5-colouring graphs with 4 crossings *

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Abstract

We disprove a conjecture of Oporowski and Zhao stating that every graph with crossing number at most 5 and clique number at most 5 is 5-colourable. However, we show that every graph with crossing number at most 4 and clique number at most 5 is 5-colourable. We also show some colourability results on graphs that can be made planar by removing few edges. In particular, we show that if a graph with clique number at most 5 has three edges whose removal leaves the graph planar, then it is 5-colourable.

1 Introduction

The crossing number of a graph G, denoted by $\mathrm{cr}(G)$, is the minimum number of crossings in any drawing of G in the plane.

The Four Colour Theorem states that if a graph has crossing number zero then it is 4-colourable. A natural question is to ask whether the chromatic number is bounded in terms of its crossing number. To answer the question, the concept of crossing cover is crucial. A *crossing cover* is a set of vertices C such that every crossing has an edge incident with a vertex in C. If C is a crossing cover then $G - C$ is planar, so $\chi(G) \leq 4 + \chi(G\langle C\rangle) \leq 4 + |C|$. Picking one vertex per crossing, we obtain a crossing cover of cardinality at most $\mathrm{cr}(G)$ so $\chi(G) \leq 4 + \mathrm{cr}(G)$.

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This upper bound is tight only for $\mathrm{cr}(G) \leq 1$. So it is natural to ask for the smallest integer $f(k)$ such that every graph G with crossing number at most k is $f(k)$-colourable? An argument similar to the one above shows that $f(k+1) \leq f(k)+1$. Settling a conjecture of Albertson [1], Schaefer [13] showed that $f(k) = O\left(k^{1/4}\right)$. This upper bound is tight up to a constant factor since $\chi(K_n) = n$ and $\mathrm{cr}(K_n) \leq \binom{|E(K_n)|}{2} = \binom{\binom{n}{2}}{2} \leq \frac{1}{8}n^4$.

Only few exact values on $f(k)$ are known. The Four Colour Theorem states $f(0) = 4$ and implies easily that $f(1) \leq 5$. Since $\mathrm{cr}(K_5) = 1$, we have $f(1) = 5$. Oporowski and Zhao [12] showed that $f(2) = 5$. Since $\mathrm{cr}(K_6) = 3$, we have $f(3) = 6$. Further, Albertson et al. [2] showed that $f(6) = 6$.

A graph G is *r-critical* if $\chi(G) = r$ and $\chi(G') < r$ for every proper subgraph G' of G. Oporowski and Zhao [12] proved that K_6 is the unique 6-critical graph with crossing number 3.

Theorem 1.1 (Oporowski and Zhao [12]). *If* $\mathrm{cr}(G) \leq 3$ *and* $\omega(G) \leq 5$ *then* $\chi(G) \leq 5$.

Oporowski and Zhao [12] asked whether the conclusion remains true even if $\mathrm{cr}(G) \in \{4, 5\}$.

Problem 1.2 (Oporowski and Zhao [12]). *If* $\mathrm{cr}(G) \leq 5$ *and* $\omega(G) \leq 5$*, is* G *5-colourable?*

We answer in the negative by showing a counterexample. The help of Zdeněk Dvořák was greatly appreciated while obtaining this result.

Theorem 1.3. *There exists a graph* G *such that* $\mathrm{cr}(G) = 5$, $\omega(G) \leq 5$ *and* $\chi(G) = 6$.

On the other hand we answer in the affirmative when $\mathrm{cr}(G) = 4$.

Theorem 1.4. *If* $\mathrm{cr}(G) \leq 4$ *and* $\omega(G) \leq 5$ *then* $\chi(G) \leq 5$.

A key notion in the proof of Theorem 1.4 is the one of dependent crossings. The *cluster* of a crossing is the set of endvertices of its two edges. Two crossings are *dependent* if their clusters intersects.

Settling a conjecture of Albertson [1], Král' and Stacho [11] showed the following.

Theorem 1.5 (Král' and Stacho [11]). *If a graph* G *has a drawing in the plane in which no two crossings are dependent, then* $\chi(G) \leq 5$.

Loosely speaking, this theorem states that if the crossings are far apart from each other then the graph is 5-colourable. On the other hand, if all the crossings are very close, that is if all their clusters share a common vertex, then the graph is also 5-colourable. In the same vein, we show that if the crossings are covered by $2k$ edges then the graph is $(4+k)$-colourable (Theorem 4.1). In particular, if the crossings are covered by three edges then the graph is 6-colourable. This bound 6 is tight since $\mathrm{cr}(K_6) = 3$ and thus one can remove three edges from K_6 to make it planar. However, by generalizing Theorem 1.1, we show that K_6 is essentially the unique obstruction for such a graph to be 5-colourable.

Theorem 1.6. *If* $\omega(G) \leq 5$ *and there exists a set* F *of at most three edges such that* $G \setminus F$ *is planar then* $\chi(G) \leq 5$.

Related open problems are discussed in the final section.

2 Preliminaries

2.1 Drawings of graphs

A *drawing* $\tilde{G}$ (in the plane or the sphere) of a graph $G = (V, E)$ consists of a bijection D from $V \cup E$ into a set $\tilde{V} \cup \tilde{E}$ such that

(i) $\tilde{V}$ is the image of V and a set of distincts points in the plane;

(ii) for any edge $e = uv$, the element $D(e) = \tilde{e}$ of $\tilde{E}$ is the image of a continuous injective mapping ϕ_e from $[0, 1]$ to the plane which is simple (i.e. does not intersect itself) such that $\phi_e(0) = D(u)$, $\phi_e(1) = D(v)$ and $\phi_e(]0, 1[) \cap \tilde{V} = \emptyset$;

(iii) every point in the plane is in at most two images of edges unless it is in $\tilde{V}$;

(iv) for two distincts edges e_1 and e_2 of E, $\tilde{e_1}$ and $\tilde{e_2}$ intersects in a finite number of points.

We shall often confound the vertex and edge sets of a graph with their image in one of its drawings.

A *crossing* in a drawing of G is a point in the plane minus $\tilde{V}$ that belongs to two edges. Formally, it is a point of $\phi_{e_1}(]0, 1[) \cap \phi_{e_2}(]0, 1[)$ for some edges e_1 and e_2. A *portion* of an edge e is a subarc of $\phi_e[0, 1]$ between two consecutives endpoints or crossings on e. A portion from a to b is called an *(a, b)-portion*.

A graph is *planar* if it has a drawing without crossings. An easy consequence of Euler's Formula is the following well known proposition.

Proposition 2.1. *If G is planar then $|E(G)| \leq 3|V(G)| - 6$.*

A drawing of G is *optimal* if it minimizes the number of crossings. Note that two edges may intersect several times, either in endvertices or crossings. However, thanks to the two following lemmas, we will only consider *nice* drawings, i.e. drawings such that two edges intersect at most once.

Lemma 2.2. *Let G be a graph. If $cr(G) \leq k$ then G has a nice drawing with at most k crossings.*

Proof. Consider an optimal drawing of G that minimizes the number of crossings between edges with a common vertex. Suppose, by contradiction, that two edges $e_1 = u_1v_1$ and $e_2 = u_2v_2$ intersect at least twice. Let a and b be two points in the intersection of e_1 and e_2. Without loss of generality we may assume that u_1, u_2, v_1, and v_2 are in the exterior of the closed curve C which is the union of the (a, b)-portion P_1 on u_1v_1 and the (a, b)-portion P_2 on u_2v_2. We may also assume that P_1 contains at least as many crossings as P_2.

Then one can redraw u_1v_1 along the (u_1, a)-portion of e_1, P_2, and the (b, v_1)-portion of e_1 slightly in the exterior of C so that e_1 and e_2 do not cross anymore. Doing so, all the crossings of P_1 including a and b (if they were crossings) disappear while a crossing is created per crossings of P_2 distinct from a and b. Since one of $\{a, b\}$ must be a crossing (there are no parallel arcs), we obtain a drawing with one crossing less, a contradiction. □

Similarly, one can show the following lemma.

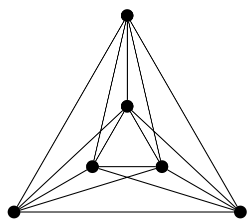

Figure 1: Drawing of K_6 with three crossings.

Lemma 2.3. *Let G be a graph. Assume that there is a set F of k edges such that $G \setminus F$ is planar. Then there exists a nice drawing of G such that each crossing contains at least one edge of F.*

In this paper, we consider only nice drawings. Thus a crossing is uniquely defined by the pair of edges it belongs to. Henceforth, we will often confound a crossing with this set of two edges.

A *face* of a drawing $\tilde{G}$ is a connected component of the space obtained by deleting $\tilde{V} \cup \tilde{E}$ from the plane. We let $F(\tilde{G})$ (or simply F) be the set of faces of $\tilde{G}$. We say that a vertex v or a portion of an edge is incident to $f \in \tilde{F}$ if v is contained in the closure of f. The boundary of f , denoted by $bd(f)$ consists of the vertices and maximum (with respect to inclusion) portions of edges incident to it. An *embedding* of a graph is the set of boundaries of the faces of some drawing of G in the plane.

Lemma 2.4. *Free to rename the vertices, there is only one embedding of K_6 using exactly three crossings. (See Figure 1.)*

Proof. Let A be an embedding of K_6 using three crossings. Let us show that it is unique. First we observe that every edge is crossed at most once. Otherwise, there will be two edges whose removal leaves the graph planar which is a contradiction to Proposition 2.1. As every cluster of a crossing contains four vertices, there must be a vertex v contained in two of them. Note that v cannot be in all three clusters since $K_6 - v$ (which is isomorphic to K_5) is not planar. Let $e_1 = vx$ and $e_2 = vy$ be the two crossed edges adjacent to v and e_3 one of the edges of the crossing whose cluster does not contain v. $K_6 \setminus \{e_1, e_2, e_3\}$ is a planar triangulation T where $\deg(v) = 3$.

We denote a, b, c the neighbours of v in T. They must induce a triangle. Without loss of generality, ab and bc are the edges crossed by e_1 and e_2, respectively.

As T is a triangulation abx and bcy form triangles. Moreover, xby is also a triangle as x and y are consecutive neighbours around b. The last two edges, which are not discussed yet, are xc and ya. They must cross inside $bxyc$ (one of them is e_3). Hence A is unique. □

Lemma 2.5. *A drawing of K_5 with all vertices incident to the same face requires 5 crossings.*

Proof. Let us number the vertices of K_5 v_1, v_2, v_3, v_4, v_5 in the clockwise order around the boundary of the face f incident to them. Then free to redraw the edges v_1v_2, v_2v_3, v_3v_4, v_4v_5 and v_5v_1, we may assume that the boundary is the cycle $v_1v_2v_3v_4v_5$ and that f is its interior. Now both v_1v_3 and v_2v_4 are in the exterior of C and thus must cross. Similarly, $\{v_2v_4, v_3v_5\}$, $\{v_3v_5, v_4v_1\}$, $\{v_4v_1, v_5v_2\}$ and $\{v_5v_2, v_1v_3\}$ are crossings. □

Lemma 2.6. *A drawing of $K_{2,3}$ such that vertices of each part are in a common face requires at least one crossing.*

Proof. Let $(\{u_1, u_2\}, \{v_1, v_2, v_3\})$ be the bipartition of $K_{2,3}$. Suppose by contradiction that $K_{2,3}$ has a drawing such that each part of the bipartition is in a common face. Then adding a vertex u_3 is the face incident to the vertices v_1, v_2 and v_3 and connecting u_3 to those vertices by new edges yields a drawing of $K_{3,3}$ with no crossing which contradicts the fact that $K_{3,3}$ is not planar. □

2.2 Properties of 6-critical graphs

Let G be a graph and a drawing of it. A *stable crossing cover* is a set which is both stable and a crossing cover.

Lemma 2.7. *If G has a stable crossing cover W then G is 5-colourable.*

Proof. Use the Four Colour Theorem on $G - W$ and extend the colouring to G by using a fifth colour on W. □

Let G be a graph and u, v be vertices of G. The operation of *identification* of u and v in G results in a graph denoted by $G/\{u, v\}$, which is obtained from $G - \{u, v\}$ by adding a new vertex w and the set of edges $\{wz \mid uz \text{ or } vz \text{ is an edge of } G\}$.

Lemma 2.8. *Let G be a graph and v be a 5-vertex of G. Let u and w be two non-adjacent neighbours of v. If $(G - v)/\{u, w\}$ is 5-colourable then so is G.*

Proof. A proper 5-colouring of $(G-v)/\{u, w\}$ corresponds to a proper 5-colouring of $G-v$ such that u and w are coloured by the same colour. So it can be extended to a proper 5-colouring of G by assigning a colour to v. □

Let G be a graph and a drawing of it in the plane. A cycle is *separating* if it has a vertex in its interior and a vertex in its exterior. A cycle C is *non-crossed* if all its edges are non-crossed. It is *regular* if any cluster of a crossing containing an edge of C contains at least three vertices of C.

Lemma 2.9. *Let G be a 6-critical graph. In every drawing of G in the plane, there is no separating regular 3-cycle.*

Proof. Suppose, by way of contradiction, that there is a regular 3-cycle C. Let G_1 be the graph induced by the vertices in C and inside C and G_2 a graph induced by the vertices in C and outside C. Since C is separating both G_1 and G_2 have less vertices than G. Hence, by 6-criticality of G, they are 5-colourings of those graphs. In addition, in both colourings, the colours of the vertices of C are distinct. So, free to permute the colours, one can assume that the two 5-colourings of G_1 and G_2 agree on C. Hence their union yields a 5-colouring of G. □

Lemma 2.10. *Let G be a 6-critical graph distinct from K_6. In every nice drawing of G, there is no separating triangle such that*

- *at most one of its edges is crossed, and*
- *there is at most one crossing in its interior.*

Proof. Suppose, by way of contradiction, that such a cycle $C = x_1x_2x_3$ exists. Then by Lemma 2.9, one of its edges, say x_2x_3, is crossed. Let uv be the edge crossing it with u inside C and v outside. By Lemma 2.9, C is not regular, so $u \neq x_1$. Moreover, $u \notin \{x_2, x_3\}$ since the drawing is nice.

Let G_1 be the graph induced by C and the vertices outside C. Then G_1 admits a 5-colouring c_1 since G is 6-critical.

Let G_2 be the graph obtained from the graph induced by C and the vertices inside C by adding the edges ux_1, ux_2 and ux_3 if they do not exist. Observe that G_2 has a planar drawing with at most 2 crossings. Indeed the edge ux_1 may be drawn along uv and then a path in the outside of C and the edges ux_2 and ux_3 may be drawn along the edges of the crossing $\{x_2x_3, uv\}$. Thus G_2 admits a 5-colouring c_2.

In both colourings, the colours of the vertices of C are distinct. So, free to permute the colours, we may assume that c_1 and c_2 agree on C. One can also choose for u a colour of $\{1, \dots, 5\} \setminus \{c_2(x_1), c_2(x_2), c_2(x_3)\}$ so that $c_2(u) \neq c_1(v)$. Then the union of c_1 and c_2 is a 5-colouring of G. □

Lemma 2.11. *Let G be a 6-critical graph. In every drawing of G in the plane, there is no non-crossed 4-cycle C such that*

- *C has a chord in its exterior,*
- *C and its interior is a plane graph, and*
- *the interior of C contains at least one vertex.*

Proof. Suppose, by way of contradiction, that there is a 4-cycle $C = tuvw$ satisfying the properties above with vt a chord in its exterior. Consider the graph G_1, which is obtained from G by removing the vertices inside C. Since G is 6-critical, G_1 admits a 5-colouring c_1 in $\{1, 2, 3, 4, 5\}$. Without loss of generality, we may assume that $c_1(v) = 5$. Hence $\{c_1(t), c_1(u), c_1(w)\} \subset \{1, 2, 3, 4\}$.

Now consider the graph G_2 which is obtained from G by removing the vertices outside C. If $c_1(u) = c_1(w)$, let H be the graph obtained from $G_2 - v$ by identifying u and w. If $c_1(u) \neq c_1(w)$, let H be the graph obtained from $G_2 - v$ by adding the edge uw if it does not already exist. In both cases H is a planar graph. Hence H admits a 4-colouring c_2 in $\{1, 2, 3, 4\}$. Moreover, by construction of H, $c_2(u) = c_2(w)$ if and only if $c_1(u) = c_1(w)$. Hence free to permute the colours, we may assume that c_1 and c_2 agree on $\{t, u, w\}$.

Hence the union of c_1 and c_2 is a 5-colouring of G. □

2.3 6-critical graphs embeddable on the torus or the Klein bottle

In the proof of Theorem 1.6, we use the list of all 6-critical graphs embeddable on the torus which was obtained by Thomassen [16] and the list of all 6-critical graphs embeddable on the Klein bottle which was obtained independently by Chenette et al. [3] and Kawarabayashi et al. [10].

Theorem 2.12 (Thomassen [16])**.** *There are four non-isomorphic 6-critical graphs embeddable on the torus. Three of them are depicted in Figure 2 and the last one is a 6-regular graph on 11 vertices.*

Theorem 2.13 (Chenette et. al. [3] ; Kawarabayashi et al. [10])**.** *There are nine non-isomorphic 6-critical graphs embeddable on the Klein bottle. They are depicted in Figure 2.*

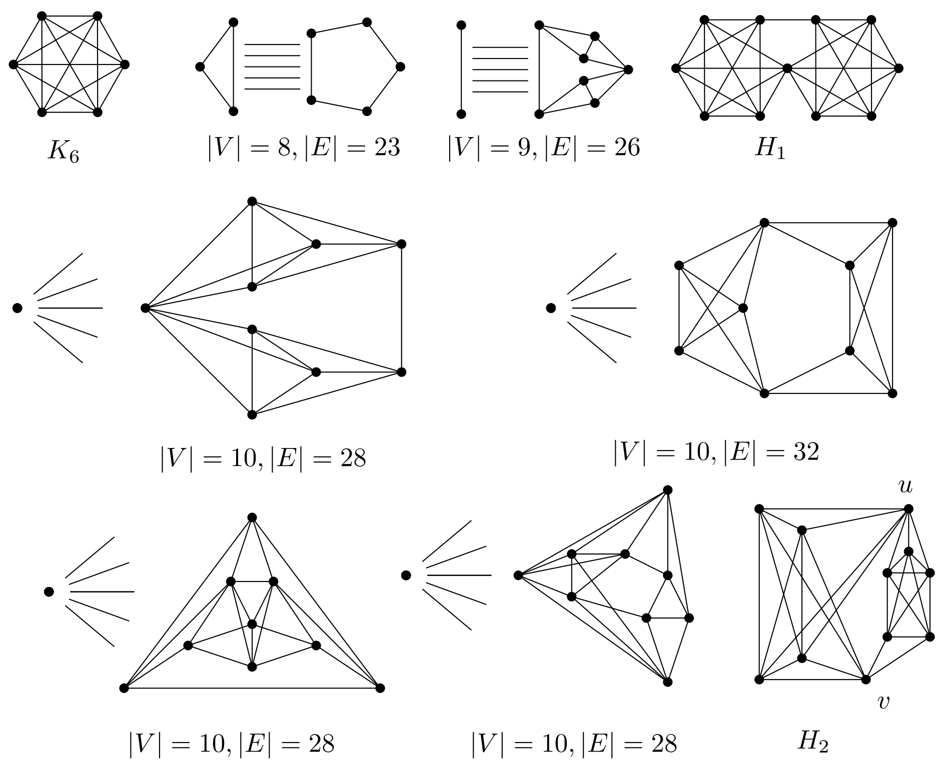

Figure 2: 6-critical graphs embeddable on the Klein bottle. The first three of them are embeddable on torus as well.

Lemma 2.14. *Let G be a 6-critical graph embeddable on the torus different from K_6. Then it is not possible to make G planar by removing three edges.*

Proof. We know the complete list of graphs which must be checked due to Theorem 2.12. For all of them except K_6, we have $|E| > 3|V| - 3$. Thus the graphs are not planar after removing three edges according to Proposition 2.1. $\square$

Lemma 2.15. *Let G be a 6-critical graph embeddable on the Klein bottle different from K_6. Then it is not possible to make G planar by removing three edges.*

Proof. We know the complete list of graphs which must be checked due to Theorem 2.13. For all of those graphs except K_6, H_1 and H_2, we have $|E| > 3|V| - 3$. Thus those graphs are not planar after removing three edges according to Proposition 2.1.

Now we need to deal with the last two graphs H_1 and H_2, see Figure 2. Let us first examine H_1. It contains two edge disjoint copies of K_6 without one edge. Each of these copies needs at least two edges to be removed by Proposition 2.1, so H_1 needs at least four edges to be removed.

Let us now examine H_2. Let F be a set of edges such that $H_2 \setminus F$ is planar. Let us denote by u and v the two vertices of the only 2-cut of H_2, see Figure 2. Observe that $H_2 - \{u, v\}$ is a disjoint union of K_5 and K_4. Since K_5 is not planar, one edge e of this K_5 is in F. But there is still a (u, v)-path P in $K_5 \setminus e$. Then the union of the graph induced by u, v, the vertices of the K_4 and the path P is a subdivision of K_6. Thus, by Proposition 2.1 for K_6, at least three of its edges must be in F. Thus $|F| \geq 4$. $\square$

3 6-critical graph of crossing number 5

We prove Theorem 1.3 by exhibiting a drawing of a 6-critical graph G using 5 crossings which is not K_6.

Theorem 3.1. *The graph G depicted in Figure 3 is 6-critical.*

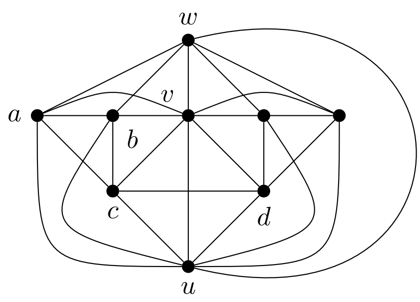

Figure 3: A 6-critical graph of crossing number 5.

Proof. We show by contradiction that G is not 5-colourable. We refer the reader to Figure 3 for names of vertices. Assume that ϱ is a 5-colouring of G. As vertices u, v and w form a triangle, they must get distinct colours. Without loss of generality, assume that $\varrho(u) = 1, \varrho(v) = 2$ and $\varrho(w) = 3$. The vertices a and b are adjacent to each other and to all the vertices of the triangle uvw, hence $\{\varrho(a), \varrho(b)\} = \{4, 5\}$. Thus $\varrho(c) = 3$ as c is adjacent to a, b, u and v. By symmetry we obtain that $\varrho(d)$ is also 3, which is a contradiction since cd is an edge.

It can be easily checked that every proper subgraph of G is 5-colourable. So G is 6-critical. □

4 Colouring graphs whose crossings are covered by few edges

Theorem 4.1. *Let G be a graph. If there is a set F of at most $2k$ edges such that $G \setminus F$ is planar then G is $(4 + k)$-colourable.*

Proof. We procced by induction on k. The result holds when $k = 0$ by the Four Colour Theorem.

Suppose that the result is true for k. Let $G = (V, E)$ be a graph with a set F of at most $2k + 2$ edges such that $G \setminus F$ is planar. Without loss of generality, we may assume that F is minimal, i.e. for any proper subset $F' \subset F$, $G \setminus F'$ is not planar.

Consider a planar drawing of $G \setminus F$. It yields a drawing of G such that each crossing contains an edge of F.

Suppose that $|F| \leq 2k+1$. Let $e = uv$ be an edge of F. By the induction hypothesis, $G - v$ is $(4 + k)$-colourable because $F \setminus e$ is a set of $2k$ edges whose removal leaves $G - v$ planar. Hence $\chi(G) \leq \chi(G - v) + 1 \leq 4 + k + 1$.

So we may assume that $|F| = 2k + 2$.

If two edges e and f of F have a common vertex v, then $G - v$ is $(4 + k)$-colourable because $F \setminus \{e, f\}$ is a set of $2k$ edges whose removal leaves $G - v$ planar. So $\chi(G) \leq \chi(G-v)+1 \leq 4+k+1$. So we may assume that the edges of F are pairwise non-adjacent.

Let $e = \{u_1, u_2\}$ and $f = \{v_1, v_2\}$ be two edges in F. Then the endvertices of these two edges induce a K_4. Suppose for contradiction that u_1 and v_1 are not adjacent. Then $G-\{u_1, v_1\}$ is $(4+k)$-colourable because $F\setminus\{e, f\}$ is a set of $2k$ edges whose removal leaves $G-\{u_1, v_1\}$ planar and u_1 and v_1 can get the same colour. So $\chi(G) \leq \chi(G-\{u_1, v_1\})+1 \leq 4 + k + 1$. Hence $X = \{u_1, u_2, v_1, v_2\}$ induces a K_4.

We further distinguish two possible cases:

$k = 0$: Let the edges of F be $e = \{u_1, u_2\}$ and $f = \{v_1, v_2\}$ and let $X = \{u_1, u_2, v_1, v_2\}$. Let C be the 4-cycle induced by X in the plane graph $G \setminus \{e, f\}$. Note that C is a separating cycle, otherwise $G \setminus e$ would be planar. We cut G along C and obtain two smaller graphs G_1 and G_2, where both of them contain X. We 5-colour them by induction. A colouring of G can be then obtained from the 5-colourings of G_1 and G_2 by permuting colours on X so that the these two colourings agree on $V(C)$.

$k \geq 1$: Note that union of all endvertices of edges from F induce a complete graph $K_{2|F|}$. A $K_{2|F|}$ must be planar after removing at most $|F|$ edges. Hence the following Euler's formula holds:

$$
\begin{aligned}
|E| &\leq 3|V| - 6 + 2k + 2 \\
\binom{4k+4}{2} &\leq 3(4k + 4) + 2k - 4 \\
8k^2 - 2 &\leq 0
\end{aligned}
$$

Hence this case is not possible.

□

Since $\mathrm{cr}(K_5) = 1$ and $\mathrm{cr}(K_6) = 3$, Theorem 4.1 is tight when $k \leq 2$. But K_6 is the only obstacle for pushing the result further as shown by the following theorem which is equivalent to Theorem 1.6.

Theorem 4.2. *Let G be a 6-critical graph distinct from K_6. Then for any set F of at most three edges, $G \setminus F$ is not planar.*

Proof. Let us consider a nice drawing of G. By Lemma 2.7, G has no stable crossing cover.

If $|F| \leq 2$ then the result is implied by Theorem 4.1. Hence we assume that $F = \{e_1, e_2, e_3\}$. Set $e_i = u_i v_i$ for $i \in \{1, 2, 3\}$.

Claim 1. *The three edges of F are pairwise vertex-disjoint.*

Proof. If there is a vertex v shared by all three edges then $\{v\}$ is a stable crossing cover, a contradiction. Hence a vertex u is shared by at most two edges of F. Let s be the number of 2-vertices in the graph induced by F.

We now derive a contradiction for each value of $s > 0$. So $s = 0$ which proves the claim.

s = 1: W.l.o.g. $u = u_1 = u_2$. None of $\{u, u_3\}$ and $\{u, v_3\}$ is a stable crossing cover so uu_3 and uv_3 are edges. We redraw the edge e_3 along the path u_3uv_3 such that it crosses only edges incident to u. See Figure 4(A). Then u is a stable crossing cover, a contradiction.

s = 2: W.l.o.g. $u = u_1 = u_2$ and $v = v_2 = v_3$. Then F induces a path. None of $\{v_1, v\}$ and $\{u, u_3\}$ is a stable crossing cover, so v_1v and uu_3 are edges. We add a handle between vertices u and v. Then we draw edges of F using the handle, see Figure 4(B). Hence G can be embedded on the torus, which is a contradiction to Lemma 2.14.

s = 3: W.l.og. $u = u_1 = u_2$ is one of the shared vertices. Let v and w be the other two. Note that F induces a triangle. By Proposition 2.1, $|E(G)| \leq 3|V(G)| - 3$. Hence there must be at least 6 vertices of degree five as the minimum degree of G is five.

Let x be a 5-vertex different from u, v and w. By minimality of G, there exists a 5-colouring ϱ of $G - x$. Free to permute the colours, we may assume that $\varrho(u) = 1$, $\varrho(v) = 2$ and $\varrho(w) = 3$. Moreover, the neighbours of x are coloured all differently. We denote by y and z the neighbours of x, which are coloured 4 and 5 respectively. We assume that G is embedded in the plane such that all crossings are covered by F. There are two consecutive neighbours of x in the clockwise order such that they have colours in $\{1, 2, 3\}$. We denote these vertices by a and b. Without loss of generality let the clockwise order around x be z, y, a, b and $\varrho(a) = 1$ and $\varrho(b) = 2$. See Figure 4(C).

Let A be the connected component of a in the graph induced by the vertices coloured 1 and 5. If A does not contain z, we can switch colours on it. Then x can be coloured by 1 and we have a contradiction. Note that the colour switch is correct even if u is in A because the new colour of u will be 5 which different from 2 and 3. Thus there must be a path between a and z of vertices coloured 1 and 5. A similar argument shows that there is a path between b and y of vertices coloured 2 and 4. These paths must be disjoint and they are not using edges of F. But they cannot be drawn in the plane without crossings, a contradiction.

□

Claim 2. *For any $i \neq j$, $i, j \in \{1, 2, 3\}$, an endvertex of e_i is adjacent to at most one endvertex of e_j.*

Proof. Suppose not. Then w.l.o.g. we may assume that u_2 is adjacent to u_1 and v_1. First we redraw the edge e_1 along the path $u_1u_2v_1$. Then every edge crossed by e_1, which is not e_3, is incident to e_2. Since $\{u_2, u_3\}$ and $\{u_2, v_3\}$ are not stable crossing covers, u_2u_3 and u_2v_3 are edges. We redraw e_3 along the path $u_3u_2v_3$. Then, again, every edge crossed by e_3, which is not e_1, is incident to e_2. Moreover, the edges e_1 and e_3 cross otherwise $\{u_2\}$ would be a stable crossing cover. See Figure 4(D).

We distinguish several cases according to the number p of neighbours of v_2 among u_1, v_1, u_3 and v_3.

$p = 0$: The vertex v_2 and a pair of two non-adjacent vertices among u_1, v_1, u_3 and v_3 would form a stable crossing cover. Hence $\{u_1, v_1, u_3, v_3\}$ induces a K_4. See Figure 4(E). By Lemma 2.9, there is no vertex inside each of the triangles $u_2u_1u_3$, $u_2u_3v_1$, $u_2v_1v_3$ and $u_2u_1v_3$. Hence all the vertices are inside the 4-cycle $u_1u_3v_1v_3$. It includes the vertex v_2. We redraw e_1 such that it is crossing only e_3 and u_2v_3. Then $\{v_3, v_2\}$ is a stable crossing cover, a contradiction. See Figure 4(F).

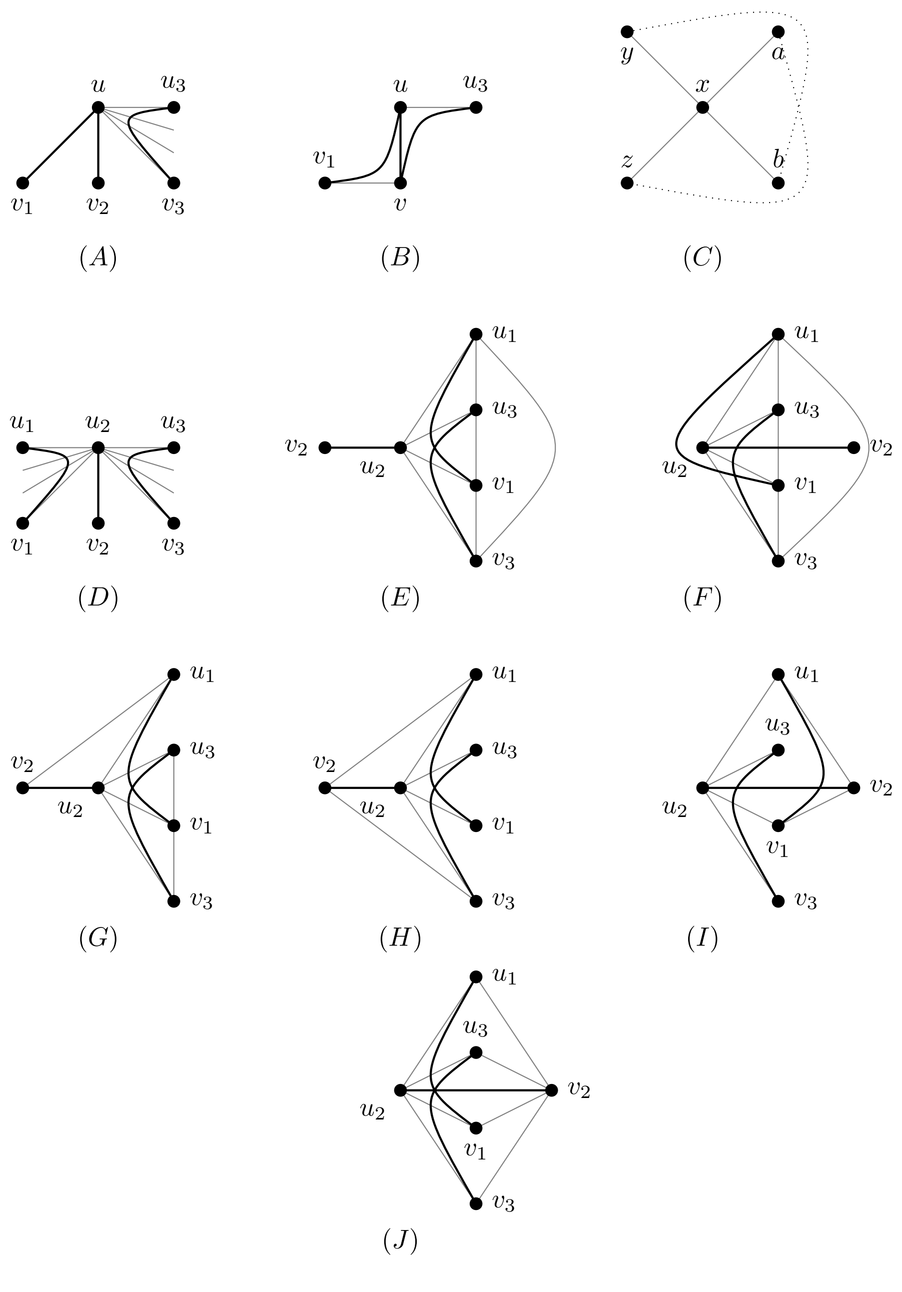

Figure 4: The three black edges are covering all the crossings.

$p = 1$: Without loss of generality we may assume that the neighbour of v_2 is u_1. None of $\{v_2, v_1, u_3\}$ and $\{v_2, v_1, v_3\}$ is a stable crossing cover so u_3v_1 and v_1v_3 are edges. By Lemma 2.9, there is no vertex inside each of the triangles $u_2u_3v_1$ and $u_2v_1v_3$. See Figure 4(G). Thus the edge e_3 could be drawn inside these triangles and the set F can be changed to $F' = \{e_1, e_2, u_2v_1\}$. Two edges of F' share an endvertex which is a contradiction to Claim 1.

$p \in \{2, 3\}$: We further distinguish two sub-cases. Either two neighbours of v_2 in $\{u_1, v_1, u_3, v_3\}$ are joined by an edge of F or not.

In the second case, without loss of generality, we may assume that the vertices adjacent to v_2 are u_1 and v_3. Now by Lemma 2.11 there is no vertex inside the 4-cycle $v_2u_1u_2v_3$. Hence e_2 can be drawn inside this cycle. See Figure 4(H). Since the removal of $\{e_1, e_3\}$ does not make G planar, v_1v_3 is inside $v_2u_1u_2v_3$. Hence the set $F' = \{e_1, e_3, u_1v_3\}$ contradicts Claim 1.

In the first case, we may assume w.l.o.g. that v_2 is adjacent to u_1 and v_1. We first redraw e_1 along the path $u_1v_2v_1$. Now all the edges crossing e_1 are incident to v_2. Thus $\{v_2, u_3\}$ or $\{v_2, v_3\}$ form a stable crossing cover. See Figure 4(I).

$p = 4$: See Figure 4(J). We repeatedly use Lemma 2.11 which implies that the 4-cycles $u_2u_3v_2u_1$, $u_2u_3v_2v_1$, $u_2v_1v_2v_3$ and $u_2v_3v_2u_1$ are not separating. This means that the graph contains only six vertices. This is a contradiction because the unique 6-critical graph on six vertices is K_6.

□

Since $\{u_1, u_2, u_3\}$ is not a stable crossing cover, it must induce at least one edge, say u_1u_2. Then Claim 2 implies that u_1v_2 and v_1u_2 are not edges. Now $\{v_1, u_2, u_3\}$ and $\{v_1, u_2, v_3\}$ are not stable crossing covers. Thus, by symmetry, we may assume that u_2u_3 and v_1v_3 are edges. $\{u_1, v_2, u_3\}$ is not a stable crossing cover so u_1u_3 is an edge; $\{v_1, v_2, u_3\}$ is not a stable crossing cover so v_1v_2 is an edge; $\{u_1, v_2, v_3\}$ is not a stable crossing cover so v_2v_3 is an edge. Hence there are two triangles $u_1u_2u_3$ and $v_1v_2v_3$, which are not separating by Lemma 2.9.

W.l.o.g. two possibilities occur. Either the edges of F do not cross each other or one pair of them is crossing. If they do not cross (Figure 5(A)), G can be embedded on the torus by adding a handle into the triangles and drawing the edges of F on the handle, which contradicts Lemma 2.14.

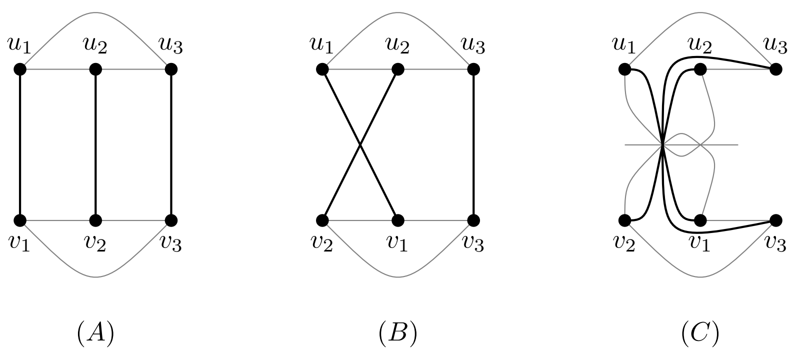

Figure 5: The last case of Theorem 1.6.

If they cross (Figure 5(B)), it is possible to draw G on the Klein bottle, see Figure 5(C), which contradicts Lemma 2.15. □

5 5-colouring graphs with 4 crossings

In this section we prove the following Theorem 5.1, which is equivalent to Theorem 1.4.

Theorem 5.1. *The unique* 6*-critical graph with crossing number at most* 4 *is* K_6*.*

Proof. Suppose, by way of contradiction, that $G = (V, E)$ is a 6-critical graph with crossing number at most 4 distinct from K_6. Moreover, one may assume that G is such a critical graph with the minimum number of vertices and with the maximum number of edges on $|V(G)|$ vertices.

Moreover, assume that we have a nice optimal drawing of G. By Theorem 1.6, there are four crossings and every edge is crossed at most once.

Since G is 6-critical, every vertex has degree at least 5. By Proposition 2.1, $|E| \leq 3|V| - 6 + \mathrm{cr}(G) \leq 3|V| - 2$. Hence there are at least four vertices of degree 5.

Let v be an arbitrary 5-vertex and v_i, $1 \leq i \leq 5$ be the neighbours of v in the counterclockwise order around v. By criticality of G, $G - v$ admits a 5-colouring ϕ. Necessarily, all the v_i are coloured differently, otherwise ϕ could be extended to v.

For any $i \leq j$, there is a path, denoted by $v_i - v_j$, from v_i to v_j such that all its vertices are coloured in $\phi(v_i)$ or $\phi(v_j)$. Otherwise, v_j is not in the connected component A of v_i in the graph induced by the vertices coloured $\phi(v_i)$ and $\phi(v_j)$. Hence by exchanging the colours $\phi(v_i)$ and $\phi(v_j)$ on A, we obtain a 5-colouring ϕ' of $G - v$ such that no neighbour of v is coloured $\phi(v_i)$. Hence by assigning $\phi(v_i)$ to v we obtain a 5-colouring of G, a contradiction.

Let q be the number of crossed edges incident to v.

Claim 3. $q \neq 0$*.*

Proof. The union of the $v_i - v_j$, $i \neq j$, is a subdivision of K_5 in $G - v$. If $q = 0$ then the v_i, $1 \leq i \leq 5$, are in one face after the removal of v. By Lemma 2.5, such a subdivision requires 5 crossings which contradicts the assumption of at most four crossings. □

Claim 4. $q \neq 1$*.*

Proof. Suppose to the opposite that $q = 1$. Without loss of generality, we may assume that the crossed edge is vv_1.

The path $v_2 - v_4$ must cross the two paths $v_1 - v_3$ and $v_3 - v_5$. Since every edge is crossed at most once then v_2v_4 is not an edge.

Let G' be the graph obtained from $G - v$ by identifying v_2 and v_4 into a new vertex v'. By Lemma 2.8, G' is not 5-colourable. Now G' has at most three crossings because we removed the crossed edge vv_1 together with v. So, by minimality of G, G' contains a subgraph H isomorphic to K_6. Moreover, H must contain v' since G contains no K_6. Since G' has only three crossings we can use Lemma 2.4. Let u_1 and u_2 be vertices of H which form a triangular face together with v' and let u_3, u_4 and u_5 be the vertices forming the other triangular face. Without loss of generality, we may assume that $u_3u_4u_5$ is inside $v'u_1u_2$ as in Figure 6(A).

Let us now consider the situation in G. Instead of discussing many rotations of K_6 we rather fix K_6 and try to investigate possible placings of v and its neighbours. We denote

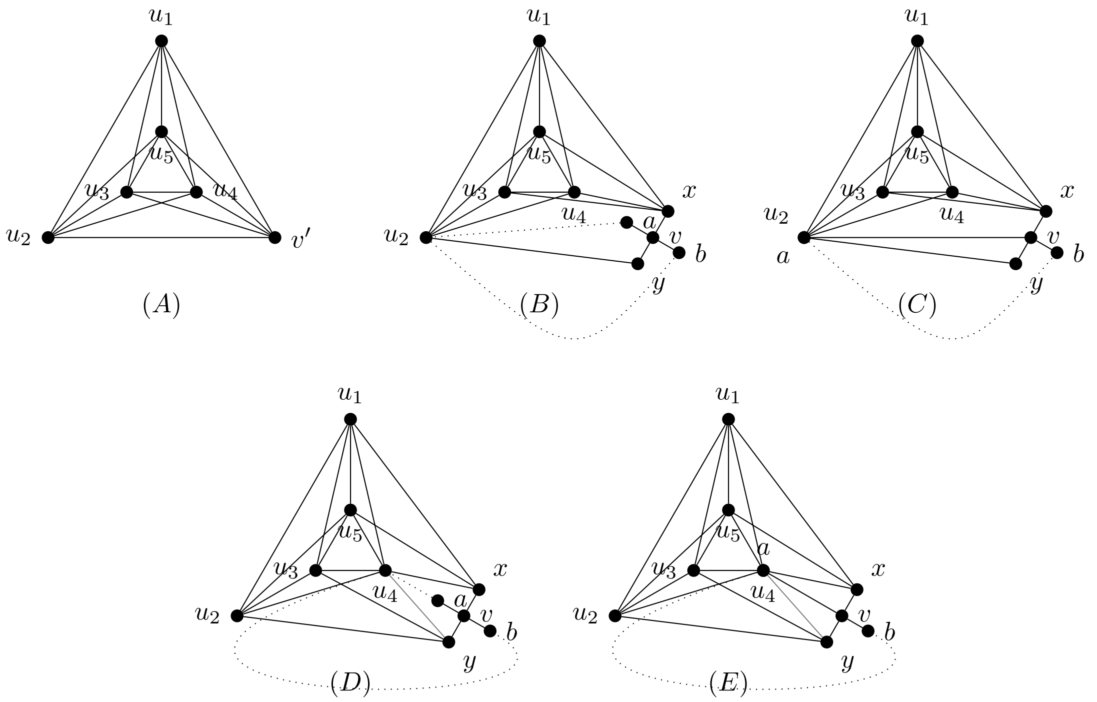

Figure 6: K_6 when identifying two neighbours of v.

the neighbours of v which were identified by x and y (i.e. $\{v_2, v_4\} = \{x, y\}$). Let a and b be the two other neighbours of v such that va and vb are not crossed ($\{a, b\} = \{v_3, v_5\}$). Moreover, we assume that in the counterclockwise order around v, the sequence is x, a, y, b. Note that the vertex v_1 may be inserted anywhere in the sequence.

One of the identified vertices, say x, is adjacent to at least two vertices of $\{u_3, u_4, u_5\}$.

1) Assume first that x is adjacent to u_3,u_4 and u_5. Then since G has no K_6, it is not adjacent to some vertex in $\{u_1, u_2\}$, say u_2. Thus $yu_2 \in E$.

   The vertex a is either inside u_2yvx or is u_2. See Figure 6(B) and (C), respectively. The path $a - b$ (represented by dotted line in the figure) necessarily uses u_2. Since colours $\phi(a)$ and $\phi(b)$ alternate on $a - b$, this path cannot contain x nor u_3, u_4 and u_5. The paths $a - b$ and avb separate x and y and there must be paths $v_1 - x$ and $v_1 - y$. Thus at least one of them must cross the path $a - b$. But none of the four crossings is available for that, a contradiction.

2) Let us now assume that x is adjacent to only two vertices of $\{u_3, u_4, u_5\}$, say u_4 and u_5. Then u_3 is adjacent to y. (Possibly u_4 and y are adjacent too.) The path $a - b$ must go through u_4 and then continue to u_1 or u_2. It cannot go through u_3 or u_5 since the colours on the path alternate. See Figure 6(D) and (E).

   The path $x - y$ must cross $a - b$. Hence either $x - y$ goes through u_3y and $a - b$ through u_4u_2 or $x - y$ goes through xu_5 and $a - b$ through u_4u_1. In both cases, one of the paths $v_1 - x$ and $v_1 - y$ must cross $a - b$. But there are no more crossings available.

This completes the proof of Claim 4. $\square$

Claim 5. *$q \neq 2$.*

Proof. Suppose to the opposite that $q = 2$.

We first prove the following assertion that will be used several times.
Assertion *Let x and y be two neighbours of v. Then x and y are adjacent if one of the following holds:*

- *vx and vy are not crossed;*
- *$\{x, y\}$ is included in the cluster of some crossing.*

Observe that $G - v$ has at most two crossings. Suppose that x and y are not adjacent. If vx and vy are not crossed, we can identify x and y along xvy without adding any new crossing. If $\{x, y\}$ is included in the cluster of some crossing, we can identify x and y along the edges of this crossing without adding any new crossing. Hence in both cases $(G - v)/\{x, y\}$ has a planar drawing with at most 2 crossings. Then Lemma 2.8 and Theorem 4.1 yield a contradiction. This proves the Assertion.

Assume that the crossed edges are consecutive, say vv_1 and vv_2. By the Assertion, v_3v_5 is an edge. See Figure 7(A). If v_3v_5 is not crossed or crosses either vv_1 and vv_2 then the cycle vv_3v_5 is regular, which contradicts Lemma 2.9. If v_3v_5 is crossed by another edge then the cycle vv_3v_5 contradicts Lemma 2.10. Henceforth, we may assume that the two crossed edges are not consecutive, say vv_2 and vv_5.

By the Assertion, v_1v_3, v_1v_4 and v_3v_4 are edges. If v_1v_3 is not crossed then the triangle vv_1v_3 is separating because v_2 and v_4 are on the opposite sides. This contradicts Lemma 2.9. If v_1v_3 is crossed it can be redrawn along the path v_1vv_3 with one crossing with vv_2. Symmetrically, we assume that v_1v_4 is crossing vv_5. See Figure 7(B).

By the Assertion, $\{v_1v_2, v_2v_3, v_4v_5, v_5v_1\} \subset E(G)$. See Figure 7(C).

Let $C = \{c_1c_2, c_3c_4\}$ and $D = \{d_1d_2, d_3d_4\}$ be the two crossings not having v in their cluster. For convenience and with a slight abuse of notation, we denote by C (resp. D) both the crossing C (resp. D) and its cluster. For $X \in \{C, D\}$, let $a(X) := |X \cap N(v)|$. Without loss of generality, we may assume that $a(C) \leq a(D)$.

A vertex u is a *candidate* if it is not adjacent to v. There is no candidate u common to both C and D otherwise $\{u, v\}$ would be a stable crossing cover. There are no non-adjacent candidate vertices $c \in C$ and $d \in D$ otherwise $\{v, c, d\}$ would be a stable crossing cover.

Assume that $a(D) = 4$. The vertex v_1 cannot be in D because it is already adjacent to all the other neighbours of v by edges not in D. Thus $D = \{v_2, v_3, v_4, v_5\}$. But then, by the Assertion, v_2v_5 is an edge. So $N(v) \cup \{v\}$ induces a K_6, a contradiction.
Hence $a(C) \leq a(D) \leq 3$.

Suppose now that $X \in \{C, D\}$ does not induce a K_4. Then two vertices x_1 and x_2 of X are not adjacent. One can add the edge x_1x_2 and draw it along the edges of the crossing such that no new crossing is created. Hence by the choice of G, the obtained graph $G \cup x_1x_2$ contains a K_6. Since K_6 has crossing number 3, one of the crossings containing v in its cluster must be used. So v belongs to the K_6 and hence the K_6 is induced by $\{v\} \cup N(v)$. In such case edges v_2v_4 and v_3v_5 cross and hence form C or D, which is not possible since $a(C) \leq a(D) \leq 3$.

Hence both C and D induce a K_4. Thus the candidates in $C \cup D$ induce a complete graph. So there are at most five of them. Since $C \cap D$ contains no candidate, we have $a(C) + a(D) \geq 3$ and so $2 \leq a(D) \leq 3$

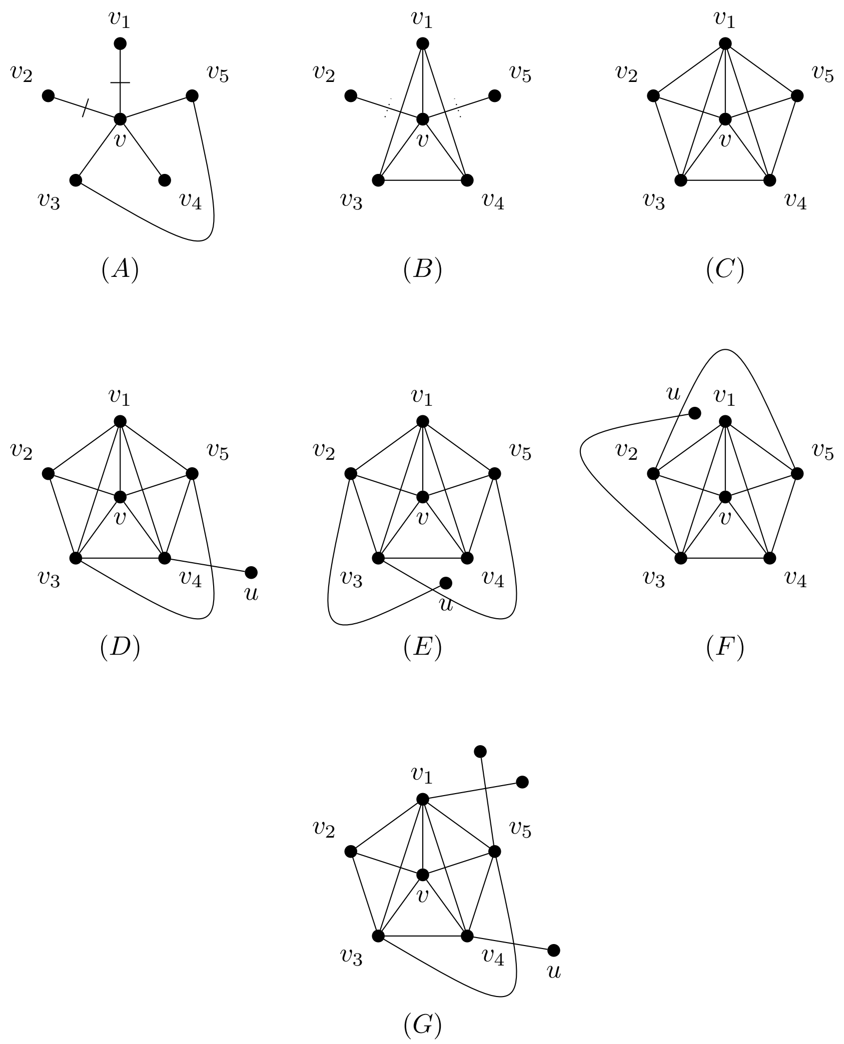

Figure 7: Two crossed edges.

Assume that $a(D) = 2$ and thus $1 \leq a(C) \leq 2$. Then C (resp. D) contains a set C' (resp. D') of two candidates. All the vertices of C' are adjacent to all the vertices of D'. But since both C and D contain a vertex in $N(v)$, drawing all the edges between these two sets requires one more crossing, a contradiction.
Hence $a(D) = 3$.

Thus, an edge of D has its two endvertices in $N(v)$ and so it is v_2v_5, v_2v_4 or v_3v_5. Let u be the unique candidate of D.

Assume first that $v_1 \in D$. Then v_1u is an edge of D. Moreover, C must be on the paths $v_2 - v_4$ and $v_3 - v_5$. Since edges are crossed at most once $D = \{v_1u, v_2v_5\}$. Let w be a candidate vertex in C. Then w is outside the cycle vv_2v_5. But the only neigbour of v_1 outside this cycle is u which is distinct from w because the crossings C and D have no candidate vertex in common. Thus $\{w, v_1\}$ is a stable crossing cover, a contradiction to Lemma 2.7.
So $v_1 \notin D$.

By symmetry, we may assume that D is either $\{v_3v_5, v_4u\}$ (Figure 7(D)) or $\{v_3v_5, v_2u\}$ (Figure 7(E)) or $\{v_2v_5, v_3u\}$ (Figure 7(F)). In the second and third cases, Lemma 2.10 is contradicted by the cycle $v_3v_4v_5$ and $v_1v_2v_5$ respectively.
Hence $D = \{v_3v_5, v_4u\}$.

The set $\{v_2, v_4\}$ is stable and covers the three crossings distinct from C. Hence $\{v_2, v_4\}$ does not intersect C, otherwise it would be a stable crossing cover. So $C \cap N(v) \subset \{v_1, v_3, v_5\}$. The edge v_1v_5 is not crossed, otherwise it could be redrawn along the edges of the crossing $\{vv_5, v_1v_4\}$ to obtain a drawing of G with less crossings. Furthermore, v_1v_3 and v_3v_5 are not in C because they are in some other crossing. Hence $a(C) \leq 2$.

Let B be the set of candidates of C. Recall that all vertices of B are adjacent to u. Moreover, every vertex $b \in B$ is adjacent to a vertex of $\{v_2, v_4\}$ otherwise $\{v_2, v_4, b\}$ is a stable crossing cover. But v_4 and u are separated by $v_3v_4v_5$, so all vertices of B are adjacent to v_2. Now the graph induced by the edges between B and $\{u, v_2\}$ is a complete bipartite graph. Moreover, its induced drawing has no crossing and the vertices of each part are in a common face. Thus, by Lemma 2.6, $|B| \leq 2$.
So $a(C) = 2$.

Recall that $C \cap N(v) \subset \{v_1, v_3, v_5\}$. Suppose that $C \cap N(v) = \{v_1, v_3\}$. The closed curve formed by the path v_3vv_1 and the two "half-edges" connecting v_1 to v_3 in C separates v_2 and u. Then the vertices of B cannot be adjacent to both u and v_2, a contradiction. Similarly, we obtain a contradiction if $C \cap N(v) = \{v_3, v_5\}$. Hence we may assume that $C \cap N(v) = \{v_1, v_5\}$. But then connecting the vertices of B to those of $\{v_2, v_4\}$ would require one more crossing. See Figure 7(G).

This completes the proof of Claim 5. □

Claim 6. $q \neq 3$.

Proof. Suppose that $q = 3$.

Let C be the crossing whose cluster does not contain v. It contains no candidate u otherwise $\{u, v\}$ would be a stable crossing cover. Hence $C \subset N(v)$.

Assume first that the three crossed edges incident to v are consecutive, say the crossed edges are vv_1, vv_2 and vv_5. By the Assertion, v_3v_4 is an edge. See Figure 8(A). Up to symmetry, the cluster of C is one of the following three sets $\{v_1, v_2, v_3, v_4\}$ or $\{v_2, v_3, v_4, v_5\}$ or $\{v_1, v_2, v_4, v_5\}$.

- $C = \{v_1, v_2, v_3, v_4\}$. Then the edges of C are not v_1v_4 and v_2v_3 because it is impossible to draw them such that each is crossed exactly once. Hence $C = \{v_1v_3, v_2v_4\}$.

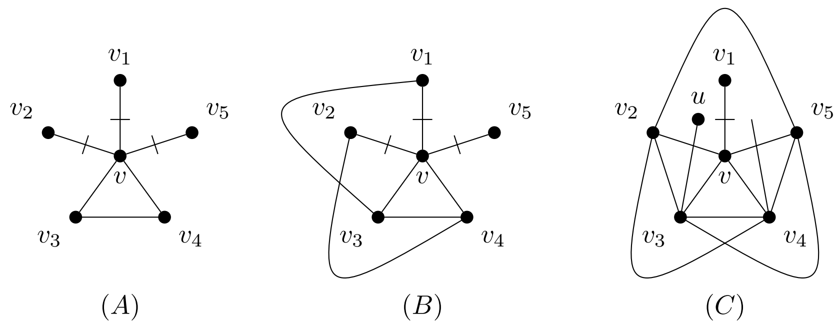

Figure 8: Three consecutive crossed edges.

The Jordan curve formed by the path v_1vv_4 and the two "half-edges" connecting v_1 to v_4 in C separates $\{v_2, v_3\}$ and v_5. See Figure 8(B). Moreover, it is crossed only once (on edge v_1v), while two crossings are needed, one for each of the disjoint paths $v_2 - v_5$ and $v_3 - v_5$, a contradiction.

- $C = \{v_2, v_3, v_4, v_5\}$. Then the edges of C are not v_2v_3 and v_4v_5 because it is impossible to draw them such that each is crossed exactly once. Hence $C = \{v_2v_4, v_3v_5\}$. Hence by the Assertion, v_2v_3, v_4v_5 and v_2v_5 are edges. The triangle vv_2v_3 has only one crossed edge. So, by Lemma 2.10, it is not separating. Thus its interior is empty and the edge crossing vv_2 is incident to v_3. Let u be the second endvertex of this edge. By symmetry, the interior of vv_4v_5 is empty and the edge crossing vv_5 is v_4t for some vertex t.

  If $u = t = v_1$, then by the Assertion v_1v_2 and v_1v_5 are edges. So $N(v) \cup \{v\}$ induces a K_6, a contradiction. Hence without loss of generality we may assume that $u \neq v_1$. See Figure 8(C).

  The interiors of the cycles vv_2v_3, vv_3v_4 and $v_2v_3v_4$ contain no vertices by Lemma 2.9. Hence v_3 is a 5-vertex. Moreover, its two neighbours u and v are not adjacent and $(G - v_3)/\{u, v\}$ has at most two crossings. Then Theorem 4.1 and Lemma 2.8 yield a contradiction.

- $C = \{v_1, v_2, v_4, v_5\}$. The crossing C is neither $\{v_1v_2, v_4v_5\}$ nor $\{v_1v_5, v_2v_4\}$ since it is impossible to draw so that every edge is crossed exactly once. Hence $C = \{v_1v_4, v_2v_5\}$. By the Assertion, $v_2v_4 \in E(G)$. Then the triangle vv_2v_4 contradicts Lemma 2.10.

Suppose now that the three crossed edges incident to v are not consecutive. Without loss of generality, we assume that these edges are vv_1, vv_3 and vv_4.

By the Assertion, v_2v_5 is an edge. If v_2v_5 is not crossed then vv_2v_5 is a separating triangle, contradicting Lemma 2.9. So v_2v_5 is crossed. It could not cross vv_3 nor vv_4 otherwise vv_2v_5 would be a regular cycle contradicting Lemma 2.9. Moreover, v_2v_5 cannot be in C otherwise vv_2v_5 would contradict Lemma 2.10. Hence v_2v_5 crosses vv_1.

By the Assertion, v_1v_2 and v_1v_5 are edges. Moreover they are not crossed, otherwise they could be redrawn along the edges of the crossing $\{vv_1, v_2v_5\}$ to obtain a drawing of G with less crossings. See Figure 9(A).

Consider the paths $v_2 - v_4$ and $v_3 - v_5$. If they cross, it is through C. Since $C \subset N(v)$, the paths $v_2 - v_4$ and $v_3 - v_5$ are actually edges. See Figure 9(B). But one can redraw

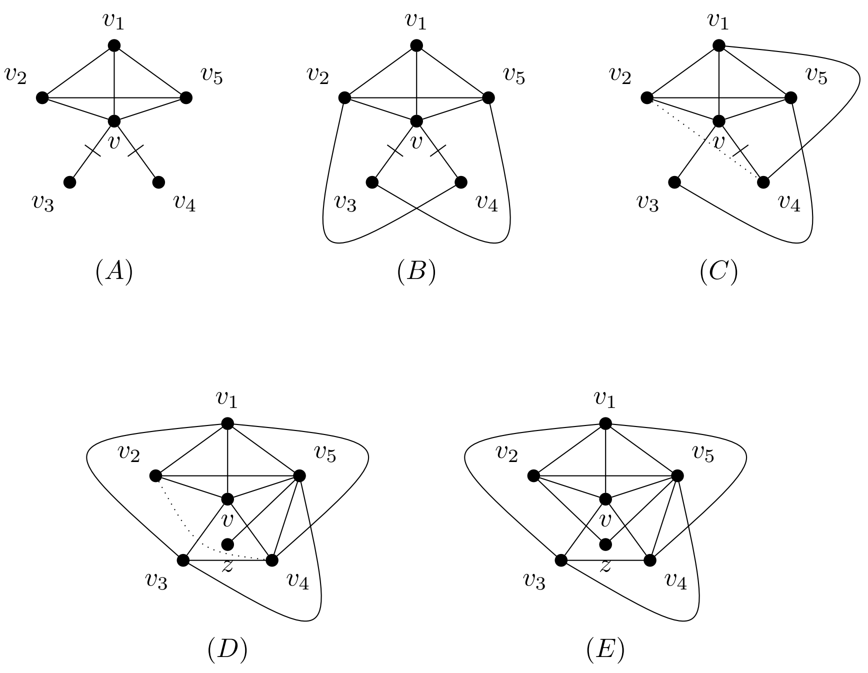

Figure 9: Three non-consecutive crossed edges.

v_2v_5 along the edges of C to obtain a drawing of G with less crossings, a contradiction.

Suppose now that $v_2 - v_4$ and $v_3 - v_5$ do not cross. By symmetry, we may assume that $v_2 - v_4$ cross vv_3. The paths $v_1 - v_4$ and $v_3 - v_5$ cross. It must be through C so v_1v_4 and v_3v_5 are both edges. See Figure 9(C). By the Assertion, v_1v_3, v_3v_4 and v_4v_5 are edges.

If v_2v_4 is also an edge, the Assertion implies that v_2v_3 is also an edge. Then $N(v) \cup \{v\}$ induces a K_6, a contradiction. Hence $v_2v_4 \notin E(G)$.

By Lemma 2.10, the cycle vv_4v_5 is not separating, so its interior contains no vertex and vv_4 is crossed by an edge with v_5 as an endvertex. Let z be the other endvertex of this edge. As an edge is crossed at most once, z is inside vv_3v_4. See Figure 9(D).

Let ab be the edge which is crossing vv_3. The sets $\{v_5, a\}$ and $\{v_5, b\}$ are not stable otherwise they would be a stable crossing cover. Hence v_5a and v_5b are both edges. Thus $ab = v_2z$. See Figure 9(E). Now v_1z is not an edge and hence $\{v_1, z\}$ is a stable crossing cover, contradicting Lemma 2.7.

This completes the proof of Claim 6. □

Claim 7. $q \neq 4$

Proof. By way of contradiction, suppose that $q = 4$. Then $\{v\}$ is a stable crossing cover, a contradiction. □

Claims 3, 4, 5, 6 and 7 yields a contradiction. This finishes the proof of Theorem 5.1. □

6 Further research

6.1 Extending our results

Theorem 4.1 states that if a graph can be made planar by removing at most $2k$ edges then it is $(4 + k)$-colourable. We believe that this is not tight. Thus a natural question is the following:

Problem 6.1. *Let k be a positive integer. What is the maximum $g(k)$ of the chromatic number over all the graphs for which there exists a set F of at most k edges such that $G \setminus F$ is planar?*

Clearly, $g(1) = g(2) = 5$ by Theorem 4.1 and because K_5 is not planar and $g(3) = 6$ by Theorem 4.1 and because $cr(K_6) = 3$. For larger value of k, we also believe that the optimal value is given by a complete graph. It is also very likely that the complete graph $K_{g(k)}$ is the unique $g(k)$-critical graphs that can be made planar by removing k edges. It is in particular the case for $k = 6$ and $k = 7$. Indeed by Proposition 2.1, at least 6 edges are needed to make K_7 planar and there is a set of 6 edges whose removal leaves K_7 planar. See Figure 10.

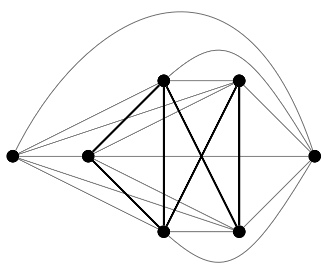

Figure 10: The graph K_7 and a set of edges (in bold) whose removal yields a planar graph.

Theorem 6.2. *Let G be a graph with $\omega(G) \leq 6$. If there is a set F of at most 7 edges such that $G \setminus F$ is planar then G is 6-colourable.*

Proof. To prove this theorem, we show that K_7 is the unique 7-critical graph for which there exists a set of at most 7 edges whose removal leaves G planar. A famous result of Dirac [4] states if G is r-critical graph and is not K_r then $2|E(G)| \geq (r-1)|V(G)|+r-3$. In particular, if $r = 7$ then $|E(G)| \geq 3|V(G)| + 2$. Hence by Proposition 2.1, we need to remove at least 8 edges to make it planar. □

One of the first problem to tackle is the following conjecture which extends both Theorem 1.6 and Theorem 1.4.

Conjecture 6.3. *If $\omega(G) \leq 5$ and there exists a set F of at most four edges such that $G \setminus F$ is planar then $\chi(G) \leq 5$.*

6.2 Critical graphs and colourability

It is easy to derive from Proposition 2.1 that for $r \geq 8$, there are only finitely many r-critical graphs that can be embedded on a fixed surface. As pointed out by Thomassen in [16], the number of 7-critical graphs that can be embedded on a fixed surface is also finite. Finally, Thomassen [17] completed the results by showing that the number of 6-critical subgraphs is finite for any fixed surface Σ. This implies in particular the $(r-1)$-colourability problem for graphs embeddable on Σ is decidable in polynomial time for any $r \geq 6$. On the other hand, deciding 3-colourability is NP-complete for planar graphs (see [7]) and thus also for graphs embeddable on any other surface. The complexity of 4-colourability remains open.

Problem 6.4. *Let Σ be a fixed surface. Does there exists a polynomial time algorithm for deciding if a graph embeddable on Σ is 4-colourable?*

The answer to Problem 6.4 is only known for the sphere by the Four Colour Theorem. An affirmative answer cannot be obtained in the same way as for $r-1 \geq 5$ because there are infinitely many 5-critical graphs as implied by a result of Fisk [6].

If $\mathrm{cr}(G) = k$ then G is embeddable in $\mathbb{S}_k$ and in $\mathbb{N}_k$ as well, where $\mathbb{S}_k$ is an orientable surface of genus k and $\mathbb{N}_k$ is a non-orientable surface of genus k. Hence for any k and $r \geq 6$, the number of r-critical graphs of crossing number k is finite and so the $(r-1)$-colourability problem for graphs of crossing number k is decidable in polynomial time. However, the design of such a polynomial time algorithm requires the knowledge of the list of 6-critical graphs.

Problem 6.5. *Let $k \geq 0$. What is the list of 6-critical graphs with crossing number at most k?*

When $k \leq 3$, then the list is empty and if $k = 4$, then the list is $\{K_6\}$. If $k = 5$, then the list contains K_6 and the graph depicted in Figure 3. But are there any other?

Similarly to graphs embeddable on a fixed surface, the complexity of 4-colourability problem for graphs with crossing number k is not known.

Problem 6.6. *Let $k \geq 0$. Does there exists a polynomial time algorithm for deciding if a given graph with crossing number k is 4-colourable?*

On the other hand we know, that it cannot be proved by listing all 5-critical graphs as there are infinitely many 5-critical graphs with crossing number one.

6.3 Choosability

A *list assignment* of a graph G is a function L that assigns to each vertex $v \in V(G)$ a list $L(v)$ of available colours. An *L-colouring* is a function $\varphi : V(G) \to \bigcup_v L(v)$ such that $\varphi(v) \in L(v)$ for every $v \in V(G)$ and $\varphi(u) \neq \varphi(v)$ whenever u and v are adjacent vertices of G. If G admits an L-colouring, then it is *L-colourable.* A graph G is *k-choosable* if it is L-colourable for every list assignment L such that $|L(v)| \geq k$ for all $v \in V(G)$. The *choose number* of G, denoted by $\mathrm{ch}(G)$, is the minimum k such that G is k-choosable.

Similarly to the chromatic number, one may seek for bounds on the choose number of a graph with few crossings or with independent crossings.

Thomassen [15] showed that every planar graph is 5-choosable. In fact, he proved a stronger result.

Definition 6.7. An *inner triangulation* is a plane graph such that every inner face of G is bounded by a triangle and its outer face by a cycle $F = (v_1 v_2 \ldots v_k v_1)$.

A list assignment L of an inner triangulation G is *suitable* if

- $|L(v_1)| = 1$ and $|L(v_2)| = 2$,

- for every $v \in V(F) \setminus \{v_1, v_2\}$, $|L(v)| \geq 3$, and

- for every $v \in V(G) \setminus V(F)$, $|L(v)| \geq 5$.

Theorem 6.8 (Thomassen [15]). *If L is a suitable list assignment of an inner triangulation G then G is L-colourable.*

Theorem 6.9. *Let G be a graph. If* $\text{cr}(G) = 1$ *then* $\text{ch}(G) \leq 5$.

Proof. Consider a plane embedding of G with one crossing $C = \{x_1y_1, x_2y_2\}$. Without loss of generality, we may assume that G is in the outer face. Free to add edges, we may assume that the outer face is bounded by the 4-cycle $x_1x_2y_1y_2x_1$ and that $G \setminus F$ is an inner triangulation.

Let L be a 5-list assignment of G. Set $c_1 \in L(x_1)$ and $c_2 \in L(x_2) \setminus \{c_1\}$. Let L' be the list assignment defined by $L'(x_1) = \{c_1\}$, $L'(x_2) = \{c_1, c_2\}$, $L'(y_i) = L(y_i) \setminus \{c_1, c_2\}$ for $i = 1, 2$ and $L'(v) = L(v)$ for every $v \in V(F) \setminus \{x_1, x_2, y_1, y_2\}$. Then L' is a suitable list assignment of $G \setminus F$. Hence $G \setminus F$ admits a proper L'-colouring, which is an L-colouring of G by the definition of G'. □

Problem 6.10. *Is every graph with crossing number* 2 *5-choosable?*

6.4 Graphs with small clique number or large girth

The celebrated Grötzsch Theorem [8] asserts that triangle-free (i.e with clique number at most 2) planar graphs are 3-colourable. (See also [18] for a short elegant proof.) Together with Theorem 1.4, this suggests that the above upper bounds may be lessen when considering graphs with small clique number. We now prove a result analogous to Theorem 1.5 for K_4-free graphs.

Theorem 6.11. *If G is a K_4-free graph which has a drawing in the plane in which no two crossings are dependent, then $\chi(G) \leq 4$.*

Proof. Let $C_i = \{u_iv_i, x_iy_i\}$, $i \in I$ be the crossings. Since G is K_4-free, without loss of generality, we may assume that for every $i \in I$, u_ix_i is not an edge. Let G' be the graph obtained from G by identifying u_i with x_i for every $i \in I$ into a vertex z_i. The graph G' is planar. Thus, by the Four Colour Theorem, G' admits a proper 4-colouring c'. Let us define c by $c(x_i) = c(x_i) = c'(z_i)$ for every $i \in I$ and $c(v) = c'(v)$ for every vertex $v \in V(G) \cap V(G')$. Since, for every $i \in I$, x_i and u_i are not adjacent, c is a proper 4-colouring of G. □

Note that Theorem 6.11 is tight because there exist K_4-free planar graphs which are not 3-colourable. But can it be improved for triangle-free graphs or is there a triangle-free graph which has a drawing in the plane in which no two crossings are dependent and which is not 3-colourable?

For triangle-free graphs, one can show an analogue to Theorem 1.6.

Theorem 6.12. *Let G be a triangle-free graph. If there is a set F of (at most)* 4 *edges such that $G \setminus F$ is planar then* $\text{ch}(G) \leq 4$.

Proof. By induction on the number n of vertices of G, the result holding trivially when $n \leq 4$. A triangle-free planar graph on n vertices has at most $2n - 4$ edges. Hence G has at most $2n$ edges. Thus either G has a vertex v of degree at most three or it is 4-regular.

In the first case, by the induction hypothesis $\text{ch}(G - v) = 4$. Let L be a 4-list-assignment of $V(G)$. $G - v$ admits an L-colouring c that can be extended to G by assigning to v a colour in its list not assigned to any of its neighbours. So G is 4-choosable.

In the second case, since G is triangle-free it contains no K_5 and thus by Brooks Theorem for list-colouring, $\text{ch}(G) \leq 4$. □

For C_3 and K_4 and more generally, for any graph or any family of graph $\mathcal{F}$, one can ask the following questions.

Problem 6.13. What is the smallest integer $f_{\mathcal{F}}(k)$ (resp. $g_{\mathcal{F}}(k)$) such that every $\mathcal{F}$-free graph G and crossing number at most k is $f_{\mathcal{F}}(k)$-colourable (resp. $g_{\mathcal{F}}(k)$-choosable)?

In particular, for $\mathcal{C}_g = \{C_i | i = 3, \ldots, g-1\}$ the family of cycles of length less than g, the $\mathcal{C}_g$-free graphs are graphs with girth at least g. Set $f_g(k) = f_{\mathcal{C}_g}(k)$. Trivially, $f_g(k) \leq f_h(k)$ if $g \geq h$. In particular for any $g \geq 3$, $f_g(k) \leq O\left(k^{1/4}\right)$ since $f_3(k) = f(k) = O\left(k^{1/4}\right)$. Erdös [5] showed that there are graphs with arbitrarily large girth and chromatic number. Hence for any fixed g, $f_g(k)$ tends to infinity when k tends to infinity. The Grötzsch graph is triangle-free, has crossing number at most 5 and chromatic number 4, so $f_3(5) \geq 4$. Thomas and Walls [14] proved that every graph of girth at least five which admits an embedding in the Klein bottle is 3-colourable. Since every graph with crossing at most 2 is embeddable in the Klein bottle, it follows that every graph of girth at least 5 and crossing number at most 2 is 3-colourable.

Jensen and Royle [9] showed a K_4-free graph with crossing number at most 6 and chromatic number 5, so $f_{K_4}(6) \geq 5$.

One can prove an analogue to Theorem 1.5 for graphs of large girth.

Proposition 6.14. *Let G be a graph having a drawing in the plane in which no two crossings are dependent.*

(i) If G has girth at least 5, then $\mathrm{ch}(G) \leq 4$.

(ii) If G has girth at least 10, then $\mathrm{ch}(G) \leq 3$.

Proof. Let us prove that G is 3-degenerate (resp. 2-degenerate) if G has girth at least 5 (resp. 10). To do so it suffices to prove that it has a vertex of degree at most 3 (resp. at most 2).

Let n be the number of vertices of G. Since no two crossings are dependent, then G has at most $n/4$ crossings. Hence there is a set F of at most $n/4$ edges such that $G \setminus F$ is planar. Moreover, $G \setminus F$ has girth at least 5 (resp. 10), so $G \setminus F$ has less than $\frac{10}{6}n$ (resp. $\frac{3}{4}n$) edges. Hence G has less than $\frac{23}{12}n < 2n$ (resp. n). Hence G has a vertex of degree at most 3 (resp. 2). □

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