t such that $|K|$ is smaller than 6. By a proof of Lemma 8 (for details see [16]), there is a subgraph K_0 of G containing edges of G, which are crossed by edges of K that are not adjacent to any of the terminals in $G^{q,\psi}$. Denote $|K_0|$ by k_0. First suppose that K_0 is a cycle. Let m denote the sum of the q-values of the faces in the interior of K_0. By Lemma 8, $|m|>k_0$. If f_5 is in the interior of K_0, then $|m|=3$, while $\ell(\{f_5\})=5$, a contradiction. Therefore K_0 is a path joining two distinct vertices of C.

From a ψ-balanced layout q we obtain that $n^s_X+n^s_Y=6$ and $n^t_X+n^t_Y=3$. This structure is the same as in the proof of Lemma 11 (see Fig. 6). The following two possibilities can occur:

Let f_5 belong to X. For the edge cut K, $|K|=k_0+n^s_Y+n^t_X+3<c(q,\psi)=6$. Since C has no chords, $k_0=2$, $n^s_Y=0$, $n^t_X=0$, $n^s_X=6$, and $n^t_Y=3$. The cycle bounding X has length 8. However, it contains only one face of odd size, which is a contradiction.

Let f_5 belong to Y. For the edge-cut K, $|K|=k_0+n^s_Y+n^t_X<c(q,\psi)=6$. For X we have $k_0+n^s_X+n^t_X\geq\ell(\{f_4\})=4$ and even. Since Y contains f_5, $k_0+n^s_X+n^t_X\geq\ell(\{f_5\})=5$ and odd. We solve the system of these constraints by a computer program. The solutions are in Table 1.

n^s_X	n^t_X	n^s_Y	n^t_Y	k_0
6	1	0	2	3
6	0	0	3	2
5	1	1	2	2
6	0	0	3	4
5	0	1	3	3
4	0	2	3	2

Table 1: Solutions in Lemma 17, Case 2.

From the first three solutions we obtain that Y is a 5-face f_5 sharing at least two sink edges with C. The other three solutions give that Y is bounded by a 7-cycle sharing at least three sink edges with C. Now we need to verify that we could find ψ satisfying the constraints in Table 1 that extends to $G-e$. In particular, by Theorem 10, we need to find a coloring where f_7 in $G-e$ does not contain all three sink edges of C.

We distinguish two cases, based on the length of the cycle bounding Y:

5-cycle: This corresponds to the first three solution in Table 1. In the second solution, f_5 contains all three sink edges of C, hence f_7 in $G-e$ also contains all three sink edges and hence the precoloring does not extend and we do not have a C-critical graph. In the first and second solution, it is possible to place the third sink edge on C such that it is not part of f_7, hence the precoloring extends to $G-e$ by Theorem 10. Resulting configuration is depicted in Fig. 4(a) for the first line and one additional edge could be shared for the third solution in Table 1. Notice that this case will work for all $e \in E(f_5) \setminus E(C)$ if it works for at least one.

7-cycle: This corresponds to the last three solutions in Table 1. Notice that we have this case no matter which edge of f_5 is chosen as e. In order to extend ψ to $G-e$, an edge from K must be removed. If no edge from K is removed, then there is still no coloring of $G-e$ since the same cut will prevent it. Hence all edges of f_5 are either in K or in C. But it is not possible to have a 5-face in a 7-cycle that contains only edges of the 7-cycle. Therefore we get no C-critical graphs.

This finishes the proof of Lemma 17. □

3.4 Three 5-faces

Lemma 18. *If G contains three 5-faces and all other faces are 4-faces, then G is described by Theorem 5(e) and depicted in Fig. 4(Bij) for all i and j.*

Proof. Let G be a C-critical graph containing three 5-faces. Let $e \in E(G) \setminus E(C)$. We want to find a 3-coloring ψ of C that does not extend to a proper 3-coloring of G, but extends to a 3-coloring of $G-e$. Note that either $G-e$ has three 5-faces and a 6-face, or $G-e$ has two 5-faces and a 7-face, or $G-e$ has one 5-faces and a 8-face. Since we know that $\ell(\{5,5,5,6\}) \geq 11$, $\ell(\{5,5,7\}) \geq 11$, and $\ell(\{5,8\}) \geq 11$, every proper 3-coloring of C extends to $G-e$, and therefore, ψ extends to $G-e$.

Without loss of generality, assume C has more source edges than sink edges in the coloring ψ. Either C contains 9 source edges and no sink edges or C contains 6 source edges and 3 sink edges.

Given $i \in \{0,1,2,3\}$, let $\ell_5(i)$ denote the smallest length of a cycle in a critical graph that may contain i of the 5-faces, namely, $\ell_5(0)=4, \ell_5(1)=5, \ell_5(2)=8$, and $\ell_5(3)=9$.

Claim 19. *There are 6 source edges in C.*

Proof. Suppose for contradiction that there are 9 source edges. Hence there is just one ψ-balanced layout q assigning -3 to every 5-face. Let K_0 and m be obtained from Lemma 8, which says $|m| > |K_0|$. Let $k = |K_0|$.

Suppose K_0 is a cycle. When i of the 5-faces are in interior of K_0, then $3i = |m| > k \geq \ell_5(i)$, which is a contradiction for all $i \in \{0, 1, 2, 3\}$.

Therefore, K_0 is a path. Let X correspond to the source component and let Y correspond to the sink component. Note that $n^s_X + n^s_Y = 9$ and $n^t_X + n^t_Y = 0$, which implies $n^t_X = n^t_Y = 0$. By Lemma 7, there must exist an edge-cut smaller than the degree of s that separates the terminals of $G^{q,\psi}$. Therefore, if X contains i of the 5-faces, then $n^s_Y + n^t_X + k + 3i \leq 8$. By the definition of ℓ_5, it follows that $n^s_Y + n^t_Y + k \geq \ell_5(3 - i)$, which gives a contradiction for all $i \in \{0, 1, 2, 3\}$. □ □

Hence C contains 6 source edges and 3 sink edges.

Claim 20. *If there is a ψ-balanced layout q, then the q values of the three 5-faces are $3, -3, -3$, and either*

A. X contains two 5-faces where the q values are $3, -3$ and Y contains one 5-face with q value -3; or

B. X contains one 5-face with q value 3 and Y contains two 5-faces where both q values are -3.

See Fig. 9 for an illustration.

Proof. In order to have a layout, the q values of the three 5-faces must be 3, -3, -3, respectively, since there are 3 more source edges than sink edges. Let K_0 and m be obtained from Lemma 8, which says $|m| > |K_0|$. Let $k = |K_0|$.

Suppose K_0 is a cycle. Denote by m the sum of the q values of the faces in the interior of K_0. When i of the 5-faces are in the interior of K_0, then $3i \geq |m| > k \geq \ell_5(i)$, which is a contradiction for all $i \in \{0, 1, 2, 3\}$.

Therefore, K_0 is a path. Note that $n^s_X + n^s_Y = 6$ and $n^t_X + n^t_Y = 3$. By Lemma 7, there must exist an edge-cut smaller than the degree of s that separates the terminals of $G^{q,\psi}$.

If Y contains three 5-faces, then $n^s_Y + n^t_X + k + 3 \leq 8$ and $n^s_Y + n^t_Y + k \geq 9$. This implies that $n^t_Y - n^t_X \geq 4$, which contradicts that $n^t_Y + n^t_X = 3$.

If X contains three 5-faces, then $n^s_Y + n^t_X + k + 6 \leq 8$ and $n^s_X + n^t_X + k \geq 9$. This implies that $n^s_X - n^s_Y \geq 7$, which contradicts that $n^s_X + n^s_Y = 6$.

If X contains only the two 5-faces where both have q values -3, then $n^s_Y + n^t_X + k + 9 \leq 8$, which is a contradiction.

If X contains exactly one 5-face where the q value is -3, then $n^s_Y + n^t_X + k + 6 \leq 8$ and $n^s_Y + n^t_Y + k \geq 8$. This implies that $n^t_Y - n^t_X \geq 6$, which contradicts $n^t_X + n^t_Y = 3$.

This finishes the proof of the claim. □ □

Claim 20 implies that there are three ψ-balanced layouts q since there are three choices for the 5-face f where $q(f) = 3$. Each of them gives a cut of type A or type B, where the types are described by Claim 20; see Fig. 9.

First we show that cuts of type A do not exist. Suppose there is a ψ-balanced layout q_1 that gives a cut of type A. In a cut of type A, there is one 5-face f in the sink component Y. Let q_2 be a ψ-balanced layout where $q_2(f) = 3$.

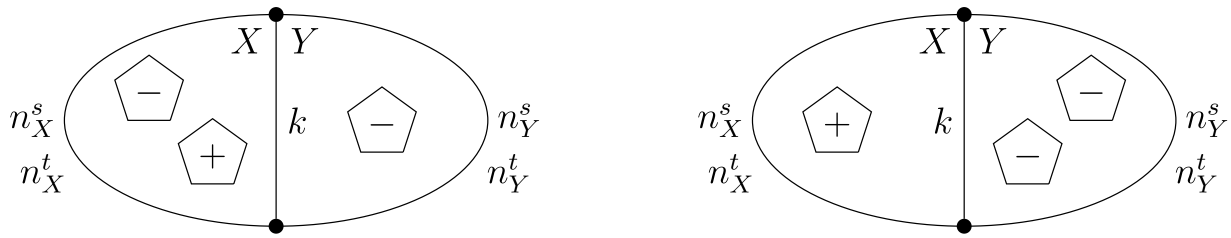


Figure 9: When C is has 6 source edges and has three 5-faces. The only possible cuts are of type A (left) and type B (right).

Let K and L be the subgraphs of G denoted by K_0 if Lemma 8 is applied to q_1 and q_2, respectively. Let $k = |K|$ and $l = |L|$. Note that we already showed that both K and L are paths joining pairs of distinct vertices of C in Claim 20. Let u_1, u_2 be the endpoints of K and let v_1, v_2 be the endpoints of L. For convenience, given $a, b, c, d \in V(C)$, let $C(a, b; c, d)$ denote the a, b-subpath of C that does not involve the vertices c and d.

Claim 21. *Layout q_2 does not give a cut of type A.*

Proof. If both cuts are of type A, then either the two cuts cross or not. For an illustration, see Fig. 10.

Suppose K and L are crossing. The situations is depicted in Fig.10 (a). Let v be an internal common vertex of K and L. We relabel the endpoints of K and L so that the clockwise order of the four vertices on C is u_1, v_1, u_2, v_2. For each $i \in \{1, 2\}$, let K_i and L_i be the v, u_i-path and v, v_i-path on K and L, respectively. Also, let $|K_i| = k_i$ and $|L_i| = l_i$. Let A be the region bounded by K_1, L_1, and $C(u_1, v_1; u_2, v_2)$, let X be the region bounded by K_1, L_2, and $C(u_1, v_2; u_2, v_1)$, let Y be the region bounded by L_1, K_2, and $C(v_1, u_2; v_2, u_1)$, and let Z be the region bounded by L_2, K_2, and $C(u_2, v_2; u_1, v_1)$.

We obtain the following constrains that must be satisfied by considering the sizes of the regions and the flow conditions.

$$k_1 + k_2 + n_X^t + n_Y^s + n_Z^t + n_A^s \leq 5 \tag{10}$$

$$l_1 + l_2 + n_X^s + n_Y^t + n_Z^t + n_A^s \leq 5 \tag{11}$$

$$n_Y^s + n_Y^t + k_2 + l_1 \text{ is } \geq 5 \text{ and odd} \tag{12}$$

$$n_Z^s + n_Z^t + k_2 + l_2 \text{ is } \geq 5 \text{ and odd} \tag{13}$$

$$n_X^s + n_X^t + k_1 + l_2 \text{ is } \geq 5 \text{ and odd} \tag{14}$$

Inequalities (10) and (11) mean that the size of the cut is less than 6. The other inequalities mean that a cycle bounding a region containing one 5-face must have an odd length of at least 5. We also use that $\min\{k_1, k_2, l_1, l_2\} \geq 1$, since v is not a vertex of C. However, the set of constraints (10)–(14) has no solution. Hence K and L cannot be crossing.

Suppose K and L are non-crossing. The situations is depicted in Fig.10 (b). Relabel the endpoints of K and L so that the clockwise order of the four vertices on C is v_1, u_1, u_2, v_2. Also, let X be the region bounded by L and $C(v_1, v_2; u_1, u_2)$, let Y be the region bounded by K and $C(u_1, u_2; v_1, v_2)$, and let Z be the region bounded by K, L, $C(v_1, u_1; v_2, u_2)$, and $C(v_2, u_2; v_1, u_1)$.

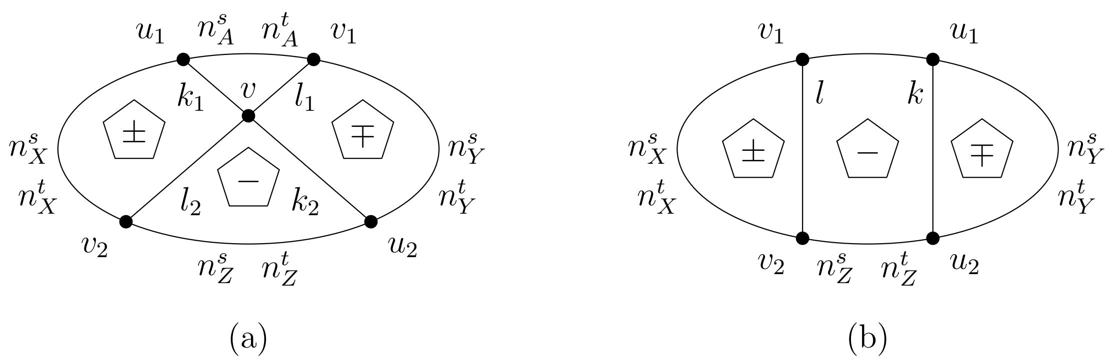


Figure 10: (a) Two crossing cuts of type A; (b) two non-crossing cuts of type A.

As in the crossing case we obtain the following set of constraints that must be satisfied.

$$k + n^t_X + n^s_Y + n^t_Z \leq 5 \tag{15}$$

$$l + n^s_X + n^t_Y + n^t_Z \leq 5 \tag{16}$$

$$n^s_Y + n^t_Y + k \text{ is } \geq 5 \text{ and odd} \tag{17}$$

$$n^s_X + n^t_X + l \text{ is } \geq 5 \text{ and odd} \tag{18}$$

Inequalities (15) and (16) are obtained since the cut size must be less than six. The other inequalities are obtained since each cycle bounding a region containing one 5-face must have an odd length of at least 5. We also include that $\min\{k, l\} \geq 2$ since G does not have chords. Recall that we assumed that C has no chords. The above set of constraints has no solution. Hence K and L cannot be non-crossing.

This finishes the proof of Claim 21. □ □

Claim 22. *Layout q_2 does not give a cut of type B.*

Proof. If q_2 is a cut of type B, then we have three cases depending on the positions of K and L. The cases are depicted in Fig. 11. If K and L are crossing, it gives the first case (AB1). If they are non-crossing, it gives two cases (AB2) and (AB3) based on the position of K and L. In order to simplify the writeup, we refer the reader to Fig. 11 for notation and description of the cases.

Depending on the case, we obtain a set of constraints that must be satisfied.

(AB1):

$$k_1 + k_2 + n^t_X + n^s_Y + n^s_A + n^t_B \leq 5 \tag{19}$$

$$l_1 + l_2 + n^s_X + n^t_Y + n^s_A + n^t_B \leq 8 \tag{20}$$

$$n^s_Y + n^t_Y + l_1 + k_2 \text{ is } \geq 5 \text{ and odd} \tag{21}$$

$$n^s_X + n^t_X + k_1 + l_2 \text{ is } \geq 8 \text{ and even}$$

$$n^s_X + n^t_X + n^s_A + n^t_A + l_1 + l_2 \text{ is } \geq 8 \text{ and even}$$

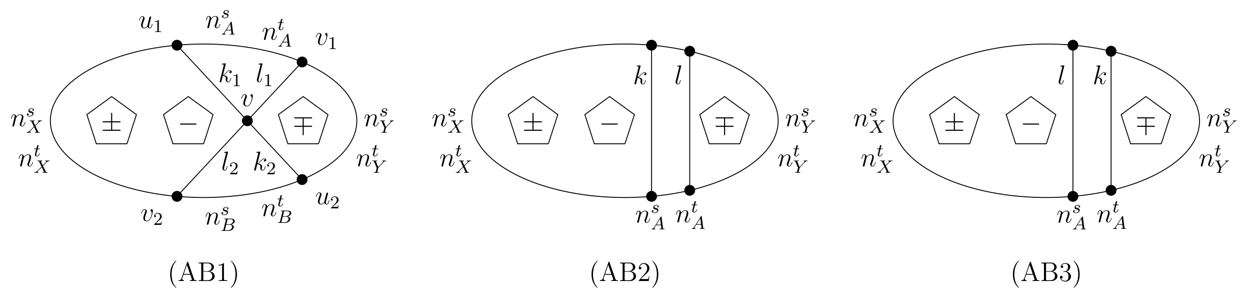


Figure 11: (AB1) Crossing cuts of types A and B; (AB2),(AB3) non-crossing cuts of types A and B.

(AB2):

$$k + n_X^t + n_Y^s + n_A^s \leq 5 \tag{22}$$

$$l + n_X^s + n_Y^t + n_A^s \leq 8 \tag{23}$$

$$n_Y^s + n_Y^t + l \text{ is } \geq 5 \text{ and odd} \tag{24}$$

$$n_X^s + n_X^t + k \text{ is } \geq 8 \text{ and even}$$

(AB3):

$$k + n_X^t + n_Y^s + n_A^t \leq 5 \tag{25}$$

$$l + n_X^s + n_Y^t + n_A^t \leq 8 \tag{26}$$

$$n_Y^s + n_Y^t + k \text{ is } \geq 5 \text{ and odd} \tag{27}$$

$$n_X^s + n_X^t + l \text{ is } \geq 8 \text{ and even}$$

Inequalities (19), (20), (22), (23), (25), and (26) mean that the size of the cut is less than $c(q_2, \psi)$. Inequalities (21), (24), and (27) mean that a region containing one 5-face must have an odd length of at least 5. All other inequalities mean that a cycle bounding a region containing two 5-faces must have an even length of at least 8. In addition, we include that $\min\{k_1, k_2, l_1, l_2\} \geq 1$ since v is not a vertex of C. We also include $\min\{k, l\} \geq 2$ since C has no chords.

None of the three sets of constraints has any solution. Hence q_2 does not give a cut of type B, and this finishes the proof of Claim 22. □ □

Claims 21 and 22 contradict each other. Hence q_1 does not give a cut of type A and every layout gives a cut of type B.

Let the 5-faces be f_1, f_2, and f_3. Let K_i be a cut of type B in layout q_i where $q_i(f_i) = 3$ and $q_i(f_j) = -3$ for $i \in \{1, 2, 3\}$ and $j \in \{1, 2, 3\} \setminus \{i\}$. Let u_i and v_i denote the endpoints of K_i and a subpath P_i of C in clock-wise direction such that P_i and K_i bound a region R_i that contains f_i.

Let $i, j \in \{1, 2, 3\}$ and $i \neq j$. The closed interior of C is cut by K_i and K_j into several closed disks. If R_i is one of the disks we call the cuts *non-crossing* and *crossing* otherwise.

See Fig. 12(a) for *non-crossing* cuts and Fig. 12(b)(c) for *crossing* cuts. In a non-crossing configuration, the order of vertices along C is u_i, v_i, u_j, v_j.

Next we show that in a crossing configuration, the order of vertices along C is u_i, u_j, v_i, v_j.

Claim 23. *If K_i and K_j are crossing, then the clock-wise direction order of the vertices along C is u_i, u_j, v_i, v_j.*

Proof. Suppose for contradiction that the order is not u_i, u_j, v_i, v_j. If the order is u_i, v_i, u_j, v_j then the cuts is not crossing. Hence, without loss of generality, the order is u_i, v_j, u_j, v_i. See Fig. 12(c) for the situation.

Let X, Y, Z, W be pairwise internally disjoint subpaths of C with endpoints $\{v_i, u_i\}$, $\{u_i, v_j\}$, $\{v_j, u_j\}$, $\{u_j, v_i\}$, respectively.

Since K_i and K_j are crossing, $K_i \cup X$ is a cycle that does not bound a region containing f_i. Therefore, it bounds a region containing two 5-faces and has even length at least 8. Analogously for K_j and Z. Together with the fact that cuts are of size at most 8, we get the following set of constraints.

$$
\begin{aligned}
k_i + n_X^s + n_Y^t + n_Z^t + n_W^t &\leq 8 \\
k_j + n_X^t + n_Y^t + n_Z^s + n_W^t &\leq 8 \\
k_i + n_X^s + n_X^t \text{ is } &\geq 8 \text{ and even} \\
k_j + n_Z^s + n_Z^t \text{ is } &\geq 8 \text{ and even}
\end{aligned}
$$

This set of constraints has no solution. □

Denote the length of the cut corresponding to q_1,q_2, and q_3 by k,l, and m, respectively. We use subscripts to denote the length of subpaths if some of the cuts are crossing. See Fig. 13.

If there is a pair of non-crossing cuts, distinguish the following cases:

(B1) all pairs of cuts are non-crossing.

(B2) one pair of cuts is crossing.

(B3) two pairs of cuts are crossing.

If all three pairs of cuts are crossing, distinguish the following cases:

(B4) P_1,P_2, and P_3 have a common edge.

(B5) There is no common edge of P_1,P_2, and P_3 and the union of R_1,R_2, and R_3 does not cover the whole disk bounded by C or there is no face of G in the intersection of R_1,R_2, and R_3.

(B6) There is no common edge of P_1,P_2, and P_3, the union of R_1,R_2, and R_3 covers the whole disk bounded by C, and there is a face of G in the intersection of all three regions.

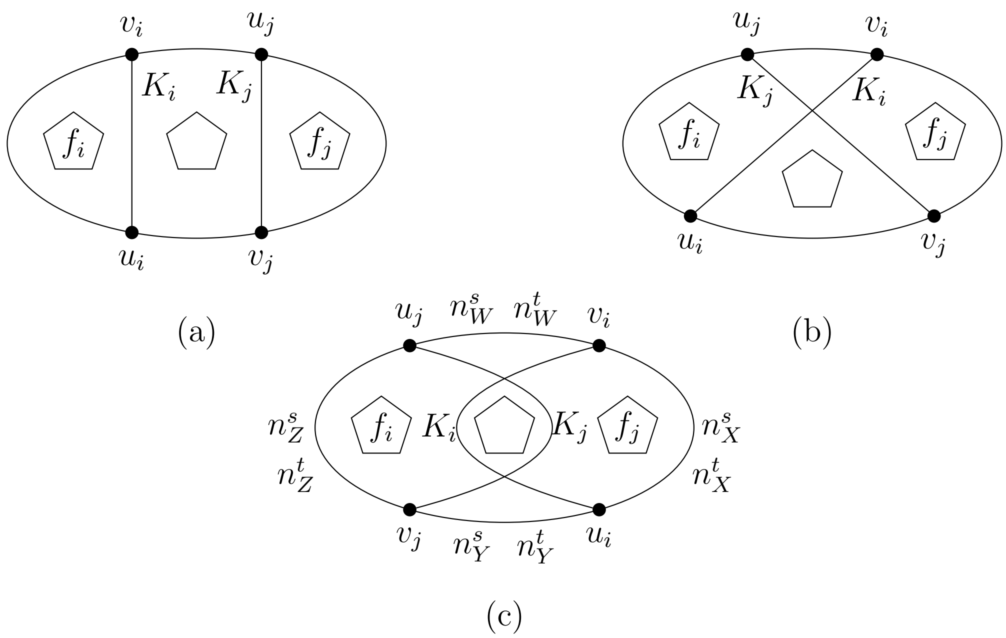


Figure 12: Possible configurations of two cuts of type B.

See Fig. 13 for an illustration of the cases (B1)–(B6). Since one layout may contain different cuts of type B, we pick K_1, K_2, K_3 such that the number of crossing pairs is minimized.

Next we give constraints for each of the cases (B1)–(B6). Solutions to these constraints were obtained by simple computer programs. Critical graphs obtained from (Bi) are depicted in Fig. 4 as (Bij) for all i, j.

Endpoints of K_1, K_2, and K_3 partition C into several internally disjoint paths. To simplify the write-up we refer the reader to Fig. 13 for the labelings of the paths.

Claim 24. *The configuration (B1) results in a critical graph where every 5-face shares at least two edges with the boundary. Moreover, in every non-extendable* 3*-coloring of the outer face, every 5-face contains two source edges.*

Proof. We refer the reader to Fig. 13 (B1) for the labellings of the regions and paths. It is enough to include constraints that all cuts are cuts of type B. That is, for f_1,

$$k + n^t_X + n^s_Y + n^s_Z + n^s_W \leq 8,$$
$$k + n^s_X + n^t_X \geq 5 \text{ and odd},$$
$$k + 9 - (n^s_X + n^t_X) \geq 8 \text{ and even}.$$

The second and third inequalities come from Lemma 6 which implies that a 5-face must be in a region bounded by an odd cycle of length at least 5 and two 5-faces must be in a region bounded by an even cycle of length at least 8. We include the same constraints also for f_2 and f_3. In addition, we also include few constraints to break the symmetry, e.g. $n^t_X + n^s_X \geq n^t_Y + n^s_Y \geq n^t_Z + n^s_Z$. All solutions are in Table 2. We obtained the solution by running a program.

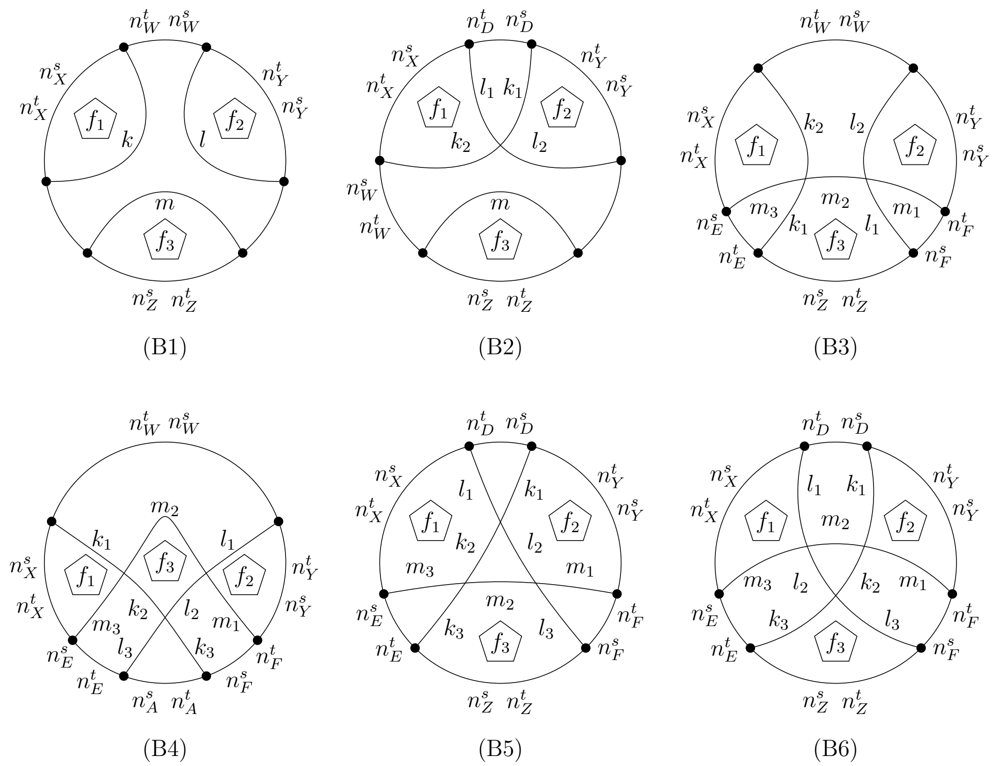


Figure 13: Possible configurations of cuts K_1, K_2, K_3.

The program is available at `http://orion.math.iastate.edu/lidicky/pub/9cyc/`. By

n^s_X	n^t_X	n^s_Y	n^t_Y	n^s_Z	n^t_Z	n^s_w	n^t_W	k	l	m
2	0	2	0	2	0	0	3	3	3	3
2	1	2	0	2	0	0	2	2	3	3
2	1	2	1	2	0	0	1	2	2	3
2	1	2	1	2	1	0	0	2	2	2

Table 2: Solutions in Lemma 17, Case 2.

inspecting the solutions from Table 2, we conclude that they satisfy the statement of the claim. □ □

For the remaining five cases, we give the sets of constraints but we skip the justifications since they add not much value. We provide the programs for solving the sets of equations and helping with checking the solutions online.

Claim 25. *Configurations (B2)–(B6) result in critical graphs (B21)–(B52). Every graph in Fig. 4 represents several graphs that can be obtained from the depicted graph by identifying* *edges and vertices and by filling every face of even size by a quadrangulation with no separating 4-cycles. Moreover, the 5-faces in (B21) and (B22) that share two edges with C can moved along C as long as it stays neighboring with a region with three sink edges.*

Proof. Outline: For each case we include constraints that all three cuts are of type B as explained in the proof of Claim 24. In addition, we use the constraint that the cycles bounding an even number of 5-faces must have sufficiently long even length and the cycles bounding an odd number of 5-faces must have sufficiently large odd length.

We also use constraints that if two cuts are crossing, then it is not possible to replace them by a non-crossing pair. For example, we add the following constraint when dealing with (B2):

$$k_1 < l_1 - n_D^t + n_D^s. \tag{28}$$

Let K_1' be a cut of type B for layout q_1 consisting of paths denoted by k_2 and l_1. Notice that now K_1' and K_2 are no longer crossing. From the minimality of the number of crossing pairs, the capacity of K_1' is strictly bigger than the capacity of K_1. The difference of the capacities is $l_1 + n_D^s - k_1 - n_D^t$, which is strictly greater than 0. This gives equation (28). A simplification of the equation is just $k_1 < l_1 + d$, which is sufficient in (B2), since all solutions of (B2) have $n_D^t = 0$ anyway.

For a path in $\{X, Y, Z, W, A, D, E, F\}$, we use its lower case letter to denote its length. We use the following sets of constraints for cases (B2)–(B6):

(B2):

$$\begin{aligned}
d &\geq 1 & l_1 &< k_1 + d\\
k_1 &< l_1 + d & x + k_2 + l_1 &\geq 5 \text{ and odd}\\
y + k_1 + l_2 &\geq 5 \text{ and odd} & z + w + k_2 + l_2 &\geq 7 \text{ and odd if } w > 0\\
x + d + y + k_2 + l_2 &\geq 8 \text{ and even} & x + y + z + w + l_1 + k_1 &\geq 9 \text{ and odd}
\end{aligned}$$

(B3):

$$\begin{aligned}
\min\{e, f\} &\geq 1 & m_1 &< l_1 + f\\
l_1 &< m_1 + f & m_3 &< k_1 + e\\
k_1 &< m_3 + e & z + k_1 + m_2 + l_1 &\geq 5 \text{ and odd}\\
y + m_1 + l_2 &\geq 5 \text{ and odd} & x + m_3 + k_2 &\geq 5 \text{ and odd}\\
y + z + f + k_1 + m_2 + l_2 &\geq 8 \text{ and even}
\end{aligned}$$

(B4):

$$\begin{aligned}
\min\{a, k_3, l_3\} &\geq 1 & y + f + k_3 + k_2 + m_2 + l_1 &\geq 8 \text{ and even}\\
k_2 + l_2 + m_2 &\geq 5 \text{ and odd} & f + y + w + x + m_3 + k_2 + k_3 &\geq 9 \text{ and odd}\\
x + k_1 + m_3 &\geq 5 \text{ and odd} & e + x + k_1 + m_2 + l_2 + l_3 &\geq 8 \text{ and even}\\
y + l_1 + m_1 &\geq 5 \text{ and odd} & y + w + x + e + l_3 + l_2 + m_1 &\geq 9 \text{ and odd}\\
& & f + y + w + x + e + l_3 + k_3 &\geq 9 \text{ and odd}
\end{aligned}$$

(B5):

$$2 \cdot \max\{k_2, l_2, m_2\} < k_2 + l_2 + m_2 \text{ or } k_2 = l_2 = m_2 = 0$$

$$\min\{d, e, f\} \geq 1 \qquad \max\{m_1, l_3\} < \min\{m_1, l_3\} + f$$
$$\max\{k_1, l_1\} < \min\{k_1, l_1\} + d \qquad \max\{k_3, m_3\} < \min\{k_3, m_3\} + e$$
$$y + k_1 + l_2 + m_1 \geq 5 \text{ and odd} \qquad x + l_1 + k_2 + m_3 \geq 5 \text{ and odd}$$
$$z + k_3 + m_2 + l_3 \geq 5 \text{ and odd} \qquad y + f + l_3 + l_2 + k_1 \geq 7 \text{ and odd}$$
$$x + e + k_3 + k_2 + l_1 \geq 7 \text{ and odd}$$

(B6):

$$\min\{d, e, f\} \geq 1 \qquad \min\{k_2, l_2, m_2\} \geq 1 \qquad y + k_1 + m_1 \geq 5 \text{ and odd}$$
$$x + l_1 + m_3 \geq 5 \text{ and odd} \qquad z + k_3 + l_3 \geq 5 \text{ and odd}$$

We enumerated all solutions to all five sets of constraints, and we checked that the resulting graphs are depicted in Fig. 4. In order to eliminate mistakes in computer programs, we have two implementations by different authors and we checked that they give identical results. Sources for programs for cases (B2)–(B6) together with their outputs can be found at `http://orion.math.iastate.edu/lidicky/pub/9cyc/`. □

This finishes the proof of Lemma 18. □

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