Semidefinite Programming and Ramsey Numbers

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April 11, 2017

Abstract

We use the theory of flag algebras to find new upper bounds for several small graph and hypergraph Ramsey numbers. In particular, we prove the exact values $R(K_4^-, K_4^-, K_4^-) = 28$, $R(K_8, C_5) = 29$, $R(K_9, C_6) = 41$, $R(Q_3, Q_3) = 13$, $R(K_{3,5}, K_{1,6}) = 17$, $R(C_3, C_5, C_5) = 17$, and $R(K_4^-, K_5^-; 3) = 12$, and in addition improve many additional upper bounds.

1 Introduction

Let $G_1, G_2, \ldots, G_k$ be graphs. Ramsey's celebrated Theorem [31] implies that for every edge coloring of a large enough complete graph K_n with colors from $\{1, 2, \ldots, k\}$ exists some i such that the K_n contains a copy of G_i with all edges colored i. The Ramsey number $R(G_1, G_2, \ldots, G_k)$ is the smallest n for which we are guaranteed to find a monochromatic copy. A Ramsey graph is a k-edge-coloring of $K_{R(G_1,G_2,\ldots,G_k)-1}$ which does not contain a copy of G_i in color i for any i.

The theory of flag algebras was developed by Razborov [32]. The easiest and most popular usage is the *plain flag algebra method*. The theory of flag algebras was applied to graphs [3, 11, 12, 33], hypergraphs [2, 18, 21, 24], graphons [20], permutations [4], discrete geometry [19, 22], and even phylogenetic trees [1], to name a few. Formally, the method works with homomorphisms from linear combinations of combinatorial structures (graphs) to real numbers. The homomorphisms can be viewed as densities of (small) graphs in a very large graph, or more precisely, a graph limit.

The core of the plain method is to use the Cauchy-Schwarz Inequality to generate valid inequalities which hold for the densities of a large number of small graphs in the extremal graph (limit). Combinations of these inequalities are used to produce bounds on the densities of small graphs. The right combination of the inequalities is usually found via semidefinite programming.

Bounding exact Ramsey numbers is a problem restricted to relatively small graphs. The flag algebra method can only find asymptotic results for very large graphs, so it seems that the method is not suitable for finding small Ramsey numbers. But this intuition is wrong, and we will develop a technique to do just that in this paper. This technique may be adapted to address other questions for smaller graphs with the flag algebra method.

We give a summary of new results in Section 2. We provide a brief introduction to the theory of flag algebra in Section 3. We describe how to use the theory to obtain bounds on Ramsey

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numbers in Section 4. For better exposition, we describe the technique on a toy example proving that $R(K_3, K_3) \leq 6$ in Section 5. In the appendix, we summarize the results of all the computations we tried.

The proofs involve extensive computations, and it is impractical to provide the actual solutions here. Even the certificates are impractically large to provide as ancillary files. Instead, we provide the computer programs we used to obtain the results. This gives the interested reader the opportunity to recreate our results, and to try the methods on related questions. The programs and brief descriptions can be found in electronic form at `http://orion.math.iastate.edu/lidicky/pub/ramsey` and as ancillary files to this preprint.

2 Results

Here, we only present the new upper bounds we achieved together with the previously best known bounds referenced in [30]. We use standard notation for all graphs and hypergraphs appearing here. In particular, K_n^- stands for a complete (hyper)graph on n vertices, minus one edge.

2.1 Graphs

We establish the following graph Ramsey numbers.

Theorem 1. *$R(K_8, C_5) = 29$.*

A Ramsey graph is the balanced complete 7-partite graph on 28 vertices. Previously, the best upper bound was 33 from [26].

Theorem 2. *$R(K_9, C_6) = 41$.*

A Ramsey graph is the balanced complete 8-partite graph on 40 vertices. We are not aware of a previous non-trivial upper bound.

Theorem 3. *$R(Q_3, Q_3) = 13$.*

Here, Q_3 stands for the graph of a 3-dimensional cube. Our flag algebra computations give an upper bound of 14, the previous lower bound was 12 from [23]. In this case, the problem is small enough for a complete enumeration, and we found the exact number and all 8063 Ramsey graphs this way.

Theorem 4. *$R(K_{3,5}, K_{1,6}) = 17$.*

The flag algebra computation gives an upper bound for the order of a Ramsey graph barely above 16. Assuming this to be the correct bound, we examine the solution more closely. The flag algebra computation gives a list of graphs on 8 vertices that are unlikely to appear in a Ramsey graph on 16 vertices, so we further assume that this graph does not contain any such subgraphs. This provides a significant restriction on the possible graphs on 9 or more vertices and we can enumerate all such graphs on up to 16 vertices. We find one Ramsey graph on 16 vertices this way, the Clebsch graph.

Theorem 5. *$R(K_4^-, K_4^-, K_4^-) = 28$.*

Previously, the best upper bound was 30 by Piwakowski [29]. A Ramsey graph (which was not known to be Ramsey at the time) was constructed by Exoo [16].

Theorem 6. *$R(C_3, C_5, C_5) = 17$.*

Here, we improve the upper bound from 21 to 17. The lower bound is by Tse [38].

We are able to improve the following bounds. Bounds without citations come from general theorems about Ramsey numbers. We denote the wheel on n vertices by W_n and a book on n vertices by B_n. That is, $W_n = K_1 + C_{n-1}$ and $B_n = K_2 + \overline{K_{n-2}}$.

Theorem 7. *New upper bounds on graph Ramsey numbers.*

	lower	*old upper*	*new upper*
$R(K_4^-, K_8^-)$	29	38 *[41]*	32
$R(K_4^-, K_9^-)$	34 *[17]*	53 *[25]*	46
$R(K_4, K_7^-)$	37 *[17]*	52 *[41]*	49
$R(K_5^-, K_6^-)$	31 *[17]*	39	38
$R(K_5^-, K_7^-)$	40 *[10]*	66 *[10]*	65
$R(K_5, K_6^-)$	43	66 *[7]*	62
$R(K_5, K_7^-)$	58	110 *[7]*	102
$R(K_6^-, K_7^-)$	59 *[17]*	135 *[41]*	124
$R(K_7, K_4^-)$	28	30 *[9]*	29
$R(K_8, K_4^-)$	29	42 *[6]*	39
$R(K_9, K_4^-)$			46
$R(K_9, C_5)$	33		36
$R(K_9, C_7)$	49		58
$R(K_{2,2,2}, K_{2,2,2})$	30 *[23]*		32
$R(K_{3,4}, K_{2,5})$		21 *[28]*	20
$R(K_{3,4}, K_{3,3})$		25 *[27]*	20
$R(K_{3,4}, K_{3,4})$		30 *[27]*	25
$R(K_{3,5}, K_{2,4})$	16 *[36]*		20
$R(K_{3,5}, K_{2,5})$	21 *[42]*		23
$R(K_{3,5}, K_{3,3})$		28 *[27]*	24
$R(K_{3,5}, K_{3,4})$		33 *[27]*	29
$R(K_{3,5}, K_{3,5})$	30 *[23]*	38 *[27]*	33
$R(K_{4,4}, K_{4,4})$	30 *[23]*	62 *[27]*	49
$R(W_7, W_4)$			21
$R(W_7, W_5)$			16
$R(W_7, W_6)$			19
$R(B_4, B_5)$	17 *[34]*	20 *[34]*	19
$R(B_3, B_6)$	17	22 *[34]*	19
$R(B_5, B_6)$	22 *[34]*	26 *[34]*	24
$R(W_5, K_6)$	33*[42]*		36
$R(W_5, K_7)$	43*[42]*		50

Theorem 8. *New upper bounds on multi-color graph Ramsey numbers.*

	lower	*old upper*	*new upper*
$R(C_3, C_6, C_6)$	15		18
$R(C_5, C_6, C_6)$	15		17
$R(C_3, C_3, C_3, C_4)$	49		59
$R(C_4, C_4, K_4)$	20 *[13]*	22 *[39]*	21
$R(C_4, K_4, K_4)$	52 *[39]*	72 *[39]*	71
$R(C_4, C_4, C_4, K_4)$	34 *[13]*	50 *[39]*	48
$R(K_3, K_4^-, K_4^-)$	21 *[37]*	27 *[37]*	22
$R(K_4, K_4^-, K_4^-)$	33 *[37]*	59 *[8]*	47
$R(K_4, K_4, K_4^-)$	55	113 *[8]*	104
$R(K_3, K_4, K_4^-)$	30	41 *[8]*	40

2.2 3-uniform hypergraphs

In a couple cases, we are able to improve bounds on Ramsey numbers for 3-uniform hypergraphs.

Theorem 9. $14 \leq R(K_4^-, K_5; 3) \leq 16$ *and* $13 \leq R(K_4^-, K_4^-, K_4^-; 3) \leq 14$.

Both lower bounds are from [15], and we are not aware of a previous upper bound for the first quantity. The second quantity was previously bounded by 16.

We establish one new hypergraph Ramsey number.

Theorem 10. $R(K_4^-, K_5^-; 3) = 12$.

To the best of our knowledge, this number has not been studied before. Using the computations for the upper bound similarly to the proof of Theorem 4, we construct the unique Ramsey 3-graph on 11 vertices. This Ramsey 3-graph R_{11} is highly symmetric, and we describe it here.

The 3-graph R_{11} has 55 edges, it is vertex and vertex-pair transitive with degree 15 and co-degree 3. In fact, every vertex link (the 2-graph spanned by the edges incident to a vertex after deleting that vertex) is isomorphic two a 10-vertex Möbius ladder, i.e., C_{10} with the 5 antipodal chords added. With vertex set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 0, A\}$, the edge set is

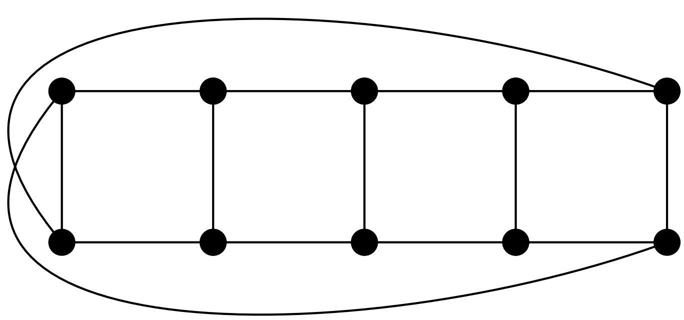


Figure 1: The vertex link in R_{11}.

$\{123, 124, 125, 136, 137, 146, 14A, 150, 15A, 169, 178, 179, 180, 18A, 190, 239, 230, 248, 240, 256, 259, 267, 26A, 278, 279, 28A, 20A, 345, 349, 34A, 356, 357, 36A, 370, 389, 380, 38A, 458, 450, 467, 468, 479, 47A, 490, 560, 578, 57A, 589, 59A, 670, 689, 680, 69A, 70A, 90A\}$.

In this case, the flag algebra computations result in a sharp bound. From this, we can use standard arguments to show that a large set of subgraphs (other than K_4^- and the complement of K_5^-) can not occur in an 11-vertex Ramsey graph. The computer is then used to enumerate all such 3-graphs up to 9 vertices, and finds that there is only one allowed 3-graph on 9 vertices. Thus, in any Ramsey graph on 11 vertices, all 9-vertex subgraphs must be isomorphic to this 3-graph. With this information, constructing R_{11} is easy, either by hand or by computer.

2.3 Tournaments, directed graphs and further directions

Erdős and Moser [14] noted that Ramsey's Theorem implies that for every k, there exists a minimum number $R(TT_k)$, such that every tournament on $R(TT_k)$ vertices contains a transitive tournament on k vertices as a subtournament. The number $R(TT_k)$ is known for $1 \leq k \leq 6$. Our method is applicable for this problem as well. We are not able to improve the upper bound on $R(TT_7)$ from [35] using flags of order 8, but it seems that with a bit of patience computing with flags of order 9 is feasible and might provide an improvement.

It is also possible to use our method for Ramsey numbers of directed graphs in tournaments but we have not explored this direction. See the appendix for all bounds we have tried to improve.

3 Flag algebra terminology

Let us now introduce the terminology related to flag algebras needed in this paper. For more details about the method, see [32]. This section is included in order to make the paper self-contained. A reader familiar with the theory may wish to skip to the next section.

For a list $\mathcal{H} = \{G_1, G_2, \ldots, G_k\}$, an edge colored graph is $\mathcal{H}$-free if it does not contain a copy of G_i as a subgraph in color i for any $1 \leq i \leq k$. Since we deal mostly with blow-ups of edge colored $\mathcal{H}$-free complete graphs, we restrict our attention to this particular case. We say that a graph is a blow-up of an edge colored complete graph if it can be obtained from an edge-colored complete graph by a blow-up of the vertices, i.e., vertices are replaced by independent sets, and edges are replaced by complete bipartite graphs between the sets, and all edges inherit the given color. For brevity, we will just write *blow-up graph* for these objects. The central notions we are going to introduce are an algebra $\mathcal{A}$ and algebras $\mathcal{A}^\sigma$, where σ is a fixed blow-up graph.

In order to precisely describe algebras $\mathcal{A}$ and $\mathcal{A}^\sigma$, we first need to introduce some additional notation. Let $\mathcal{F}$ be the set of all finite blow-up graphs. Next, for every $\ell \in \mathbb{N}$, let $\mathcal{F}_\ell \subset \mathcal{F}$ be the set of blow-up graphs on exactly ℓ vertices. For $H \in \mathcal{F}_\ell$ and $H' \in \mathcal{F}_{\ell'}$, recall that $p(H, H')$ is the probability that a randomly chosen subset of ℓ vertices in H' induces a subgraph isomorphic to H. Note that $p(H, H') = 0$ if $\ell' < \ell$. Let $\mathbb{R}\mathcal{F}$ be the set of all formal linear combinations of elements of $\mathcal{F}$ with real coefficients. Furthermore, let $\mathcal{K}$ be the linear subspace of $\mathbb{R}\mathcal{F}$ generated by all linear combinations of the form

$$H - \sum_{H' \in \mathcal{F}_{v(H)+1}} p(H, H') \cdot H'. \tag{1}$$

Finally, we define $\mathcal{A}$ to be the space $\mathbb{R}\mathcal{F}$ factorized by $\mathcal{K}$.

The space $\mathcal{A}$ has naturally defined linear operations of addition and scalar multiplication by real numbers. To introduce a multiplication inside $\mathcal{A}$, we first define it on the elements of $\mathcal{F}$ in the following way. For $H_1, H_2 \in \mathcal{F}$, and $H \in \mathcal{F}_{v(H_1)+v(H_2)}$, we define $p(H_1, H_2; H)$ to be the probability that a randomly chosen subset of $V(H)$ of size $v(H_1)$ and its complement induce in H subgraphs isomorphic to H_1 and H_2, respectively. We set

$$H_1 \times H_2 = \sum_{H \in \mathcal{F}_{v(H_1)+v(H_2)}} p(H_1, H_2; H) \cdot H.$$

The multiplication on $\mathcal{F}$ has a unique linear extension to $\mathbb{R}\mathcal{F}$, which yields a well-defined multiplication also in the factor algebra $\mathcal{A}$. A formal proof of this can be found in [32, Lemma 2.4].

Let us now move to the definition of an algebra $\mathcal{A}^\sigma$, where $\sigma \in \mathcal{F}$ is an arbitrary blow-up graph with a fixed labelling of its vertex set. The labelled graph σ is usually called a *type* within the flag algebra framework. Without loss of generality, we will assume that the vertices of σ are labelled by $1, 2, \ldots, v(\sigma)$. Now we follow almost the same lines as in the definition of $\mathcal{A}$. We define $\mathcal{F}^\sigma$ to be the set of all finite blow-up graphs H with a fixed *embedding* of σ, i.e., an injective mapping θ from $V(\sigma)$ to $V(H)$ such that $\mathrm{im}(\theta)$ induces in H a subgraph isomorphic to σ. The elements of $\mathcal{F}^\sigma$ are usually called *σ-flags* and the subgraph induced by $\mathrm{im}(\theta)$ is called the *root* of a σ-flag.

Again, for every $\ell \in \mathbb{N}$, we define $\mathcal{F}^\sigma_\ell \subset \mathcal{F}^\sigma$ to be the set of the σ-flags from $\mathcal{F}^\sigma$ that have size ℓ (i.e., the σ-flags with the underlying blow-up graph having ℓ vertices). Analogously to the case for $\mathcal{A}$, for two blow-up graphs $H, H' \in \mathcal{F}^\sigma$ with the embeddings of σ given by θ, θ', we set $p(H, H')$ to be the probability that a randomly chosen subset of $v(H) - v(\sigma)$ vertices in $V(H') \setminus \theta'(V(\sigma))$ together with $\theta'(V(\sigma))$ induces a subgraph that is isomorphic to H through an isomorphism f that preserves the embedding of σ. In other words, the isomorphism f has to satisfy $f(\theta') = \theta$. Let $\mathbb{R}\mathcal{F}^\sigma$ be the set of all formal linear combinations of elements of $\mathcal{F}^\sigma$ with real coefficients, and let $\mathcal{K}^\sigma$ be the linear subspace of $\mathbb{R}\mathcal{F}^\sigma$ generated by all the linear combinations of the form

$$H - \sum_{H' \in \mathcal{F}^\sigma_{v(H)+1}} p(H, H') \cdot H'.$$

We define $\mathcal{A}^\sigma$ to be $\mathbb{R}\mathcal{F}^\sigma$ factorized by $\mathcal{K}^\sigma$.

We now describe the multiplication of two elements from $\mathcal{F}^\sigma$. Let $H_1, H_2 \in \mathcal{F}^\sigma$, $H \in \mathcal{F}^\sigma_{v(H_1)+v(H_2)-v(\sigma)}$, and θ be the fixed embedding of σ in H. As in the definition of multiplication for $\mathcal{A}$, we define $p(H_1, H_2; H)$ to be the probability that a randomly chosen subset of $V(H) \setminus \theta(V(\sigma))$ of size $v(H_1) - v(\sigma)$ and its complement in $V(H) \setminus \theta(V(\sigma))$ of size $v(H_2) - v(\sigma)$, extend $\theta(V(\sigma))$ in H to subgraphs isomorphic to H_1 and H_2, respectively. This definition naturally extends to $\mathcal{A}^\sigma$.

Now consider an infinite sequence $(G_n)_{n\in\mathbb{N}}$ of blow-up graphs of increasing orders. We say that the sequence $(G_n)_{n\in\mathbb{N}}$ is *convergent* if the probability $p(H, G_n)$ has a limit for every $H \in \mathcal{F}$. A standard compactness argument (e.g., using Tychonoff's theorem) yields that every such infinite sequence has a convergent subsequence. All the following results can be found in [32]. Fix a convergent increasing sequence $(G_n)_{n\in\mathbb{N}}$ of blow-up graphs. For every $H \in \mathcal{F}$, we set $\phi(H) = \lim_{n\to\infty} p(H, G_n)$ and linearly extend ϕ to $\mathcal{A}$. We usually refer to the mapping ϕ as to the limit of the sequence. The obtained mapping ϕ is a homomorphism from $\mathcal{A}$ to $\mathbb{R}$. Moreover, for every $H \in \mathcal{F}$, we obtain $\phi(H) \geq 0$. Let $\mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$ be the set of all such homomorphisms, i.e., the set of all homomorphisms ψ from the algebra $\mathcal{A}$ to $\mathbb{R}$ such that $\psi(H) \geq 0$ for every $H \in \mathcal{F}$. It is interesting to see that this set is exactly the set of all limits of convergent sequences of blow-up graphs [32, Theorem 3.3].

Let $(G_n)_{n\in\mathbb{N}}$ be a convergent sequence of blow-up graphs and $\phi \in \mathrm{Hom}^+(\mathcal{A},\mathbb{R})$ be its limit. For $\sigma \in \mathcal{F}$ and an embedding θ of σ in G_n, we define G_n^θ to be the blow-up graph rooted on the copy of σ that corresponds to θ. For every $n \in \mathbb{N}$ and $H^\sigma \in \mathcal{F}^\sigma$, we define $p_n^\theta(H^\sigma) = p(H^\sigma, G_n^\sigma)$. Picking θ at random gives rise to a probability distribution $\mathbf{P}_\mathbf{n}^\sigma$ on mappings from $\mathcal{A}^\sigma$ to $\mathbb{R}$, for every $n \in \mathbb{N}$. Since $p(H, G_n)$ converges (as n tends to infinity) for every $H \in \mathcal{F}$, the sequence of these probability distributions on mappings from $\mathcal{A}^\sigma$ to $\mathbb{R}$ also converges [32, Theorems 3.12 and 3.13]. We denote the limit probability distribution by $\mathbf{P}^\sigma$. In fact, for any σ such that $\phi(\sigma) > 0$, the homomorphism ϕ itself fully determines the random distribution $\mathbf{P}^\sigma$ [32, Theorem 3.5]. Furthermore, any mapping ϕ^σ from the support of the distribution $\mathbf{P}^\sigma$ is in fact a homomorphism from $\mathcal{A}^\sigma$ to $\mathbb{R}$ such that $\phi^\sigma(H^\sigma) \geq 0$ for all $H^\sigma \in \mathcal{F}^\sigma$ [32, Proof of Theorem 3.5].

The last notion we introduce is the *averaging* (or downward) operator $[\![\cdot]\!]_\sigma : \mathcal{A}^\sigma \to \mathcal{A}$. It is a linear operator defined on the elements of $H^\sigma \in \mathcal{F}^\sigma$ by $[\![H^\sigma]\!]_\sigma = p_H^\sigma \cdot H^\emptyset$, where $H^\emptyset$ is the (unlabeled) blow-up graph from $\mathcal{F}$ corresponding to H^σ, and p_H^σ is the probability that a random injective mapping from $V(\sigma)$ to $V(H^\emptyset)$ is an embedding of σ in $H^\emptyset$ yielding a σ-flag isomorphic to H^σ. The key relation between ϕ and ϕ^σ is the following:

$$\forall H^\sigma \in \mathcal{A}^\sigma, \quad \phi\left([\![H^\sigma]\!]_\sigma\right) = \phi([\![\sigma]\!]_\sigma) \cdot \int \phi^\sigma(H^\sigma),$$

where the integration is over the probability space given by the random distribution $\mathbf{P}^\sigma$ on ϕ^σ. Therefore, if $\phi^\sigma(A^\sigma) \geq 0$ almost surely for some $A^\sigma \in \mathcal{A}^\sigma$, then $\phi\left([\![A^\sigma]\!]_\sigma\right) \geq 0$. In particular,

$$\forall A^\sigma \in \mathcal{A}^\sigma, \quad \phi\left(\left[\!\!\left[(A^\sigma)^2\right]\!\!\right]_\sigma\right) \geq 0. \tag{2}$$

The plain method is a tool from the flag algebra framework that, for a given density problem of the form

$$\min_{\phi\in\mathrm{Hom}^+(\mathcal{A},\mathbb{R})} \phi(A),$$

where $A \in \mathcal{A}$, systematically searches for 'best possible' inequalities of the form (2). If we fix in advance an upper bound on the size of graphs in the terms of inequalities we will be using, we can find the best inequalities of the form (2) using semidefinite programming.

To reduce the size of $\mathcal{A}$ and with it the size of all required computations, it is often benefitial to use a partially color-blind setting. In this setting, the colors are partitioned into classes, and two blow-up graphs are considered to be the same if they differ only by a permutation of colors inside the classes. All of the theory described in this chapter naturally works for this setting as well.

4 Using flag algebra to bound Ramsey numbers

For some $n < R(G_1, G_2, \ldots, G_k)$, start with a $\{G_1, G_2, \ldots, G_k\}$-free k-edge-coloring H of a K_n. Now replace every vertex by a large independent set of size N, say. If this blow-up graph contains a copy of G_i in color i, then two of the vertices in this copy are in the same independence set. Making N larger and larger, this graph sequence becomes an object that can be analysed by the plain flag algebra method.

Formally, we consider the model of blow-ups of k-edge-colored complete graphs, for which every copy of G_i in color i contains two vertices in the same independence set. This model can easily be described in the theory of flag algebras. If you prefer the language of graph limits, we look at the k-colored graphon of H, i.e. a step function $W : [0,1]^2 \to \{0,1\}^k$, where every $W(x,y)$ contains exactly one 1 for off-diagonal steps, and all 0s for the diagonal steps.

In this model, we find a lower bound δ_2 for the density of non-edges via the plain flag algebra method. The minimum is achieved exactly by a balanced blow-up of any Ramsey graph. Therefore, if δ_2 is a lower bound for the density of non-edges, then

$$R(G_1, G_2, \ldots, G_k) = n + 1 \le \frac{1}{\delta_2} + 1.$$

More generally, we can look at lower bounds δ_ℓ for the density of independent sets of size ℓ. Again, the minimum is achieved exactly by a balanced blow-up of any Ramsey graph, and it follows that

$$R(G_1, G_2, \ldots, G_k) = n + 1 \le \delta_\ell^{-\frac{1}{\ell-1}} + 1.$$

Notice that we can make use of the integrality of $R(G_1, G_2, \ldots, G_k)$. If we want to show that $R(G_1, G_2, \ldots, G_k) \le s$ for some $s \in \mathbb{N}$, all we need to show is that $\delta_\ell > \frac{1}{s^{\ell-1}}$ for some ℓ. In most cases, we found the same bounds by using different ℓ, but in some cases, the bounds were different.

The application of the plain flag algebra method requires the enumeration of all small graphs in the model. A computer is used to enumerate the small graphs, set up the inequalities and then solve the resulting semidefinite program. The semidefinite program can be solved by state of the art solvers CSDP [5] and SDPA [40].

These solvers use floating point arithmetic, and in most applications of the plain flag algebra method the following rounding step requires some thought, sometimes ingenuity, to turn the results into a proof. In our application, though, we are usually not interested in sharp bounds as we can use the integrality of $R(G_1, G_2, \ldots, G_k)$, and the rounding is easy. Round the result to a desired level of precision, while keeping the resulting matrix positive semidefinite. Due to continuity, the resulting bounds are almost unchanged. We end up with a certificate consisting of several (sometimes very large) rational positive semidefinite matrices.

5 Illustration of the method: $R(K_3, K_3) = 6$

In this section we illustrate our method on the smallest non trivial Ramsey number $R(K_3, K_3) = 6$. This may be the most complicated proof of this fact ever published. In fact, at an early point in the proof we determine all 2-colorings of K_4 without monochromatic triangles, from which it is easy to find the unique Ramsey graph on five vertices. For larger Ramsey numbers a similar complete enumeration is not feasible, and our method, which only uses relatively small graphs, can find new upper bounds.

Recall that there is a 2-edge-coloring of K_5 without monochromatic triangles, see Figure 2, so all we need to show is that $R(K_3, K_3) \le 6$.

Proof of $R(K_3, K_3) \le 6$. Let $k \ge 5$, and suppose that G is a 2-edge-colored K_k with no monochromatic triangle. Let G_n be a blow-up of G on n vertices where every vertex of G is replaced by an independent set of size I_i for $1 \le i \le k$. Clearly, $\sum_i I_i = n$. The number of non-edges in G is $\sum_{i=1}^k \binom{I_i}{2}$. This is minimized if $I_i \in \{\lfloor n/k \rfloor, \lceil n/k \rceil\}$ for all $1 \le i \le k$. Hence, the number of non-edges is at least $\frac{n}{2}(\frac{n}{k} - 1)$, which gives an asymptotic density of non-edges of at least $1/k$.

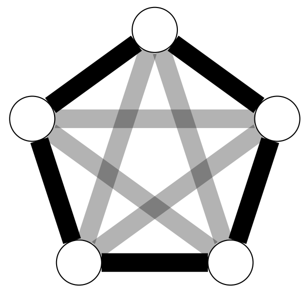
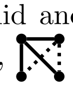
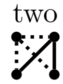
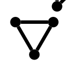
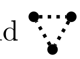
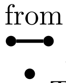
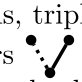


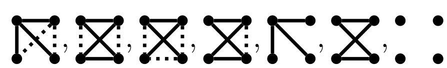
Figure 2: A 2-edge coloring of K_5 with no monochromatic triangle. At the same time, it can be viewed as a blow-up graph where every circle in the picture represents an independent set.

Denote by δ the minimum asymptotic density of non-edges over all 2-colored blow-up graphs with no monochromatic triangles. Therefore, $k \leq 1/\delta$ and hence $R(K_3, K_3) \leq 1/\delta + 1$. Notice that these asymptotic densities are always $1/m$ for some natural number m. Hence in order to prove that the largest graph with no monochromatic triangles has at most k vertices, it is enough to show that $\delta > 1/(k+1)$. If there was a complete graph on $k+1$ vertices with no monochromatic triangle, then there would be a blow-up graph with $\delta \leq 1/(k+1)$. In our case, it is enough to show that $\delta > \frac{1}{6}$.

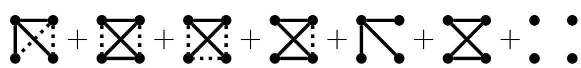
We work in $\mathcal{B}$: the class of 2-colored blow-up graphs with no monochromatic triangles. In figures, we will use solid and dotted lines to distinguish the two colors. We use the color-blind setting, so for example,  is considered the same graphs as  .

Forbidden subgraphs in $\mathcal{B}$ are monochromatic triangles  (and  , but this already follows from color-blindness). Since all graphs in $\mathcal{B}$ are blow-up graphs, triples inducing exactly one edge  and triples inducing exactly two edges with different colors  are also forbidden.

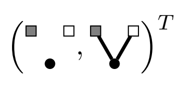
This leaves exactly seven graphs on 4 vertices in $\mathcal{B}$, taking color-blindness into account:



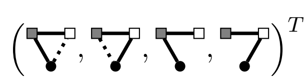
With a slight abuse of notation, we use the drawing of a graph H also for the asymptotic density $\phi(H)$, making our equalities and inequalities much more intuitive. As a first equality, we have in $\mathcal{B}$:

 $= 1.$

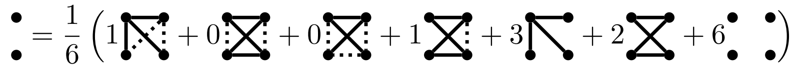
We use two types of size two. The first type σ_0 is a non-edge and the second type σ_1 is an edge. Due to the color-blind setting, there are no other types. We use flags of size three for both types. In the figures, we use a gray square and a white square to distinguish the two labeled vertices. We have two flags for σ_0 in a vector

$F_0 =$ 

and we have four flags for σ_1 in a vector

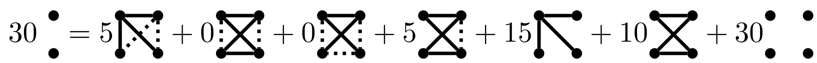
$F_1 =$  .

Using (1), we have



(3)

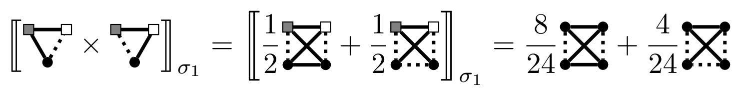
and we want to show that  $\geq \frac{1}{5}$. In order to avoid fractions, we show 30  ≥ 6 and we use



Let us define two matrices M_0 and M_1 (we will discuss later how to find them),

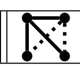
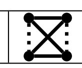
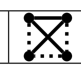
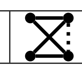
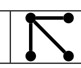
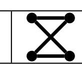
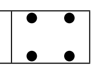
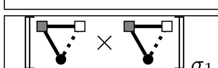
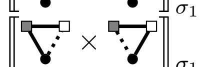
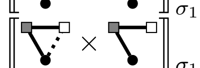
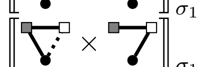
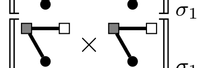
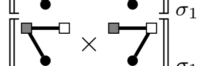
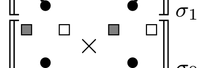
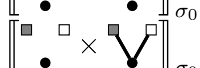
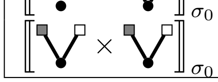
$$M_0 = \begin{pmatrix} 16 & -4 \\ -4 & 1 \end{pmatrix} \qquad M_1 = \begin{pmatrix} 126 & -48 & -73 & -5 \\ -48 & 126 & -5 & -73 \\ -73 & -5 & 64 & 14 \\ -5 & -73 & 14 & 64 \end{pmatrix}.$$

The matrices M_0 and M_1 are positive semidefinite since they have sets of eigenvalues $\{0, 17\}$ and $\{112 - 2\sqrt{2117}, 112 + 2\sqrt{2117}, 156, 0\}$. We plan to use the inequalities $0 \leq [\![F_0^T M_0 F_0]\!]_{\sigma_0}$ and $0 \leq [\![F_1^T M_1 F_1]\!]_{\sigma_1}$. They involve taking products of flags and using the downward operator. As an example for the required computations, consider the following.



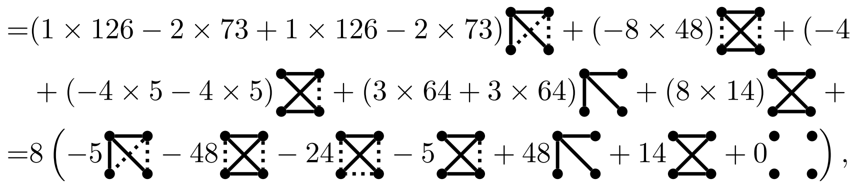
The last step in the previous computation requires us to randomly label two of the four vertices, and to compute the probability that this leads to the desired flag.

Performing similar computations for all required multiplications, we get the following table, in which we omitted all zeros and multiplied all entries by 24 to avoid the fractions.

							
	1						
		8	4				
	2						
				4			
					3		
						8	
							12
					3		
	2			2		4	

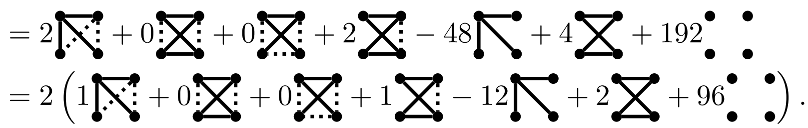
This gives

$$0 \leq 24 \cdot \left[\!\left[F_1^T \begin{pmatrix} 126 & -48 & -73 & -5 \\ -48 & 126 & -5 & -73 \\ -73 & -5 & 64 & 14 \\ -5 & -73 & 14 & 64 \end{pmatrix} F_1 \right]\!\right]_{\sigma_1}$$

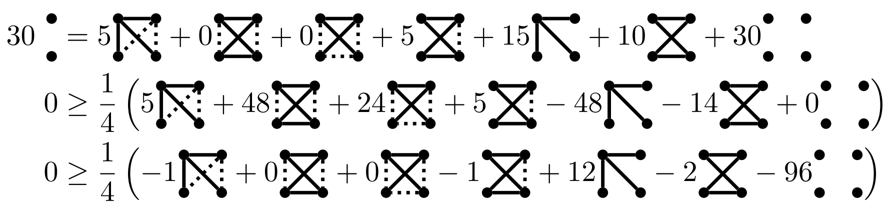


and

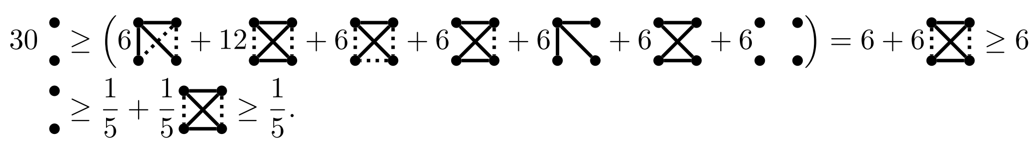
$$0 \leq 24 \cdot \left[\!\left[F_0^T \begin{pmatrix} 16 & -4 \\ -4 & 1 \end{pmatrix} F_0 \right]\!\right]_{\sigma_0}$$



Finally we add the equations:

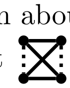


and obtain



(4)

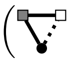
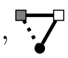
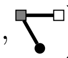
$\square$

Since the inequality is tight, (4) can be used to obtain additional information about the graph on five vertices with no monochromatic triangles. In particular, it shows that  $= 0$ in this graph. From this, it is fairly simple to see that the Ramsey graph is unique.

In the previous proof, we used the positive semidefinite matrices M_0 and M_1, and it just so happened that all the inequalities worked together perfectly. This is no accident. The two matrices were found through a semidefinite program with the help of the computer. The bigger the flags we work with, the bigger the semidefinite programs become we have to solve, so there is a computational limit on the sizes of flags we can work with. Further, the solvers work with floating point arithmetic, and sometimes it is difficult to round correctly.

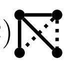
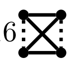
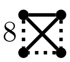
But recall the observation, that it in order to prove $R(K_3, K_3) \leq 6$ it is actually sufficient to show that  $> \frac{1}{6}$. This results in significantly easier computations, and there is no need for exact values but fairly simple rounding will do. To illustrate this, we will show the same statement again with this simpler approach.

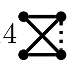
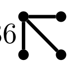
Second proof of $R(K_3, K_3) \leq 6$*.* In order to show that $\delta > \frac{1}{6}$, we do not need F_0 and we can remove one entry from F_1 and consider only

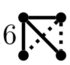
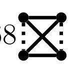
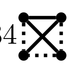
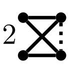
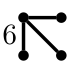
$F_1' = \big($  ,  ,  $\big)^T$.

We also use a different matrix M', see below. The smallest eigenvalue is greater than 0.000133, hence the matrix is positive definite.

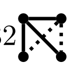
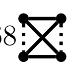
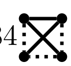
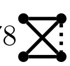
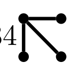
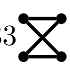
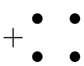
$$
\begin{aligned}
0 \leq & 24 \cdot \left[\!\left[F_1'^T M' F_1' \right]\!\right]_{\sigma_1} \\
= & 24 \cdot \left[\!\left[F_1'^T \begin{pmatrix} 0.0744 & -0.0223 & -0.0520 \\ -0.0223 & 0.0238 & -0.0014 \\ -0.0520 & -0.0014 & 0.0536 \end{pmatrix} F_1' \right]\!\right]_{\sigma_1}
\end{aligned}
$$

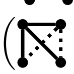
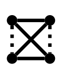
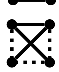
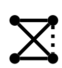
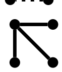
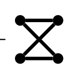
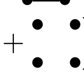
$=(0.0744 \times 2 - 0.0520 \times 4 + 0.0238 \times 2)$  $- 0.0223 \times 16$  $- 0.0223 \times 8$ 

$\quad - 8 \times 0.0014$  $+ 6 \times 0.0536$ 

$= - 0.0116$  $- 0.3568$  $- 0.1784$  $- 0.0112$  $+ 0.3216$ .

We subtract the result from (3) and obtain

 ≥ 0.1782  $+ 0.3568$  $+ 0.1784$  $+ 0.1778$  $+ 0.1784$  $+ 0.33$  $+$ 

$> 0.17 \big($  $+$  $+$  $+$  $+$  $+$  $+$  $\big)$

$$= 0.17 > \frac{1}{6}.$$

□

Note that the last inequality is far from tight and there is no need for exact rounding to transform the resulting matrix of the semidefinite program into M'. However, we cannot extract any further information about the Ramsey graph.

Acknowledgements

The research reported in this paper is partially supported by the HPC@ISU equipment at Iowa State University, some of which has been purchased through funding provided by NSF under MRI grant number CNS 1229081 and CRI grant number 1205413.

This work used computer clusters at the Center for Computational Mathematics, University of Colorado Denver, purchased thanks to NSF Grants AGS 0835579 (Gross) and CNS 0958354 (Colibri).

This work utilized the Janus supercomputer, which is supported by the NSF (award number CNS-0821794) and the University of Colorado Boulder. The Janus supercomputer is a joint effort of the University of Colorado Boulder, the University of Colorado Denver and the National Center for Atmospheric Research.

We are also grateful to the Charles University in Prague for providing to us access to the computers Kaminka, and Kamenozrout.

The first author gratefully acknowledges the support and hospitality and access to computing resources provided by the IMA during his visit which took place from Sep, 2014 to Dec, 2014.

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A All attempted bounds

The following table provides a summary of computations we performed, where shaded rows correspond to improved upper bounds. The purpose of the table is to illustrate the size of the computations, and to also show our attempts where the method provided an upper bound that did not improve on the best known one. The basic parameter of computations is the order n of unlabeled flags. A bigger value of n typically gives a better result. On the other hand, the number of graphs of order n grows quickly and becomes unmanageable soon.

One of the main issues is the memory needed by CSDP when solving the semidefinite program. The memory demands grow quickly with the number of unlabeled flags. If the number of unlabeled flags is around 10,000, the instance is solvable on a desktop. Numbers under 100,000 will fit in 128G of memory, which requires a supercomputer. Numbers above 100,000 require high memory super computers. All instances we have tried fit in about 300G of memory. Even larger instances than we tried could be solvable as very high memory nodes may have even terabytes of memory, but one would have to be very patient.

The running time also depends heavily on the number of unlabeled flags. CSDP solver runs in iterations and it took 30 to 60 iterations to solve most of the problems in this class. The larger instances compute a few iterations per day to a couple of days per iteration on the supercomputers we use. Let us mention that we obtained a significant speedup ($10\times$) of the CSDP solver by compiling it with Intel Math Kernel Library.

Previous bounds	**Order**	**Flags**	**Our upper bound**
$R(K_3, K_6) = 18$	8	1418	$\lfloor 18.54 \rfloor$
$R(K_3, K_7) = 23$	10	37133	$\lfloor 23.96 \rfloor$
$R(K_3, K_8) = 28$	10	38322	$\lfloor 29.99955 \rfloor$
$R(K_3, K_9) = 36$	10	38440	$\lfloor 38.224 \rfloor$
$40 \leq R(K_3, K_{10}) \leq 42$	10	38450	$\lfloor 54.85 \rfloor$
$R(K_4, K_5) = 25$	9	134037	$\lfloor 28.31 \rfloor$
$36 \leq R(K_4, K_6) \leq 41$	8	11667	$\lfloor 44.12 \rfloor$
$49 \leq R(K_4, K_7) \leq 61$	8	11765	$\lfloor 67.54 \rfloor$
$59 \leq R(K_4, K_8) \leq 84$	8	11773	$\lfloor 150.33 \rfloor$
$43 \leq R(K_5, K_5) \leq 48$	8	8722	$\lfloor 53.45 \rfloor$
$58 \leq R(K_5, K_6) \leq 87$	8	18503	$\lfloor 96.38 \rfloor$
$80 \leq R(K_5, K_7) \leq 143$	8	18601	$\lfloor 183.72 \rfloor$
$102 \leq R(K_6, K_6) \leq 165$	8	9795	$\lfloor 205.0016 \rfloor$
$29 \leq R(K_4^-, K_8^-) \leq 38$	9	23398	$\lfloor 32.997 \rfloor$
$34 \leq R(K_4^-, K_9^-) \leq 53$	9	23427	$\lfloor 46.29 \rfloor$
$30 \leq R(K_4, K_6^-) \leq 33$	8	11372	$\lfloor 33.3 \rfloor$
	9	150078	in progress
$37 \leq R(K_4, K_7^-) \leq 52$	8	11747	$\lfloor 49.77 \rfloor$
$31 \leq R(K_5^-, K_6^-) \leq 39$	8	14889	$\lfloor 38.7 \rfloor$
$40 \leq R(K_5^-, K_7^-) \leq 66$	8	15286	$\lfloor 65.007 \rfloor$
$R(K_5^-, K_8^-) \leq 100$	8	15311	$\lfloor 113.21 \rfloor$
$30 \leq R(K_5, K_5^-) \leq 33$	8	14169	$\lfloor 35.22 \rfloor$
$43 \leq R(K_5, K_6^-) \leq 66$	8	18186	$\lfloor 62.96 \rfloor$
$58 \leq R(K_5, K_7^-) \leq 110$	8	18583	$\lfloor 102.81 \rfloor$
$45 \leq R(K_6^-, K_6^-) \leq 70$	8	9478	$\lfloor 71.09 \rfloor$
$59 \leq R(K_6^-, K_7^-) \leq 135$	8	19339	$\lfloor 124.48 \rfloor$
$37 \leq R(K_6, K_5^-) \leq 53$	8	15206	$\lfloor 55.92 \rfloor$
$58 \leq R(K_6, K_6^-) \leq 110$	8	19259	$\lfloor 111.09 \rfloor$
$R(K_6, K_7^-) \leq 205$	8	19656	$\lfloor 245.64 \rfloor$
$28 \leq R(K_7, K_4 -) \leq 30$	9	23315	$\lfloor 29.92 \rfloor$
$51 \leq R(K_7, K_5^-) \leq 83$	8	15304	$\lfloor 86.52 \rfloor$
$80 \leq R(K_7, K_6^-) \leq 192$	8	19357	$\lfloor 210.36 \rfloor$
$29 \leq R(K_8, K_4^-) \leq 42$	9	23419	$\lfloor 39.18 \rfloor$
$R(K_9, K_4^-)$	9	23427	$\lfloor 46.29 \rfloor$
$R(K_{3,4}, K_{2,5}) \leq 21$	8	16649	$\lfloor 20.988 \rfloor$

$R(K_{3,4}, K_{3,3}) \leq 25$	8	14529	$\lfloor 20.97 \rfloor$
$R(K_{3,4}, K_{3,4}) \leq 30$	8	8836	$\lfloor 25.14 \rfloor$
$15 \leq R(K_{3,5}, K_{1,6})$	8	14113	$\lfloor 17.01 \rfloor$ (tight)
$16 \leq R(K_{3,5}, K_{2,4})$	8	12327	$\lfloor 20.86 \rfloor$
$21 \leq R(K_{3,5}, K_{2,5})$	8	17591	$\lfloor 23.87 \rfloor$
$R(K_{3,5}, K_{3,3}) \leq 28$	8	15471	$\lfloor 24.35 \rfloor$
$R(K_{3,5}, K_{3,4}) \leq 33$	8	18600	$\lfloor 29.04 \rfloor$
$30 \leq R(K_{3,5}, K_{3,5}) \leq 38$	8	9778	$\lfloor 33.77 \rfloor$
$30 \leq R(K_{4,4}, K_{4,4}) \leq 62$	8	9837	$\lfloor 49.49 \rfloor$
$29 \leq R(K_8, C_5) \leq 33$	9	15067	$\lfloor 29.75 \rfloor$ (tight)
$33 \leq R(K_9, C_5)$	9	15076	$\lfloor 36.23 \rfloor$
	10	74556	in progress
$41 \leq R(K_9, C_6)$	9	25482	$\lfloor 41.70 \rfloor$ (tight)
$49 \leq R(K_9, C_7)$	9	49758	$\lfloor 58.69 \rfloor$
$46 \leq R(K_{10}, C_5)$	10	74566	in progress
$R(W_7, W_4)$	8	10114	$\lfloor 21.22 \rfloor$
$R(W_7, W_5)$	8	10361	$\lfloor 16.31 \rfloor$
$R(W_7, W_6)$	8	13780	$\lfloor 19.56 \rfloor$
$R(W_7, W_7)$	8	8048	$\lfloor 19.81 \rfloor$
$R(W_8, W_4)$	8	11391	$\lfloor 26.79 \rfloor$
$R(W_8, W_5)$	8	11748	$\lfloor 17.78 \rfloor$
$R(W_8, W_6)$	8	15217	$\lfloor 26.76 \rfloor$
$R(W_8, W_7)$	8	17547	$\lfloor 21.05 \rfloor$
$R(W_8, W_8)$	8	9519	$\lfloor 25.80 \rfloor$
$17 \leq R(B_4, B_5) \leq 20$	8	14456	$\lfloor 19.75 \rfloor$
$17 \leq R(B_3, B_6) \leq 22$	8	9568	$\lfloor 19.25 \rfloor$
$22 \leq R(B_5, B_6) \leq 26$	8	18543	$\lfloor 24.01 \rfloor$
$33 \leq R(W_5, K_6)$	8	12024	$\lfloor 36.86 \rfloor$
$43 \leq R(W_5, K_7)$	8	12122	$\lfloor 50.30 \rfloor$
$R(W_6, K_6)$	8	15439	$\lfloor 40.75 \rfloor$
$R(W_6, K_7)$	8	15591	$\lfloor 55.81 \rfloor$
$12 \leq R(Q_3, Q_3)$	9	116054	$\lfloor 14.041 \rfloor$ (tight)[1]
$30 \leq R(K_{2,2,2}, K_{2,2,2})$	8	8792	$\lfloor 32.89 \rfloor$
	9	147411	in progess
$R(K_3, K_3, K_4) = 30$	7	120737	$\lfloor 32.50 \rfloor$
$45 \leq R(K_3, K_3, K_5) \leq 57$	7	141516	$\lfloor 57.32 \rfloor$
$55 \leq R(K_3, K_4, K_4) \leq 77$	6	15625	$\lfloor 85.35 \rfloor$
$89 \leq R(K_3, K_4, K_5) \leq 158$	6	16272	$\lfloor 406.80 \rfloor$
$51 \leq R(K_3, K_3, K_3, K_3) \leq 62$	6	18571	$\lfloor 65.17 \rfloor$
$17 \leq R(C_3, C_5, C_5) \leq 21$	7	102305	$\lfloor 17.14 \rfloor$ (tight)
$15 \leq R(C_3, C_6, C_6)$	7	7283	$\lfloor 18.72 \rfloor$
$15 \leq R(C_5, C_6, C_6)$	6	11193	$\lfloor 17.92 \rfloor$

[1] We provide the tight bound 13 in this paper, but it was not obtained by direct FA computation.

$24 \leq R(C_3, C_4, C_4, C_4) \leq 27$	6	120853	$\lfloor 29.23 \rfloor$
$30 \leq R(C_3, C_3, C_4, C_4) \leq 36$	6	155664	$\lfloor 37.77 \rfloor$
$49 \leq R(C_3, C_3, C_3, C_4)$	6	88612	$\lfloor 59.22 \rfloor$
$20 \leq R(C_4, C_4, K_4) \leq 22$	7	192287	$\lfloor 21.78 \rfloor$
$27 \leq R(K_3, C_4, K_4) \leq 32$	6	9928	$\lfloor 32.93 \rfloor$
$52 \leq R(C_4, K_4, K_4) \leq 72$	6	9386	$\lfloor 71.56 \rfloor$
$34 \leq R(C_4, C_4, C_4, K_4) \leq 50$	6	170041	$\lfloor 48.22 \rfloor$
$43 \leq R(C_3, C_4, C_4, K_4) \leq 76$	5	4418	$\lfloor 157.25 \rfloor$
$28 \leq R(K_4^-, K_4^-, K_4^-) \leq 30$	6	2589	$\lfloor 28.51 \rfloor$ (tight)
$21 \leq R(K_3, K_4^-, K_4^-) \leq 27$	7	145774	$\lfloor 22.70 \rfloor$
$33 \leq R(K_4, K_4^-, K_4^-) \leq 59$	6	9476	$\lfloor 47.39 \rfloor$
$55 \leq R(K_4, K_4, K_4^-) \leq 113$	6	11410	$\lfloor 91.981 \rfloor$
$28 \leq R(C_4, K_4, K_4^-) \leq 36$	6	15170	$\lfloor 36.85 \rfloor$
$30 \leq R(K_3, K_4, K_4^-) \leq 41$	6	12554	$\lfloor 40.36 \rfloor$
$R(K_4, K_4; 3) = 13$	7	16169	$\lfloor 15.35 \rfloor$
$14 \leq R(K_4^-, K_5; 3)$	7	5802	$\lfloor 16.41 \rfloor$
$13 \leq R(K_4^-, K_4^-, K_4^-; 3) \leq 16$	6	1345	$\lfloor 14.65 \rfloor$
$R(K_4^-, K_5^-; 3)$	8	1432	$\lfloor 12.00 \rfloor$ (tight)
$32 \leq R(TT_7) \leq 54$	8	5790	$\lfloor 56.67 \rfloor$
	9	126456	in progress
$R(TT_8) \leq 108$	8	5848	$\lfloor 128.756 \rfloor$
	9	132045	in progress