Fine structure of 4-critical triangle-free graphs I. Planar graphs with two triangles and 3-colorability of chains

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January 10, 2017

Abstract

Aksenov proved that in a planar graph G with at most one triangle, every precoloring of a 4-cycle can be extended to a 3-coloring of G. We give an exact characterization of planar graphs with two triangles in which some precoloring of a 4-cycle does not extend. We apply this characterization to solve the precoloring extension problem from two 4-cycles in a triangle-free planar graph in the case that the precolored 4-cycles are separated by many disjoint 4-cycles. The latter result is used in followup papers to give detailed information about the structure of 4-critical triangle-free graphs embedded in a fixed surface.

1   Introduction

The interest in the 3-coloring properties of planar graphs was started by a celebrated theorem of Grötzsch [11], who proved that every planar triangle-free graph is 3-colorable. While in general, deciding 3-colorablity of a planar graph is an NP-complete problem [9], there are many other sufficient conditions guaranteeing 3-colorability, see e.g. the survey of Montassier [15].

For a long time, the question of the complexity of deciding whether a triangle-free graph embedded in a fixed surface (other than the sphere) is 3-colorable was open. The question was resolved for the projective plane by the result of Gimbel and Thomassen [10], and in a far reaching generalization, Dvořák, Král' and Thomas [6] proved that there exists a linear-time algorithm for this problem for any fixed surface, even if a bounded number of vertices have prescribed colors. In order to design their algorithm, they show in [5] that every triangle-free

graph embedded in a fixed surface exhibits a special structure that determines its 3-coloring properties.

Before we describe their structural result, let us recall some definitions. A *surface* is a two-dimensional manifold, possibly with boundary. By the surface classification theorem, each surface can be (up to homeomorphism) obtained from the sphere by adding a finite number of handles and crosscaps, and in the case of surfaces with boundary, by drilling a finite number of holes. The *disk* is the sphere with a hole and the *cylinder* is the sphere with two holes. An *embedding* of a graph G in a surface Σ is a function α that maps vertices of G to distinct points in Σ and edges of G to simple curves in Σ intersecting only in their endpoints, such that for each $uv \in E(G)$, $\alpha(u)$ and $\alpha(v)$ are the endpoints of the curve $\alpha(uv)$ and no other vertices are mapped to points in this curve. Throughout the paper, graphs will generally be embedded in some surface; we usually keep the embedding implicit and we use terms such as vertex and edge to refer to both the elements of the graph and the points or curves in the surface that represent them. A *face* of a graph G embedded in Σ is a connected component of the surface after removing the points and curves of the embedding of G; in particular, if Σ is a surface with boundary and a cycle C in G traces a component of the boundary, then C does not necessarily bound a face. A closed walk in G is *contractible* if the closed curve tracing this walk in the surface is null-homotopic. A contractible cycle bounds a disk in Σ, unique unless Σ is the sphere; the cycle is *facial* if the interior of such a disk is a face of G.

We are now ready to state the structural result of Dvořák et al. [6]. Let G be a graph embedded in a surface Σ so that every contractible 4-cycle is facial, and suppose G intersects the boundary of Σ in a set X of k vertices (which we view as precolored). Then Σ can be cut in to a bounded number of pieces along curves tracing closed walks in G, so that

- the total number of vertices contained in the boundaries of the pieces is bounded by a constant depending only on Σ and k,

- each piece Π and the subgraph H of G drawn in Π satisfies one of the following:

  1. every 3-coloring of the vertices of H contained in the boundary of Π extends to a 3-coloring of H; or,

  2. a 3-coloring of the vertices of H contained in the boundary of Π extends to a 3-coloring of H if and only if it satisfies a specific condition ("the winding number constraint"); or,

  3. Π is homeomorphic to the cylinder (whose boundary consists of two cycles in H of arbitrary length) and all faces of H have length 4; or,

  4. Π is homeomorphic to the cylinder whose boundary consists of two cycles in H of length 4.

Thus, to determine whether a precoloring of X extends to a 3-coloring G, we can try all the (constantly many) extensions to 3-colorings of the boundary vertices

of the pieces, and then test whether one of them extends to all the pieces. In the first two possibilities for the pieces of the structure we have a complete information about which colorings of the boundaries of the pieces extend, and the last part is thus trivial. However, in the last two cases the information is much more limited. While this is sufficient for the purposes of the algorithm of [6], it would be preferable to have a more detailed structural theorem where the 3-coloring properties of all the pieces are known. This is the main aim of this series of papers.

In this paper we focus on the last subcase of a graph embedded in the cylinder with boundary cycles of length 4. Note that if such a graph contains only a bounded number of separating 4-cycles, then we can further cut the surface along them and by using the ideas of [5], we can subdivide the pieces to a bounded number of subpieces satisfying the conditions of one of the first two well-understood cases of the structure theorem. Hence, it is interesting to study the graphs in cylinder with many separating 4-cycles, and this is the topic of this paper.

Let us now give a few definitions enabling us to state the main result more precisely. In this paper, we generally consider graphs embedded in the sphere, the disk, or the cylinder. Suppose that G is a graph embedded in a surface Σ with boundary and consider a component Θ of the boundary (Θ is a simple closed curve bounding a hole in the surface). Let Σ_Θ denote the surface obtained from Σ by closing the hole, i.e., by identifying Θ with the boundary of an open disk Λ_Θ disjoint from Σ. We say that Θ is *surrounded* in G if the embedding of G in Σ_Θ has a face homeomorphic to an open disk containing Λ_Θ and bounded by a cycle R. Equivalently, the part of the surface Σ between R and Θ intersects the drawing of G exactly in R and all non-contractible simple curves contained in this part are homotopically equivalent to the closed curve tracing R. In that case, we say that R is the *ring* surrounding the hole.

From now on, we always assume that if a graph G is embedded in a surface with boundary, then all the holes of the surface are surrounded. Note that the ring may or may not trace the boundary of the hole it surrounds, and in particular the rings surrounding different holes do not have to be disjoint (or even distinct, in case that G is just a cycle). A face f of G is a *non-ring* face if f is not contained in any of the parts of the surface between the holes and the rings that surround them.

We construct a sequence of graphs T_1, T_2, ..., which we call *Thomas-Walls graphs* (Thomas and Walls [16] proved that they are exactly the 4-critical graphs that can be drawn in the Klein bottle without contractible cycles of length at most 4). Let T_1 be equal to K_4. For $n \geq 1$, let $u_1 u_3$ be any edge of T_n that belongs to two triangles and let T_{n+1} be obtained from $T_n - u_1 u_3$ by adding vertices x, y and z and edges $u_1 x$, $u_3 y$, $u_3 z$, xy, xz, and yz. The first few graphs of this sequence are drawn in Figure 1. For $n \geq 2$, note that T_n contains unique 4-cycles $C_1 = u_1 u_2 u_3 u_4$ and $C_2 = v_1 v_2 v_3 v_4$ such that $u_1 u_3, v_1 v_3 \in E(G)$. Let $T'_n = T_n - \{u_1 u_3, v_1 v_3\}$. We also define T'_1 to be a 4-cycle $C_1 = C_2 = u_1 v_1 u_3 v_3$. We call the graphs T'_1, T'_2, ... *reduced Thomas-Walls graphs*, and we say that $u_1 u_3$ and $v_1 v_3$ are their *interface pairs*. Note that T'_n has an embedding in the
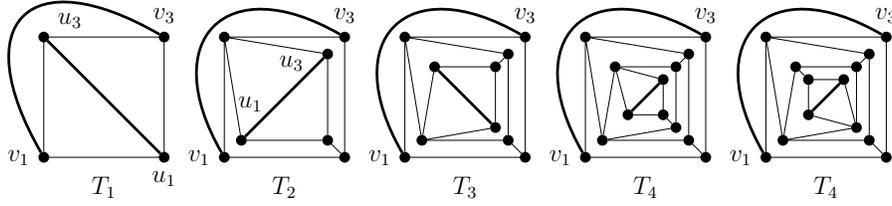
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Figure 1: Some Thomas-Walls graphs (with two different drawings of T_4).

cylinder with rings C_1 and C_2.

A *patch* is a graph F drawn in the disk with ring C of length 6 that traces the boundary of the disk, such that C is an induced cycle in F, every face of F has length 4, and every 4-cycle in F is facial. Let G be a graph embedded in the sphere, possibly with holes. Let G' be any graph which can be obtained from G as follows. Let S be an independent set in G such that every vertex of S has degree 3. For each vertex $v \in S$ with neighbors x, y and z, remove v, add new vertices a, b and c and a 6-cycle $C = xaybzc$ (where a, b, and c are drawn very close to the original location of v and the edges of C are drawn very close to the curves representing the edges vx, vy, and vz), and draw any patch with ring C in the disk bounded by C. We say that any such graph G' is obtained from G by *patching*. This operation was introduced by Borodin et al. [3] in the context of describing planar 4-critical graphs with exactly 4 triangles.

Consider a reduced Thomas-Walls graph $G = T'_n$ for some $n \geq 1$, with interface pairs $u_1 u_3$ and $v_1 v_3$. A *patched Thomas-Walls graph* is any graph obtained from such a graph G by patching, and $u_1 u_3$ and $v_1 v_3$ are its interface pairs (note that u_1, u_3, v_1, and v_3 have degree two in G, and thus they are not affected by patching).

Let G be a graph embedded in the sphere with n holes ($n \in \{1, 2\}$), with rings $C_i = x_i y_i z_i w_i$ of length 4, for $1 \leq i \leq n$. Let y'_i be either a new vertex or y_i, and let w'_i be either a new vertex or w_i. Let G' be obtained from G by adding 4-cycles $x_i y'_i z_i w'_i$ forming the new rings. We say that G' is obtained by *framing on pairs* $x_1 z_1$, ..., $x_n z_n$.

Let G be a graph embedded in the cylinder with rings C_1 and C_2 of length three, such that every non-ring face of G has length 4. We say that such a graph G is a $3, 3$-*quadrangulation*. Let G' be obtained from G by subdividing at most one edge in each of C_1 and C_2. We say that such a graph G' is a *near* $3, 3$-*quadrangulation*.

We say that a graph G embedded in the cylinder is *tame* if G contains no contractible triangles, and all triangles of G are pairwise vertex-disjoint. Let G be a tame graph embedded in the cylinder with rings of length at most 4. We say that G is a *chain* of graphs G_1, ..., G_n, if there exist non-contractible (≤ 4)-cycles C_0, ..., C_n in G such that

- the cycles are pairwise vertex-disjoint except that for $(i, j) \in \{(0, 1), (n, n-$

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1)$\}$, C_i can intersect C_j if C_i is a 4-cycle and C_j is a triangle,

- for $0 \leq i < j < k \leq n$ the cycle C_j separates C_i from C_k,

- the cycles C_0 and C_n are the rings of G,

- every triangle of G is equal to one of $C_0, \ldots, C_n$, and

- for $1 \leq i \leq n$, the subgraph of G drawn between C_{i-1} and C_i is isomorphic to G_i.

We say that $C_0, \ldots, C_n$ are the *cutting cycles* of the chain. The main result of this paper is the following.

Theorem 1.1. *There exists an integer $c \geq 0$ such that the following holds. Let G be a tame graph embedded in the cylinder with rings C_1 and C_2 of length at most 4. If G is a chain of at least c graphs, then*

- *every precoloring of $C_1 \cup C_2$ extends to a 3-coloring of G, or*

- *G contains a subgraph H obtained from a patched Thomas-Walls graph by framing on its interface pairs, with rings C_1 and C_2, or*

- *G contains a near $3,3$-quadrangulation H with rings C_1 and C_2 as a subgraph.*

Aksenov [2] proved the following strengthening of Grötzsch's theorem (fixing a previous flawed proof of this fact by Grünbaum [12]).

Theorem 1.2 (Aksenov [2]). *Every planar graph with at most 3 triangles is 3-colorable.*

In the course of the proof, Aksenov also established another interesting fact.

Theorem 1.3 (Aksenov [2]). *Let G be a graph drawn in the cylinder with a ring C of length at most 4. If every triangle in G is non-contractible, then every precoloring of C extends to a 3-coloring of G.*

As a part of the proof of Theorem 1.1, we need to prove a strengthening of Theorem 1.3 and describe the 3-coloring properties of graphs embedded in the disk with a ring C of size 4 and with exactly two triangles. To state this result (Theorem 1.4 below) of independent interest, it is convenient to introduce the notion of a critical graph.

Let C be the union of the rings of a graph G embedded in a surface (C is empty when G is embedded in a surface without boundary). By a *precoloring* of C, we mean any proper 3-coloring of C. We say that G is *critical* if $G \neq C$ and for every proper subgraph G' of G such that $C \subseteq G'$, there exists a precoloring of C that extends to a 3-coloring of G', but not to a 3-coloring of G; that is, removing any edge or vertex not belonging to C affects the set of precolorings of C that extend to a 3-coloring of the graph.
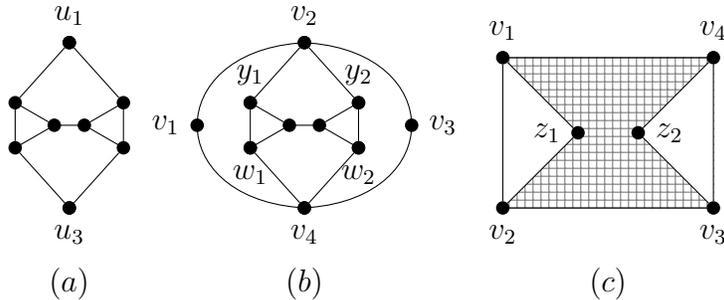
Figure 2: Graphs related to Theorem 1.4.

In particular, consider any graph H embedded in a surface, let C be the union of its rings, and let G be an inclusionwise-minimal subgraph of H such that $C \subseteq G$ and every precoloring of C that extends to a 3-coloring of G also extends to a 3-coloring of H. Then either $G = C$ or G is critical, and G carries all the information regarding which precolorings of C extend to H; we say that G is a *critical skeleton* of H. Consequently, it suffices to consider the properties of critical graphs, and we will do so in Theorem 1.4 as well as in many of the further results.

We need another construction related to Thomas-Walls graphs. For $n \geq 1$, let $G = T_n'$ be a reduced Thomas-Walls graph with rings $C_1 = u_1 u_2 u_3 u_4$ and $C_2 = v_1 v_2 v_3 v_4$ and interface pairs $u_1 u_3$ and $v_1 v_3$. Consider the embedding of T_n' in the disk, obtained by closing the hole in the face bounded by C_1. Let G' be a graph obtained from G by either adding the edge $u_1 u_3$, or the subgraph depicted in Figure 2(a) (this graph is often called *Havel's quasiedge*, since Havel [13] used it to disprove a conjecture by Grünbaum that every planar graph without intersecting triangles is 3-colorable). A *patched Havel-Thomas-Walls graph* is any graph obtained from such a graph G' by patching, and $v_1 v_3$ is its *interface pair*.

Let a *tent* be a graph embedded in the disk with the ring $v_1 v_2 v_3 v_4$, containing vertices z_1 adjacent to v_1 and v_2, and z_2 adjacent to v_3 and v_4, such that all faces other than $v_1 v_2 z_1$ and $v_3 v_4 z_2$ have length 4, and $v_1 v_3, v_2 v_4 \notin E(G)$, see Figure 2(c).

Theorem 1.4. *Let G be a graph embedded in the disk with at most two triangles and with the ring of length 4. If G is critical, then G is either a tent, or obtained from a patched Havel-Thomas-Walls graph by framing on its interface pair.*

It is important to note that the precolorings of the rings which extend to graphs appearing in the conclusions of Theorems 1.1 and 1.4 can be precisely described, as we show in Section 2 (Lemma 2.7 for framed patched Thomas-Walls graphs, Lemma 2.11 for near 3, 3-quadrangulations, Corollary 2.6 for tents, and Lemma 2.8 for framed patched Havel-Thomas-Walls graphs). In particular,

in the structure theorem we aim for, it is satisfactory to have pieces that are patched reduced Thomas-Walls graphs.

Furthermore, in Theorem 1.1, all non-ring faces f of the graphs H are bounded by contractible (≤ 5)-cycles, and the subgraph $G[f]$ of G drawn in the closure of f contains no triangles. Consequently, every precoloring of the cycle bounding f extends to a 3-coloring of $G[f]$ (see e.g. Lemma 2.4 below). We conclude that a precoloring of the rings of G extends to a 3-coloring of G if and only if it extends to a 3-coloring of H.

The rest of the paper is structured as follows. In Section 2, we describe possible colorings of the graphs appearing in Theorems 1.1 and 1.4. Section 3 is devoted to examining a chain G of graphs, many of which are not quadrangulations, showing that either all precolorings of the rings of G extend, or that the rings of G have length 4 and all their precolorings which extend to a 3-coloring of the reduced Thomas-Walls graph T_4' also extend to G. This allows us to prove Theorem 1.4 in Section 4. In Section 5, we continue by examining chains that contain a long subchain consisting only of quadrangulations and show that either every precoloring of the rings extends or the chain is a near $3, 3$-quadrangulation. We combine these results and prove Theorem 1.1 in Section 6.

2   Colorings of the special graphs

In this section we study which precolorings of rings extend in the special graphs appearing in Theorems 1.1 and 1.4. We examine patched Thomas-Walls graphs in Lemma 2.7, patched Havel-Thomas-Walls graphs in Lemma 2.8 and tame almost $3, 3$-quadrangulations in Lemma 2.11.

We need the following result of Aksenov on the extendability of the precoloring of a 5-cycle.

Theorem 2.1 (Aksenov [2]). *Let G be a graph embedded in the disk with a ring $C = v_1 v_2 v_3 v_4 v_5$ tracing its boundary. Suppose that G contains exactly one triangle T. If G is critical, then all faces of G other than the one bounded by T have length 4. Furthermore, if ψ is the 3-coloring of C given by $\psi(v_1) = \psi(v_3) = 1$, $\psi(v_2) = \psi(v_4) = 2$ and $\psi(v_5) = 3$ and ψ does not extend to a 3-coloring of G, then v_2 and v_3 are incident with T.*

We also need the following result of Gimbel and Thomassen [10] on the extendability of the precoloring of a (≤ 6)-cycle.

Theorem 2.2 (Gimbel and Thomassen [10]). *Let G be a triangle-free graph drawn in the disk with a ring C of length at most 6 tracing its boundary. If G is critical, then $|C| = 6$ and every face of G has length 4.*

Furthermore, suppose G' is a graph drawn in the disk with a ring $C' = v_1 \ldots v_6$ of length 6 tracing its boundary, such that all faces of G' have length 4. Then a precoloring ψ of C' does not extend to a 3-coloring of G' if and only if one of the following conditions holds.

- *There exists $i \in \{1, 2, 3\}$ such that $v_i v_{i+3} \in E(G')$ and $\psi(v_i) = \psi(v_{i+3})$, or*

- *$\psi(v_1) = \psi(v_4) \neq \psi(v_2) = \psi(v_5) \neq \psi(v_3) = \psi(v_6) \neq \psi(v_1)$.*

Theorem 2.2 has an important consequence for patches.

Corollary 2.3. *Let G be a patch with ring $xaybzc$, and let H be the graph with vertex set $\{x, y, z, v\}$ and edges xv, yv and zv. Then a 3-coloring of $\{x, y, z\}$ extends to a 3-coloring of G if and only if it extends to a 3-coloring of H.*

Proof. Let ψ be a 3-coloring of $\{x, y, z\}$ and let $C = xaybzc$ be the ring of G. If ψ assigns three different colors to the vertices x, y, and z, so that ψ does not extend to a 3-coloring of H, then ψ extends uniquely to a 3-coloring ψ' of C, and ψ' does not extend to a 3-coloring of the patch G by the second part of Theorem 2.2. Hence, ψ does not extend to a 3-coloring of G.

Suppose now that ψ extends to a 3-coloring of H, and thus by symmetry we can assume that $\psi(x) = \psi(y)$. There exists a 3-coloring ψ' of C extending ψ such that $\psi'(a) \neq \psi'(z)$. Since x and y have the same color and one of them is incident to each of b and c, we have $\psi'(x) \neq \psi'(b)$ and $\psi'(y) \neq \psi'(c)$. Again by the second part of Theorem 2.2, ψ' (and thus also ψ) extends to a 3-coloring of G. □

That is, replacing a vertex of degree three by a patch does not affect the 3-colorability of the graph. We also often use the following mild strenthening of Theorem 2.2.

Lemma 2.4. *Let G be a graph drawn in a surface Σ. Let K be a closed walk of length at most 6 in G forming the boundary of an open disk $\Lambda \subset \Sigma$, such that no contractible triangle of G is contained in the closure of Λ. Let G' be the subgraph of G drawn in the closure of Λ. If a 3-coloring ψ of K does not extend to a 3-coloring of G', then $|K| = 6$, G' has a subgraph containing K whose faces in Λ all have length 4, $K = v_1 \ldots v_6$, and either there exists $i \in \{1, 2, 3\}$ such that $v_i v_{i+3} \in E(G')$ and $\psi(v_i) = \psi(v_{i+3})$, or $\psi(v_1) = \psi(v_4) \neq \psi(v_2) = \psi(v_5) \neq \psi(v_3) = \psi(v_6) \neq \psi(v_1)$.*

Proof. Let Δ be a closed disk and let θ_0 be a homeomorphism from the interior of Δ to Λ that extends to a continuous function θ from Δ to the closure of Λ. Let $G_\Lambda = \theta^{-1}(G')$ and $K' = \theta^{-1}(K)$. Note that G_Λ is embeded in Δ and K' is its ring of length $|K|$ tracing the boundary of Δ. Furthermore, if K is a cycle, then G_Λ is isomorphic to G', and if K is not a cycle ($|K| = 6$ and K is a union of two intersecting triangles), then G_Λ is obtained from G' by splitting the vertices appearing multiple times in K in the natural way.

Observe that $\psi' = \psi \circ \theta$ is a 3-coloring of K' that extends to a 3-coloring of G_Λ if and only if ψ extends to a 3-coloring of G. Let G'_Λ be a critical skeleton of G_Λ. If $G'_\Lambda = K'$, then ψ' extends to a 3-coloring of G. Otherwise, Theorem 2.2 implies that $|K'| = 6$ and all faces of G'_Λ have length 4, and thus $\theta(G'_\Lambda)$ is a subgraph of G' whose faces in Λ all have length 4. Furthermore, if ψ' does

not extend to a 3-coloring of G'_Λ (and equivalently, ψ does not extend to a 3-coloring of G'), then ψ must satisfy one of the conditions from the statement of Lemma 2.4 by the second part of Theorem 2.2. □

Let us give an observation about critical graphs that is often useful.

Lemma 2.5. *Let G be a critical graph drawn in the sphere with holes.*

- *Every vertex $v \in V(G)$ that does not belong to the rings has degree at least three.*

- *If K is a (≤ 5)-cycle in G forming the boundary of an open disk Λ, and the closure of Λ does not contain any contractible triangle of G, then Λ is a face of G.*

- *If K is a closed walk of length 6 in G forming the boundary of an open disk Λ, and the closure of Λ does not contain any contractible triangle of G, then either Λ is a face of G, or all faces of G contained in Λ have length 4.*

Proof. Let C be the union of the rings of G, and consider any vertex $v \in V(G) \setminus V(C)$. Suppose for a contradiction that v has degree at most 2. Let ψ be any 3-coloring of C that extends to a 3-coloring φ of $G - v$. Then ψ also extends to a 3-coloring of G, by giving the vertices of $V(G) \setminus \{v\}$ the same color as in the coloring φ and by choosing a color of v distinct from the colors of its neighbors. This contradicts the assumption that G is critical.

The second and third claim follow similarly using Lemma 2.4. □

Furthermore, colorings of tents can be described using Theorem 2.2. Consider a 4-cycle $C = u_1u_2u_3u_4$ and its 3-coloring ψ. We say that ψ is u_1-*diagonal* if $\psi(u_1) \neq \psi(u_3)$, and that it is *bichromatic* if $\psi(u_1) = \psi(u_3)$ and $\psi(u_2) = \psi(u_4)$. Note that every 3-coloring of C is u_1-diagonal, u_2-diagonal, or bichromatic. The following claim is proved analogously to Corollary 2.3.

Corollary 2.6. *If G is a tent with the ring $C = v_1v_2v_3v_4$, then exactly the v_1-diagonal and v_2-diagonal colorings of C extend to 3-colorings of G.*

Let G_0 be either a reduced Thomas-Walls graph, or a Havel-Thomas-Walls graph. Let S be an independent set of vertices of G_0 of degree three. Let $C_0 = v_1v_2v_3v_4$ be a ring of G_0, with the interface pair v_1v_3. Let G_1 be obtained from G_0 by replacing the vertices of S by patches, and let G be obtained from G_1 by framing on its interface pairs. Let $C = v_1v_2'v_3v_4'$ be the ring of G corresponding to C_0. We say that the ring C is *strong* if $G \neq C$, $v_2, v_4 \notin S$, $v_2' = v_2$, and $v_4' = v_4$, that is, v_2 and v_4 are not affected by patching or created by framing. Otherwise, we say that C is *weak*. A 3-coloring ψ of C is *dangerous* if either ψ is v_1-diagonal, or C is strong and ψ is bichromatic.

Let us first deal with Thomas-Walls graphs.

Lemma 2.7. *Let $n \geq 1$ be an integer, let H be a patched Thomas-Walls graph obtained from T'_n by patching, and let G be a graph obtained by framing on interface pairs u_1u_3 and v_1v_3 of H. Let $C_1 = u_1u_2u_3u_4$ and $C_2 = v_1v_2v_3v_4$ be the rings of G and let ψ be a precoloring of $C_1 \cup C_2$. If ψ extends to a 3-coloring of G, then it is dangerous on at most one of C_1 and C_2. Furthermore, if $n \geq 4$ and ψ is not dangerous on both C_1 and C_2, then ψ extends to a 3-coloring of G.*

Proof. Firstly, suppose that G is the reduced Thomas-Walls graph T'_n. We proceed by induction on n. The claims obviously hold when $n = 1$. Hence, assume that $n \geq 2$, and thus both C_1 and C_2 are strong. Let G be obtained from a copy G' of T'_{n-1} with rings C_1 and $C'_2 = v'_1v'_2v'_3v_4$ (with interface pairs u_1u_3 and v'_2v_4) by adding the ring C_2 and the edge $v_2v'_2$. Let ψ be a 3-coloring of $C_1 \cup C_2$.

Suppose for a contradiction that ψ is dangerous on both C_1 and C_2 and extends to a 3-coloring φ of G; then $\psi(v_2) = \psi(v_4)$, and because of the edge $v_2v'_2$, φ is v'_2-diagonal on C'_2, and thus it is dangerous on C'_2. This is a contradiction by the induction hypothesis for G'. Therefore, if ψ extends to a 3-coloring of G, then it is dangerous on at most one of C_1 and C_2.

Suppose now that ψ is dangerous on at most one of C_1 and C_2, and $n \geq 4$. We need to show that ψ extends to a 3-coloring of G. By a straightforward case analysis, this is true when $n = 4$, and thus assume that $n \geq 5$. By symmetry, we can assume that ψ is not dangerous on C_2, and thus $\psi(v_2) \neq \psi(v_4)$. Color v'_2 by $\psi(v_4)$ and give v'_1 and v'_3 distinct colors; the obtained coloring of C'_2 is v'_1-diagonal, and thus it is not dangerous on C'_2. By the induction hypothesis, we can extend the coloring to G'. This gives a 3-coloring of G extending ψ.

Therefore, the claim holds for reduced Thomas-Walls graphs. Suppose that G is obtained from $G_0 = T'_n$ by patching on an independent set S and framing on the interface pairs. Let S_1 be the subset of S consisting of vertices not incident with the rings. Let G_1 be the graph such that G_1 is obtained from G_0 by patching on S_1 and G is obtained from G_1 by patching on $S \setminus S_1$ and framing on the interface pairs. By Corollary 2.3 and the previous analysis of T'_n, the graph G_1 satisfies the conclusions of Lemma 2.7.

Since the vertices of the interface pairs of G_0 have degree two, they do not belong to S. We define a coloring ψ' of the rings of G_1 as follows. Suppose that $C'_1 = u_1u'_2u_3u'_4$ is a ring of G_1, where u'_2 has degree three. Let $\psi'(u_1) = \psi(u_1)$ and $\psi'(u_3) = \psi(u_3)$. If ψ is u_1-diagonal on C_1, or if $u'_2 \notin S$, $u_2 = u'_2$, and $u_4 = u'_4$, then let $\psi'(u'_2) = \psi(u_2)$ and $\psi'(u'_4) = \psi(u_4)$. Otherwise, choose $\psi'(u'_2)$ and $\psi'(u'_4)$ so that ψ'_i is u'_2-diagonal on C'_1 and so that $\psi'(u'_i) = \psi(u_i)$ for all $i \in \{2, 4\}$ such that $u_i = u'_i$. Define ψ' on the other ring C'_2 of G_1 analogously. Note that for $i \in \{1, 2\}$, C'_i is strong in G_1, and if $C'_i \neq C_i$, then C_i is weak in G; hence, ψ is dangerous on C_i if and only if ψ' is dangerous on C'_i.

Suppose first for a contradiction that ψ is dangerous both on C_1 and C_2, and that it extends to a 3-coloring φ of G.

- If C_1 is weak, then ψ is u_1-diagonal on C_1, and $\psi'(u'_i) = \varphi(u'_i) = \psi(u_i)$ for $i \in \{2, 4\}$. Let z be the neighbor of u'_2 distinct from u_1 and u_3. If $u'_2 \notin S$, then u'_2z is an edge of G, and $\varphi(z) \neq \psi'(u'_2)$. If $u'_2 \in S$, then

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$\varphi(z) \neq \psi'(u_2')$ by the second part of Theorem 2.2 applied to the patch P replacing u_2' and the 3-coloring of P given by φ.

- If C_1 is strong, then $u_2' = u_2$, $u_4' = u_4$, and $u_2'z$ is an edge of G.

Together with a similar argument applied at C_2, we conclude that $\psi' \cup (\varphi \upharpoonright V(G_1))$ is a 3-coloring of G_1 extending ψ', which is a contradiction since ψ' is dangerous both on C_1' and C_2' and the conclusions of Lemma 2.7 are satisfied by G_1 as we argued before.

Next, suppose that ψ is not dangerous on one of the rings, say on C_1, and that $n \geq 4$. Then ψ' is not dangerous on C_1', and thus it extends to a 3-coloring φ' of G_1. If $u_2' \notin S$, then let φ_1 be an empty coloring. If $u_2' \in S$, then by Theorem 2.2, there exists a 3-coloring φ_1 of the patch replacing u_2' such that $\varphi_1(x) = \psi(x)$ for $x \in \{u_1, u_2, u_3\}$ and $\varphi_1(z) = \varphi'(z)$, where z is the neighbor of u_2' distinct from u_1 and u_3. Let φ_2 be defined analogously at C_2. Observe that $\psi \cup \varphi_1 \cup \varphi_2 \cup (\varphi' \upharpoonright V(G))$ is a 3-coloring of G extending ψ. $\qquad\square$

Using this lemma, we can easily handle Havel-Thomas-Walls graphs as well.

Lemma 2.8. *Let G be a graph obtained by framing on the interface pair v_1v_3 of a patched Havel-Thomas-Walls graph, with ring $C = v_1v_2v_3v_4$. A 3-coloring ψ of C extends to a 3-coloring of G if and only if ψ is not dangerous on C.*

Proof. Consider first the case that G is a Havel-Thomas-Walls graph, obtained from the reduced Thomas-Walls graph T_n' with rings C and $C' = u_1u_2u_3u_4$ and interface pairs v_1v_3 and u_1u_3 by either adding the edge u_1u_3, or the graph in Figure 2(a). Consider any 3-coloring φ of G. Note that both the edge u_1u_3 and the graph from Figure 2(a) ensure that $\varphi(u_1) \neq \varphi(u_3)$, and thus φ is u_1-diagonal on C'. Consequently, φ is dangerous on C', and by Lemma 2.7, it is not dangerous on C. Therefore, if ψ extends to a 3-coloring of G, then ψ is not dangerous on C.

Conversely, if ψ is not dangerous on C, then we can extend it to a 3-coloring of T_n' that is u_1-diagonal on C' (for $n \geq 4$, this follows by Lemma 2.7; for $1 \leq n \leq 3$, it is easy to construct the desired colorings), and further extend the coloring to the graph from Figure 2(a) if present in G.

The case that G is obtained from a Havel-Thomas-Walls graph by patching and framing is handled in the same way as in the proof of Lemma 2.7. $\qquad\square$

Next, we consider the colorability of near $3,3$-quadrangulations. We need some additional definitions. Given a 3-coloring $\psi : V(G) \to \{0, 1, 2\}$ of a graph G, let us define an orientation G_ψ of G by orienting every edge $uv \in E(G)$ towards v if and only if $\psi(v) - \psi(u) \in \{1, -2\}$. Let $W = v_1v_2 \ldots v_k$ be a walk in a graph G. We define $\omega(W, \psi)$ to be the difference between the number of forward and backward edges of W in G_ψ, i.e.,

$$\omega(W, \psi) = |\{i : 1 \leq i \leq k-1, v_iv_{i+1} \in E(G_\psi)\}| - |\{i : 1 \leq i \leq k-1, v_{i+1}v_i \in E(G_\psi)\}|.$$

Suppose that G is embedded in the cylinder, and let C be a non-contractible cycle in G. We fix one orientation around the cylinder as *positive*. Let W (with

$v_1 = v_k)$ be a closed walk tracing C in the positive direction. We define the *winding number* of ψ on C as $\omega(W, \psi)/3$. Observe that the winding number is an integer of the same parity as the length of C. We need the following result concerning the colorability of quadrangulations, see e.g. the Propositions 4.1 and 4.2 in [4].

Lemma 2.9. *Let G be a graph embedded in the cylinder with rings C_1 and C_2, such that all non-ring faces of G have length 4. For any 3-coloring φ of G, the winding number of φ on C_1 is equal to the winding number of φ on C_2.*

We also need a strengthening of Lemma 2.9 for quadrangulations of the cylinder with rings of length at most 4.

Lemma 2.10. *Let G be a tame graph embedded in the cylinder with rings C_1 and C_2 of length at most 4. If all faces of G have length 4, the distance between C_1 and C_2 is at least $|C_1|$, and ψ is a precoloring of $C_1 \cup C_2$ that does not extend to a 3-coloring of G, then $|C_1| = |C_2| = 3$ and ψ has opposite winding numbers on C_1 and C_2, i.e. ψ has winding number $+1$ on one of them and -1 on the other one.*

Proof. Note that C_1 and C_2 have the same parity, and thus $|C_1| = |C_2|$. Suppose first that $|C_1| = 4$. In this case, G is bipartite. Let $C_1 = v_1 v_2 v_3 v_4$, with the labels chosen so that $\psi(v_1) = \psi(v_3)$. For $0 \le i \le 3$, let S_i denote the set of vertices of G at distance exactly i from $\{v_2, v_4\}$. Let G' be the graph obtained from $G - (S_0 \cup S_1 \cup S_2)$ by identifying all vertices in S_3 to a single vertex x. Note that G' is also bipartite, and in particular it has no loops and it is triangle-free. Furthermore, since the distance between C_1 and C_2 is at least four, C_2 is a cycle in G'. By Lemma 2.4, the graph G' has a 3-coloring φ' that matches ψ on C_2. Then, we can obtain a 3-coloring of G that extends ψ by coloring every vertex $v \in V(G') \setminus \{x\}$ by the color $\varphi'(v)$, all vertices in S_3 by the color $\varphi'(x)$, all vertices in S_2 by a color distinct from $\varphi'(x)$ and $\psi(v_1)$, all vertices in S_1 by the color $\psi(v_1)$, and v_2 by $\psi(v_2)$ and v_4 by $\psi(v_4)$.

Suppose now that $|C_1| = 3$. Note that the winding number of ψ on each of C_1 and C_2 is either $+1$ or -1. By Lemma 2.9, if the winding numbers of ψ on C_1 and C_2 are opposite, then ψ does not extend to a 3-coloring of G. Hence, we can assume that the winding numbers of ψ on C_1 and C_2 are the same. For $i \in \{1, 2\}$, let $C_i = v_{i,1} v_{i,2} v_{i,3}$, with the labels chosen so that $\psi(v_{i,j}) = j$ for $j \in \{1, 2, 3\}$. For $j \in \{1, 2, 3\}$, let $S_{i,j}$ denote the set of vertices of G adjacent to $v_{i,j}$ that do not belong to $V(C_i)$. Since G is tame and the distance between C_1 and C_2 is at least three, these sets are pairwise disjoint. Let G' be the graph obtained from G by, for $i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$, identifying all vertices in $S_{i,j}$ to a single vertex $x_{i,j}$, and suppressing the resulting faces of length two. Note that $K_i = x_{i,1} x_{i,2} x_{i,3}$ is a non-contractible triangle in G'. If ψ extends to a 3-coloring of G', then it also extends to a 3-coloring of G, obtained by giving each vertex in $S_{i,j}$ the color of $x_{i,j}$.

Let G_0 be the subgraph of G' drawn between K_1 and K_2. By Theorem 1.3, there exists a 3-coloring φ_1 of G_0 such that $\varphi_1(x_{1,j}) = (j \bmod 3) + 1$ for $j \in$

$\{1, 2, 3\}$. By permuting the colors in φ_1, we obtain a 3-coloring φ_2 of G_0 such that $\varphi_2(x_{1,j}) = ((j+1) \bmod 3) + 1$ for $j \in \{1, 2, 3\}$ and $\varphi_1(v) \neq \varphi_2(v)$ for every $v \in V(G_0)$. Suppose that $\varphi_1 \cup \psi$ is not a 3-coloring of G'. By Lemma 2.9, the winding numbers of φ_1 on K_1 and K_2 are the same, and thus $\varphi_1(x_{2,j}) = \psi(v_{2,j})$ for $j \in \{1, 2, 3\}$. Then, it follows that $\psi \cup \varphi_2$ is a 3-coloring of G'. We conclude that G', and thus also G, has a 3-coloring which extends ψ. $\qquad \square$

Let G be a near 3,3-quadrangulation, let C be one of the rings of G and let ψ be a 3-coloring of C. Let us discuss several cases:

- If C shares an edge with a triangle T in G (where possibly $T = C$ if C is a triangle), then ψ uniquely extends to a 3-coloring φ of T. Let w be the winding number of φ on T. In this case, we say that ψ on C *causes winding number w*.

- If C does not share an edge with a triangle and we can label the vertices of C as $v_1 v_2 v_3 v_4$ so that the path $v_1 v_2 v_3$ is a part of the boundary of a 5-face f and $\psi(v_1) \neq \psi(v_3)$, then draw an edge between v_1 and v_3 in f, and let w be the winding number of ψ on the triangle $v_1 v_3 v_4$. In this case, we also say that ψ on C *causes winding number w*.

- Otherwise, we say that ψ on C *does not cause fixed winding number*.

If C_1 and C_2 are the rings of G and ψ is their precoloring, we say that ψ is *inconsistent* if ψ causes winding numbers on both C_1 and C_2 and these winding numbers are opposite. Otherwise, ψ is *consistent*.

Lemma 2.11. *Let G be a tame near 3,3-quadrangulation embedded in the cylinder with rings C_1 and C_2. If a precoloring ψ of $C_1 \cup C_2$ extends to a 3-coloring of G, then it is consistent. Furthermore, if the distance between C_1 and C_2 is at least 9, then every consistent precoloring of $C_1 \cup C_2$ extends to a 3-coloring of G.*

Proof. Let G_0 be a graph and ψ_0 a 3-coloring of its rings obtained from G and ψ as follows. For $i = 1, 2$, if C_i shares an edge with a triangle T, then remove all vertices between C_i and T (excluding T, but including $V(C_i) \setminus V(T)$), and let ψ_0 restricted to T be the unique 3-coloring of T that matches ψ on $V(T) \cap V(C_i)$. If $C_i = v_1 v_2 v_3 v_4$, $v_1 v_2 v_3$ is a part of the boundary of a 5-face f, and $\psi(v_1) \neq \psi(v_3)$, then remove v_2 and add the edge $v_1 v_3$ drawn inside f, and let ψ_0 restricted to $v_1 v_3 v_4$ match ψ. Finally, if $\psi(v_1) = \psi(v_3)$, then do not alter G at C_i and let ψ_0 restricted to C_i match ψ. Let C_1' and C_2' be the rings of G_0 obtained from C_1 and C_2, respectively.

Using Theorem 1.3, observe that ψ extends to a 3-coloring of G if and only if ψ_0 extends to a 3-coloring of G_0. Note that if ψ is inconsistent, then G_0 is a 3,3-quadrangulation and the winding numbers of ψ_0 on C_1' and C_2' are opposite, and thus by Lemma 2.9, ψ_0 does not extend to a 3-coloring of G_0.

Therefore, it suffices to consider the case that ψ is consistent and the distance between C_1 and C_2 in G is at least 9, and to show that in this case, ψ_0 extends

to a 3-coloring of G_0. If ψ on C_1 or C_2 causes winding number, then let w be this winding number. Otherwise, let $w = 1$.

We now modify G_0 and ψ_0 into another auxiliary graph G_1 with a precoloring ψ_1 of its rings. For each $i = 1, 2$ such that $|C_i'| = 4$, we modify the graph as follows. Let $C_i' = v_1 v_2 v_3 v_4$, where $v_1 v_2 v_3 u_4 u_5$ is a 5-face and $\psi_0(v_1) = \psi_0(v_3)$. Let φ be the unique 3-coloring of the 5-cycle $K = v_1 u_5 u_4 v_3 v_4$ matching ψ_0 on v_1, v_3, and v_4, such that φ has winding number w on K. If necessary, exchange the labels of v_1 with v_3 and of u_4 with u_5 so that $\varphi(v_4) = \varphi(u_5)$. If v_1 is contained in a triangle T, then remove from G_1 all vertices between C_i' and T (excluding T, but including $V(C_i') \setminus V(T)$), and let ψ_1 restricted to T be the 3-coloring with winding number w such that $\psi_0(v_1) = \psi_1(v_1)$ (since T does not share an edge with C_i', $\varphi(v_1) = \varphi(v_3)$, and $\varphi(v_4) = \varphi(u_5)$, the coloring φ extends to a 3-coloring of the subgraph of G_0 drawn between K and T by Theorem 2.1, and the restriction of this 3-coloring to T matches ψ_1 by Lemma 2.9). If v_1 is not contained in any triangle, then remove v_1 and v_2 and identify all remaining neighbors of v_1 to a single vertex x, and let ψ_1 restricted to $x u_4 v_3$ be the 3-coloring such that $\psi_1(v_3) = \varphi(v_3)$, $\psi_1(u_4) = \varphi(u_4)$ and $\psi_1(x) = \varphi(v_4) = \varphi(u_5)$. For each $i = 1, 2$ such that $|C_i'| = 3$, we do not modify the graph and we let ψ_1 restricted to C_i' match ψ_0.

Observe that G_1 is a 3, 3-quadrangulation and that if ψ_1 extends to a 3-coloring of G_1, then we can obtain a 3-coloring of G_0 that extends ψ_0. Let C_1'' and C_2'' be the rings of G_1. Let T_1 and T_2 be non-contractible triangles in G_1 such that the subgraph G_1' drawn between T_1 and T_2 is tame and G_1' is as large as possible, with labels chosen so that T_1 separates C_1'' from T_2. Since G is tame and the distance between C_1 and C_2 in G is at least 9, the construction of G_0 and G_1 ensures that the distance between T_1 and T_2 in G_1' is at least three.

By Theorem 1.3, ψ_1 extends to a 3-coloring φ_1 of the subgraphs of G_1 drawn between C_1'' and T_1, and between C_1'' and T_2, and by Lemma 2.9, φ_1 has winding number w both on T_1 and T_2. By Lemma 2.10, the restriction of φ_1 to $T_1 \cup T_2$ extends to a 3-coloring φ_2 of G_1'. Hence, ψ_1 extends to a 3-coloring $\varphi_1 \cup \varphi_2$ of G_1. This also gives a 3-coloring of G extending ψ. $\qquad\square$

3 Basic cylinders

We now make the first step towards the proof of Theorem 1.1, by studying the coloring properties of chains of graphs. Let G and G' be graphs embedded in a surface with the same rings, and let C be the union of the rings. We say that G *dominates* G' if every precoloring of C that extends to a 3-coloring of G also extends to a 3-coloring of G'. We aim to show that for any chain G with sufficiently many graphs that are not quadrangulations, either all precolorings of the rings of G extend, or G is dominated by the reduced Thomas-Walls graph T_4'. To do so, we contract 4-faces in the graphs of the chain as long as possible, ending up with easily enumerated and analyzed "basic" graphs.

Let us give a few more definitions. We say that a graph H embedded in the cylinder with rings K_1 and K_2 is a *quadrangulation* if all its non-ring faces have

length 4. We say that H is *quadrangulated* if it contains a quadrangulation with rings K_1 and K_2 as a subgraph.

Let H be a graph embedded in the cylinder without contractible triangles, and let $T_y = xy_1y_2$ and $T_z = xz_1z_2$ be triangles in H with the vertices listed in the positive direction around the cylinder. By *collapsing* T_y and T_z, we mean removing the vertices and edges of H that are separated from the rings by $T_y \cup T_z$, and identifying y_i with z_i for $i = 1, 2$. Let H' be obtained from H by collapsing T_y and T_z. By Lemma 2.4 (applied to the subgraph of H drawn between T_y and T_z), every 3-coloring of H' extends to a 3-coloring of H, and thus H' dominates H.

Let G be a tame chain of graphs $G_1, \ldots, G_n$ with the sequence of cutting cycles $\mathcal{C} = C_0, \ldots, C_n$. We say that the vertices of the cutting cycles are *special*. Let $f = x_1x_2x_3x_4$ be a 4-face in G_i for some $i \in \{1, \ldots, n\}$. We call the identification of x_1 and x_3 to a new vertex x *legal* if

- at most one of x_1 and x_3 is special, and

- G_i does not contain (not necessarily distinct) paths $x_1z_1z_2x_3$ and $x_1z_1'z_2'x_3$ such that $\{x_1, z_1, z_2, x_3\} \cap V(C_{i-1}) \neq \emptyset$ and $\{x_1, z_1', z_2', x_3\} \cap V(C_i) \neq \emptyset$.

If a graph H is obtained from a graph G by a legal identification, then H dominates G since every 3-coloring of H extends to a 3-coloring of G. Note that if the indentification of x_1 with x_3 creates parallel edges, we suppress them, i.e., keep only one (arbitrary) of them. Since G is tame, x_1 and x_3 are not adjacent, and thus the identification does not create loops. Also, by the second condition of the legality, we can collapse the triangles possibly created by the identification while keeping the rings of G_i disjoint.

Let G be a tame graph embedded in the cylinder with rings of length at most 4. We say that G is *basic* if either G contains no contractible 4-cycle, or it is one of the graphs depicted in Figure 3 (the rings of each depicted basic graph are the cycle bounding its outer face and the cycle bounding its central face of length 3 or 4). We aim to show that it suffices to consider the chains of basic graphs.

We need the following result concerning extension of a precoloring of a (≤ 9)-cycle.

Theorem 3.1 (Thomassen [17]). *Let G be a graph of girth at least 5 drawn in the disk with the ring C of length at most 8. If G is critical, then $|C| = 8$ and $G = C + e$ for an edge e with both ends in C.*

Borodin et al. [1] investigated precolorings of 7-faces in planar triangle-free graphs, and their result has the following corollary.

Theorem 3.2 (Borodin et al. [1]). *Let G be a triangle-free graph embedded in the disk with the ring C of length 7 tracing its boundary. If G is critical, then it has exactly one 5-face f, all faces other than f have length 4, and the cycle bounding f intersects C in a path of length at least two. Furthermore, if ψ is a precoloring of C that does not extend to a 3-coloring of G and xyz is a subpath of C such that $\psi(x) = \psi(z)$, then y is incident with f.*
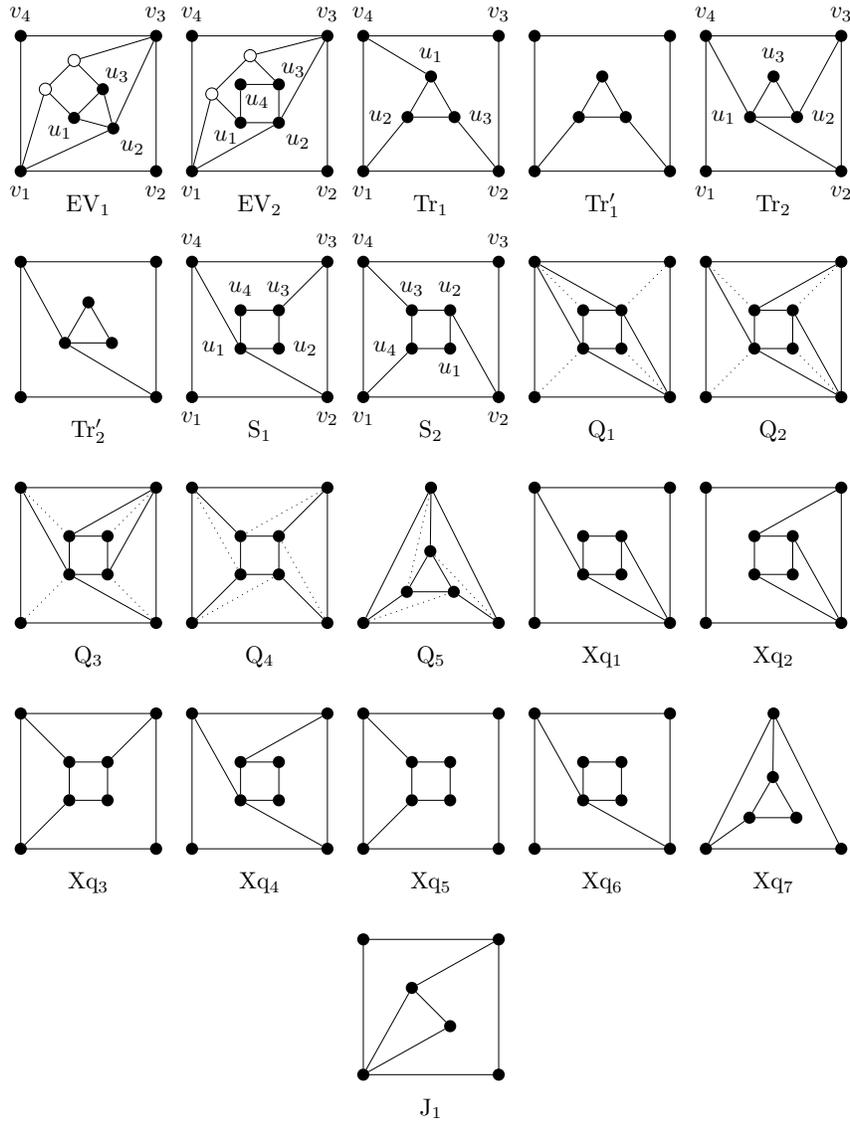
Figure 3: Basic graphs with contractible 4-cycles. The dotted lines indicate how to identify vertices in quadrangulations.

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We now turn our attention to legal identifications. Let G be a chain of critical graphs $G_1, \ldots, G_n$ with the sequence of cutting cycles $\mathcal{C} = C_0, \ldots, C_n$. Let $r(G, \mathcal{C})$ denote the number of the graphs among $G_1, \ldots, G_n$ that are not quadrangulations.

Lemma 3.3. *Let G be a tame graph embedded in the cylinder with rings of length at most 4. Suppose that G is chain of critical graphs $G_1, \ldots, G_n$ with cutting cycles $\mathcal{C} = C_0, \ldots, C_n$. If there exists $i \in \{1, \ldots, n\}$ such that G_i contains a 4-face allowing a legal identification, then there exists a tame graph H embedded in the cylinder with the same rings, such that*

- *$|V(H)| < |V(G)|$ and H dominates G, and*

- *H is a chain of at least n critical graphs with cutting cycles $\mathcal{C}'$ satisfying $r(H, \mathcal{C}') \geq r(G, \mathcal{C})$.*

Proof. Let $f = x_1 x_2 x_3 x_4$ be a 4-face allowing a legal identification of x_1 and x_3 in G_i. Let a chain H with cutting cycles $\mathcal{C}'$ be obtained from G as follows. First, identify x_1 with x_3 to a new vertex x, obtaining a graph H'. Collapse all intersecting non-contractible triangles in H' to a single triangle, obtaining a graph H''. Note that H'' contains at most one triangle T not belonging to $\mathcal{C}$. Furthermore, by the legality of the identification, the cycles C_{i-1} and C_i in H'' are vertex-disjoint and at most one of them (of length 4) intersects T. If T does not exist, then let $\mathcal{C}' = \mathcal{C}$. Otherwise, let $\mathcal{C}' = C_0', \ldots, C_{n'}'$ be the sequence of cutting cycles in H'' obtained from $\mathcal{C}$ by adding T and removing the 4-cycle distinct from C_0 and C_n that intersects T, if any. For $j = 1, \ldots, n'$, let H_j'' denote the subgraph of H'' drawn between C_{j-1}' and C_j'. Finally, replace H_j'' by its maximal critical subgraph H_j for $j = 1, \ldots, n'$, obtaining the graph H.

Suppose for a contradiction that H' contains a contractible triangle $x z_1 z_2$. Then $x_1 x_2 x_3 z_1 z_2$ is a contractible 5-cycle in G_i, and by Lemma 2.5, this 5-cycle bounds a face of G_i. Hence, x_2 has degree two in G_i, and by Lemma 2.5, x_2 is special. But then also both neighbors of x_2 would be special, which contradicts the assumption that the identification of x_1 and x_3 is legal. Therefore, H' contains no contractible triangles, and consequently, H'' and H are tame. Clearly, H dominates G and $|V(H)| < |V(G)|$. Note that $\mathcal{C}'$ contains all the triangles of H, and that $|\mathcal{C}'| \geq |\mathcal{C}|$. Hence, if $r(H, \mathcal{C}') \geq r(G, \mathcal{C})$, then H satisfies all the conditions of Lemma 3.3. Let us argue that we can choose the face f and its labelling so that the identification of x_1 and x_3 is legal and $r(H, \mathcal{C}') \geq r(G, \mathcal{C})$. We distinguish several cases.

Suppose first that $f = x_1 x_2 x_3 x_4$ is a face of G_i such that at most one of x_1 and x_3 is special and G_i contains no path $x_1 z_1 z_2 x_3$ of length three disjoint with $\{x_2, x_4\}$. Note that this implies that the identification of x_1 with x_3 is legal. Consider the procedure described at the beginning of the proof. In this case $H'' = H'$ and $\mathcal{C}' = \mathcal{C}$, and H is obtained from G by replacing G_i with H_i. Suppose that H_i is a quadrangulation. For every 4-face h of H_i, either h corresponds to a 4-face of G_i, or h corresponds to a contractible 6-cycle K in G_i containing the path $x_1 x_2 x_3$. In the latter case, note that K does not

bound a face, since then x_2 would have degree two and Lemma 2.5 would imply that x_1, x_2 and x_3 are special vertices, contrary to the assumption that the identification of x_1 and x_3 is legal. Hence, Lemma 2.5 implies that all faces of G_i inside K have length 4. We conclude that if H_i is a quadrangulation, then G_i is a quadrangulation as well, and thus $r(H, \mathcal{C}') \geq r(G, \mathcal{C})$.

Let us remark that if the identification of x_1 with x_3 does not create a new triangle, then the legality only requires that not both x_1 and x_3 are special. Therefore, it remains to consider the case that the following holds.

For every 4-face $y_1y_2y_3y_4$ in G_i such that not both y_1 and y_3 are special, there exists a path in G_i of length three joining y_1 with y_3 and disjoint with $\{y_2, y_4\}$.
$$\tag{1}$$

Next, suppose that $f = x_1x_2x_3x_4$ is a face in G_i such that x_3 and x_4 are not special. By (1), G contains paths of length three between x_1 and x_3, and between x_2 and x_4, and the two paths must intersect, forming a triangle. Since all triangles are contained in $\mathcal{C}$, this triangle is equal to say C_i and contains x_1 and x_2. Hence, G contains paths $x_1zy_1x_3$ and $x_2zy_2x_4$, where $C_i = x_1x_2z$. At least one of the 4-cycles $zx_2x_3y_1$ and $zx_1x_4y_2$ (say the former) is contractible. By Lemma 2.5, $zx_2x_3y_1$ is a face. Note that since G is tame and x_3 and x_4 are not special, y_1 is not adjacent to y_2 or x_4. Consequently, G contains no path of length three between y_1 and x_2 disjoint with $\{z, x_3\}$, and by (1), it follows that $y_1 \in V(C_{i-1})$. Therefore, the 4-cycle $zx_1x_4y_2$ is also contractible, and by symmetry, $y_2 \in V(C_{i-1})$. However, then G_i is isomorphic to the graph EV_1 (if $y_1 \neq y_2$) or Q_5 (if $y_1 = y_2$) from Figure 3, and contains no 4-face allowing a legal identification. This contradicts the assumptions of Lemma 3.3.

By symmetry, it follows that every edge of a 4-face is incident with a special vertex. In particular, the following holds.

For every 4-face $y_1y_2y_3y_4$ in G_i, there exists $k \in \{1, 2\}$ such that both y_k and y_{k+2} are special.
$$\tag{2}$$

Consider now a 4-face $f = x_1x_2x_3x_4$ in G_i such that x_1 is not special. By (2), x_2 and x_4 are special. If $x_2, x_4 \in V(C_j)$ for some $j \in \{i-1, i\}$, then x_1 and x_3 are in different components of $G_i - \{x_2, x_4\}$, which contradicts (1). Hence, we have a strengthening of (2):

For every 4-face $y_1y_2y_3y_4$ in G_i, there exists $k \in \{1, 2\}$ such that one of y_k and y_{k+2} belongs to C_{i-1} and the other one to C_i.
$$\tag{3}$$

Let us now consider any 4-face $f = x_1x_2x_3x_4$ in G_i such that the identification of x_1 with x_3 is legal. By symmetry, we can assume that x_1 is not special, and by (3), we can assume that $x_2 \in V(C_{i-1})$ and $x_4 \in V(C_i)$. Since x_1 is not contained in a triangle, (3) implies that x_2 and x_4 are the only special neighbors of x_1.

Let us now distinguish two subcases depending on whether x_3 is special or not. Let us first consider the subcase that x_3 is not special. By symmetry, x_2 and x_4 are the only special neighbors of x_3. In this case, we claim that the

graph H constructed at the beginning of the proof satisfies $r(H, \mathcal{C}') \geq r(G, \mathcal{C})$, and thus Lemma 3.3 holds. By (1), the identification creates a new triangle T. Since x_1 and x_3 have no special neighbors other than x_2 and x_4, even after the collapse of the triangles during the construction of H'', T is vertex-disjoint with C_{i-1} and C_i. Hence, H is obtained from G by replacing G_i with two subgraphs H_j and H_{j+1} for some $j \in \{1, \ldots, n'-1\}$ such that H_j and H_{j+1} intersect in T. Clearly, $r(H, \mathcal{C}') \geq r(G, \mathcal{C})$, unless both H_j and H_{j+1} are quadrangulations and G_i is not a quadrangulation. As we argued before, every 4-face of $H_j \cup H_{j+1}$ corresponds to either a 4-face in G, or to a contractible walk of length 6 bounding a quadrangulated disk in G. Hence, all the (≥ 5)-faces of G are destroyed by collapsing the triangles, that is, G_i contains two paths $x_1 y_1 y_2 x_3$ and $x_1 y_1' y_2' x_3$ such that the open disk Λ bounded by the closed walk $x_1 y_1 y_2 x_3 y_2' y_1'$ contains a face of G_i of length greater than 4, and all faces of G_i not contained in Λ have length 4. But then the edge $y_1 y_2$ is contained in a 4-face in G_i. Since y_1 and y_2 are neighbors of x_1 and x_3, respectively, they are not special. This contradicts (2).

Finally, we consider the subcase that x_3 is special. By symmetry, we can assume that $x_3 \in V(C_i)$. If C_i is a triangle, then since the identification of x_1 and x_3 is legal, we conclude that no path $x_1 y_1 y_2 x_3$ intersects C_{i-1}, and by the same argument as in the previous paragraph, we show that $r(H, \mathcal{C}') \geq r(G, \mathcal{C})$. Therefore, suppose that $|C_i| = 4$.

By (1), G_i contains a path $x_1 y_1 y_2 x_3$ disjoint from $\{x_2, x_4\}$. Since the identification of x_1 with x_3 is legal, we have $y_2 \notin V(C_{i-1})$. If $y_2 \notin V(C_i)$, then let W be the contractible closed walk of length 7 consisting of $x_4 x_1 y_1 y_2 x_3$ and the path of length three between x_3 and x_4 in C_i. If $y_2 \in V(C_i)$, then let W be the contractible closed walk of length 5 consisting of $x_4 x_1 y_1 y_2$ and a path of length two between y_2 and x_4 in C_i. In both cases, let Λ be the open disk bounded by W. Since $y_2 \notin V(C_{i-1})$, (3) implies that Λ does not contain any 4-face of G_i, and by Theorem 3.1, we conclude that Λ is a face. Since this holds for every path of length three between x_1 and x_3, it follows that $x_1 y_1 y_2 x_3$ is the only such path, and thus H' contains only one new triangle $T = xy_1 y_2$. Let H_j be the subgraph of H between C_{i-1} and T. Note that H_j is not a quadrangulation, since otherwise G_i would contain a 4-cycle incident with x_1 and y_1, contrary to (2). Consider the other subgraph H_{j+1} in the chain that contains T. Since Λ is a face, it follows that y_1 has degree two in H_{j+1}, and since H is tame, the face of H_{j+1} incident with y_1 has length greater than 4 and H_{j+1} is not a quadrangulation. Therefore, $r(H, \mathcal{C}') \geq r(G, \mathcal{C})$ as required. $\square$

Now, let us show that legal identification is always possible in a non-basic graph.

Lemma 3.4. *Let G be a tame critical graph embedded in the cylinder with rings C_1 and C_2 of length at most 4, such that every triangle of G is equal to one of the rings. Assume furthermore that the rings of G are either vertex-disjoint, or one of them has length 3 and the other one length 4. If G has no 4-face admiting a legal identification (with respect to the sequence C_1, C_2 of cutting cycles), then G is basic.*

Proof. Since no 4-face of G admits a legal identification,

for any 4-face $x_1x_2x_3x_4$ of G such that $x_1 \notin V(C_1 \cup C_2)$, there exists a path of length three between x_1 and x_3 disjoint with $\{x_2, x_4\}$.

$$(4)$$

Suppose that G contains a 4-face $x_1x_2x_3x_4$ such that $x_3, x_4 \notin V(C_1 \cup C_2)$. By (4), G contains paths of length three between x_1 and x_3, and between x_2 and x_4, and the two paths must intersect, forming a triangle. This triangle is equal to say C_1 and contains x_1 and x_2. Hence, G contains paths $x_1zy_1x_3$ and $x_2zy_2x_4$, where $C_1 = x_1x_2z$. At least one of the 4-cycles $zx_2x_3y_1$ and $zx_1x_4y_2$ (say the former) is contractible. By Lemma 2.5, $zx_2x_3y_1$ is a face. Note that since G is tame and $x_3, x_4 \notin V(C_1 \cup C_2)$, y_1 is not adjacent to y_2 or x_4. Consequently, G contains no path of length three between y_1 and x_2 disjoint with $\{z, x_3\}$, and by (4), it follows that $y_1 \in V(C_2)$. Therefore, the 4-cycle $zx_1x_4y_2$ is also contractible, bounds a face, and by symmetry, $y_2 \in V(C_2)$. However, then G is isomorphic to the basic graph Q_5 or EV_1.

Hence, we can assume that each edge of a 4-face of G is incident with a vertex of $C_1 \cup C_2$, or equivalently

for every 4-face $y_1y_2y_3y_4$ in G, there exists $k \in \{1, 2\}$ such that both y_k and y_{k+2} belong to $V(C_1 \cup C_2)$.

$$(5)$$

Suppose that $x_1x_2x_3x_4$ is a 4-face in G such that $x_1 \notin V(C_1 \cup C_2)$. By (5), $x_2, x_4 \in V(C_1 \cup C_2)$. If $x_2, x_4 \in V(C_j)$ for some $j \in \{1, 2\}$, then x_1 and x_3 are in different components of $G - \{x_2, x_4\}$, which contradicts (4). Hence, we have a strengthening of (5):

For every 4-face $y_1y_2y_3y_4$ in G_i, there exists $k \in \{1, 2\}$ such that one of y_k and y_{k+2} belongs to C_1 and the other one to C_2.

$$(6)$$

Suppose that $x_1x_2x_3x_4$ is a 4-face such that $x_1 \notin V(C_1 \cup C_2)$. By symmetry, we can assume that $x_2 \in V(C_1)$ and $x_4 \in V(C_2)$. Since x_1 is not contained in a triangle, (6) implies that x_2 and x_4 are the only neighbors of x_1 in $V(C_1 \cup C_2)$.

If x_3 is not special, then by symmetry, x_2 and x_4 are the only neighbors of x_3 in $V(C_1 \cup C_2)$. However, then no path $x_1z_1z_2x_3$ intersects C_1 or C_2, and thus the identification of x_1 and x_3 is legal, which is a contradiction.

Hence,

every 4-face of G intersects $C_1 \cup C_2$ in at least three vertices.

$$(7)$$

Suppose that C_1 and C_2 are not disjoint, and thus say $|C_1| = 3$ and $|C_2| = 4$. If C_1 and C_2 share an edge, then Lemma 2.5 implies that $G = C_1 \cup C_2$ has no 4-face, and thus it is basic. Let us consider the case that C_1 and C_2 share only one vertex. If G contains an edge $e \notin E(C_1 \cup C_2)$ with both ends in $V(C_1 \cup C_2)$, then since G is tame, we conclude that G is the basic graph J_1. If G contains no such edge and contains a 4-face, then by (7), such a 4-face shares two edges with $C_1 \cup C_2$. However, then Theorem 3.2 gives a contradiction with (6) or (7).

Therefore, we can assume that C_1 and C_2 are vertex-disjoint. Suppose that $x_1x_2x_3x_4$ is a 4-face such that $x_1 \notin V(C_1 \cup C_2)$. By (6) and (7), we can assume that $x_2 \in V(C_1)$ and $x_3, x_4 \in V(C_2)$. Recall that x_2 and x_4 are the only neighbors of x_1 in $V(C_1 \cup C_2)$. Since the identification of x_1 and x_3 is not legal, there exists a path $x_1y_1y_2x_3$ disjoint with $\{x_2, x_4\}$ such that $y_2 \in V(C_1)$. Since $x_2 \in C_1$, $x_3 \in V(C_2)$, C_1 and C_2 are vertex-disjoint, and G contains no non-ring triangles, it follows that x_2x_3 is not contained in a triangle. It follows that y_2 is not adjacent to x_2, and thus $|C_1| = 4$. Let $C_1 = x_2z_1y_2z_2$, with the labels chosen so that $x_2x_3y_2z_1$ is a contractible 4-cycle. By Lemma 2.5, $x_2x_3y_2z_1$ and $x_2z_2y_2y_1x_1$ are faces. Also by Lemma 2.5, y_1 has degree at least three. By (7), the edge x_1y_1 is not incident with a 4-face, and thus by Theorems 2.2 and 3.2 applied to the cycle consisting of $x_4x_1y_1y_2x_3$ and the path of length $|C_2| - 1$ in C_2 between x_3 and x_4 implies that $|C_2| = 4$ and the edge y_1y_2 is incident with a 4-face. By (7), y_1 has a neighbor in C_2, and thus G is isomorphic to the basic graph EV_2.

It follows that every 4-face in G has all vertices contained in $V(C_1 \cup C_2)$. Note that since G is tame, C_1 and C_2 are induced cycles. If G has no 4-face, then by Lemma 2.5 it contains no contractible 4-cycle, and thus it is basic. Hence, it suffices to consider the case that G contains a 4-face $f = x_1x_2x_3x_4$. By symmetry, we can assume that $x_1, x_2 \in V(C_1)$ and $x_4 \in V(C_2)$. Let h be the face of $C_1 \cup C_2 \cup \{x_2x_3, x_3x_4, x_4x_1\}$ distinct from f and the rings. Note that $|h| = |C_1| + |C_2| \le 8$. If G has no face of length 4 other than f, then by Theorem 3.1 applied to the subgraph of h drawn in the closure of h, we conclude that either h is a face of G and G is one of the basic graphs Tr_1', Tr_2', Xq_5, Xq_6, or Xq_7; or $|C_1| = |C_2| = 4$ and one edge of G is drawn in h, and G is a basic graph S_1 or S_2.

Let us consider the case that G has a face f' of length 4 distinct from f; hence, an edge of G splits h into a 4-face f' and a $(|C_1| + |C_2| - 2)$-face h'. If G contains no 4-face other than f and f', then by Theorem 3.1, h' is a face of G, and thus G is one of the basic graphs Xq_1, Xq_2, Xq_3, Xq_4, Q_5, Tr_1, or Tr_2. Finally, G might also have four 4-faces, and then G is one of the basic graphs Q_1, Q_2, Q_3, or Q_4. $\hfill\square$

By combining Lemmas 3.3 and Lemma 3.4, we obtain the following.

Lemma 3.5. *Let G be a tame graph embedded in the cylinder with rings of length at most 4. Suppose that G is a chain of $n \ge 1$ critical graphs with the sequence $\mathcal{C}$ of cutting cycles. Then there exists a tame graph G' embedded in the cylinder with the same rings, such that G' dominates G and G' is a chain of at least n basic graphs with cutting cycles $\mathcal{C}'$, with $r(G', \mathcal{C}') \ge r(G, \mathcal{C})$.*

Proof. We prove the claim by induction on $|V(G)|$. Let G be a chain of critical graphs $G_1, \ldots, G_n$ with cutting cycles $\mathcal{C} = \{C_0, \ldots, C_n\}$. If all of $G_1, \ldots, G_n$ are basic, then the claim is trivialy true with $G' = G$. Otherwise, by Lemma 3.4, there exists $i \in \{1, \ldots, n\}$ such that a 4-face in G_i admits a legal identification. Then, the claim follows by the induction hypothesis applied to the graph H obtained by Lemma 3.3. $\hfill\square$

Dvořák and Lidický [8] gave an exact description of critical graphs embedded in the cylinder with rings of length at most 4 and without contractible (≤ 4)-cycles—in addition to the infinite family of reduced Thomas-Walls graphs, there are only 95 such critical graphs. In particular, their examination gives the following.

Theorem 3.6 (Dvořák and Lidický [8])**.** *Let G be a graph embedded in the cylinder with rings of length at most 4, such that G contains no contractible (≤ 4)-cycles. If G is critical and the distance between its rings is at least 6, then G is a reduced Thomas-Walls graph.*

We need the following observation about basic graphs.

Lemma 3.7. *Let G be a tame chain of basic graphs, such that each of them either contains no contractible 4-cycles, or is isomorphic to EV_1, EV_2, Tr_1, Tr_2, S_1, or S_2, with pairwise vertex-disjoint cutting cycles C_0, ..., C_n. If $n \geq 5$, then G is dominated by a graph G' with the same vertex-disjoint rings C_0 and C_n and without contractible (≤ 4)-cycles. Furthermore, if $|C_0| = 4$ and e_0 is any edge of C_0, then G' can be chosen so that e is not contained in a triangle in G'.*

Proof. Let C_0, ..., C_n be the cutting cycles of the chain, and for $i = 1, \ldots, n$, let G_i denote the subgraph of G drawn between C_{i-1} and C_i.

We start with several observations about graphs dominating EV_1, EV_2, Tr_1, Tr_2, S_1, or S_2. Let H be one of these graphs, with rings $u_1 u_2 \ldots$ and $v_1 v_2 \ldots$ labelled as in Figure 3. In all the identifications described below, we suppress arising parallel edges.

- If $H = EV_1$ or $H = EV_2$, then

  - let H' be the graph isomorphic to one of the graphs I_1 or I_2 depicted in Figure 4, obtained from H by identifying u_2 with v_2 and u_1 with v_1 and by removing the vertices that do not belong to the rings, and

  - let H'' be the graph isomorphic to one of the graphs I_1 or I_2, obtained from H by identifying u_2 with v_2 and u_3 with v_3 and by removing the vertices that do not belong to the rings.

  Let $\mathcal{R}(H) = \{H', H''\}$ and $\mathcal{D}(H) = \{u_2, v_2\}$.

- If $H = Tr_1$, then

  - let H' be the graph isomorphic to the graph I_1 depicted in Figure 4, obtained from H by identifying u_2 with v_2 and u_1 with v_1, and

  - let H'' be the graph isomorphic to the graph I_1, obtained from H by identifying u_2 with v_4 and u_3 with v_1.

  Let $\mathcal{R}(H) = \{H', H''\}$ and $\mathcal{D}(H) = \{u_2, v_1\}$.

- If $H = Tr_2$, then

– let H' be the graph isomorphic to the graph I_1 depicted in Figure 4, obtained from H by identifying u_1 with v_1 and u_2 with v_2, and

– let H'' be the graph isomorphic to the graph I_4 depicted in Figure 4, obtained from H by identifying u_1 with v_3.

Let $\mathcal{R}(H) = \{H', H''\}$ and $\mathcal{D}(H) = \{u_1\}$.

- If $H = S_1$, then let H' be the graph isomorphic to the graph I_3 depicted in Figure 4, obtained from H by identifying u_1 with v_1. Let $\mathcal{R}(H) = \{H'\}$ and $\mathcal{D}(H) = \{u_1, v_1\}$.

- If $H = S_2$, then let H' be the graph isomorphic to the graph I_3 depicted in Figure 4, obtained from H by identifying u_4 with v_4. Let $\mathcal{R}(H) = \{H'\}$ and $\mathcal{D}(H) = \{u_4, v_4\}$.

Observe that every graph in $\mathcal{R}(H)$ dominates H and contains no contractible (≤ 4)-cycle. For every edge e of a ring of H, there exists at least one graph $H_e \in \mathcal{R}(H)$ such that the corresponding edge in H_e is not shared by both rings. Furthermore, if a vertex v of a ring of H does not belong to $\mathcal{D}(H)$, then there exists at least one graph $H_v \in \mathcal{R}(H)$ such that the corresponding vertex in H_v is not shared by both rings.
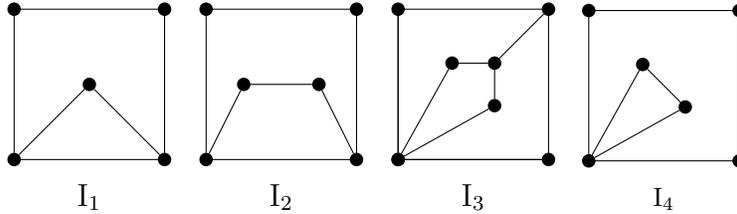

Figure 4: Results of identifications in Lemma 3.7.

For $i = 1, \ldots, n$, let G_i' be the graph obtained as follows. Let v_i be a vertex of C_{i-1} not belonging to $\mathcal{D}(G_i)$, chosen so that

- if $i = 1$, then v_i is incident with e_0,

- if $i \geq 2$ and the intersection Z_i of C_0 and C_{i-1} in $G_1' \cup \ldots \cup G_{i-1}'$ is not a subset of $\mathcal{D}(G_i)$, then $v_i \in Z_i \setminus \mathcal{D}(G_i)$, and

- if $i \geq 2$, $|C_{i-1}| = 4$ and the intersection Y_i of C_{i-2} and C_{i-1} in G_{i-1}' is not a subset of $\mathcal{D}(G_i)$, then $v_i \in Y_i \setminus \mathcal{D}(G_i)$.

If G_i does not contain any contractible 4-cycle, then let $G_i' = G_i$. Otherwise, let G_i' be an element of $\mathcal{R}(G_i)$, chosen so that the vertex of G_i' corresponding to v_i is not shared by both rings of G_i'.

Note that this ensures that no two triangles of $G' = G_1' \cup \ldots \cup G_n'$ share an edge, and thus G' contains no contractible (≤ 4)-cycles. Furthermore, the edge corresponding to e_0 is not contained in a triangle of G'.

The graph G' dominates G. Note that $|Z_2| \leq 2$ and $|Z_i| \leq 1$ for $i \geq 3$, and thus the rings of G' share at most one vertex. If the rings of G' are vertex-disjoint, then G' satisfies the conclusion of the lemma. Hence, suppose the rings of G' share a unique vertex z. This is only possible if each of G_1, $\ldots$, G_n is isomorphic to EV_1, EV_2, Tr_1, Tr_2, S_1, or S_2. For $i = 0, \ldots, n$, let z_i denote the vertex of C_i corresponding to z.

Consider the graph $F_1 = G_1' \cup G_2' \cup G_3'$, whose rings C_0 and C_3 intersect only in z. Observe that every edge of F_1 with both ends in $V(C_0 \cup C_3)$ is contained in $E(C_0 \cup C_3)$. Since $|C_0| + |C_3| \leq 8$ and F_1 contains no contractible (≤ 4)-cycle, Theorem 3.1 shows that every precoloring of $C_0 \cup C_3$ that assigns the same color to z_0 and z_3 extends to a 3-coloring of F_1. Since z is contained in both rings of G', the choice of G_4' implies that z_3 belongs to $\mathcal{D}(G_4)$. Hence, $F_1 \cup G_4$ is dominated by a graph obtained from G_4 by adding the cycle C_0 and identifying z_0 with z_3. By inspecting all choices of $G_4 \in \{\mathrm{EV}_1, \mathrm{EV}_2, \mathrm{Tr}_1, \mathrm{Tr}_2, \mathrm{S}_1, \mathrm{S}_2\}$ and $z_3 \in \mathcal{D}(G_4)$, we conclude that the following holds.

- if $|C_4| = 3$, then $F_1 \cup G_4$ is dominated by the graph F_2 obtained from the disjoint union of C_0 and C_4 by adding an edge between z_0 and a vertex of C_4, and

- if $|C_4| = 4$, then either

  - $F_1 \cup G_4$ is dominated by the graph F_2' isomorphic to Xq_6 or Tr_2' obtained from the disjoint union of C_0 and C_4 by adding edges between z_0 and two non-adjacent vertices of C_4 distinct from z_4, or

  - G_4 is isomorphic to Tr_1.

If $|C_4| = 3$, then $F_2 \cup G_5' \cup \ldots \cup G_n'$ satisfies the conclusions of the lemma. Let us consider the case that $|C_4| = 4$. If G_4 is isomorphic to Tr_1 (and thus $|C_3| = 3$), note that none of the conditions for specifying v_4 applies, and thus we can choose G_4' so that z does not correspond to any vertex of $\mathcal{D}(G_5)$. However, then the choice of v_5 ensures that C_0 and C_5 are vertex-disjoint in G', which is a contradiction.

Hence, we can assume that $F_1 \cup G_4$ is dominated by the graph F_2'. By inspection of all choices of $G_5 \in \{\mathrm{EV}_1, \mathrm{EV}_2, \mathrm{Tr}_1, \mathrm{Tr}_2, \mathrm{S}_1, \mathrm{S}_2\}$ and $z_4 \in \mathcal{D}(G_4)$, we conclude that $F_2' \cup G_5$ is dominated by a graph F_3 obtained from the vertex-disjoint union of cycles C_0 and C_5 by adding an edge between them, and thus $F_3 \cup G_6' \cup \ldots \cup G_n'$ satisfies the conclusions of the lemma. $\square$

Let us now prove a variation on Theorem 3.6.

Lemma 3.8. *Let G be a tame chain of n basic graphs, such that each of them either contains no contractible 4-cycles, or is isomorphic to J_1, EV_1, EV_2, Tr_1, Tr_2, S_1, or S_2. If $n \geq 32$, then either every precoloring of the rings of G extends*

to a 3-coloring of G, or both rings of G have length 4 and G is dominated by the reduced Thomas-Walls graph T_4'.

Proof. Let C_0, $\ldots$, C_n be the cutting cycles of the chain. Let $s_0 = 0$ if C_0 and C_1 are vertex-disjoint, and $s_0 = 1$ otherwise. Let $s_1 = n$ if C_{n-1} and C_n are vertex-disjoint, and $s_1 = n-1$ otherwise. Let F be the union of subgraphs of G drawn between C_0 and C_{s_0}, and between C_{s_1} and C_n. Note that if a component of F contains a contractible 4-cycle, then it is isomorphic to J_1. Identify two opposite vertices in each 4-face of F, obtaining a graph F'.

For $i = 1, \ldots, 5$, let G_i denote the subgraph of G drawn between C_{s_0+5i-5} and C_{s_0+5i}. Let G_6 denote the subgraph of G_i drawn between C_{s_0+25} and C_{s_1}. By the definition of a chain, J_1 does not appear in the subchain bounded by C_{s_0} and C_{s_1}. For $i = 1, \ldots, 6$, Lemma 3.7 implies that G_i is dominated by a graph G_i' with the same vertex-disjoint rings, such that the graph $G' = F' \cup G_1' \cup \ldots \cup G_6'$ contains no contractible (≤ 4)-cycles—this is ensured by choosing e_0 to be an edge of C_{s_0+5i-5} contained in a triangle in G_{i-1}' (if any) for $i = 2, \ldots, 6$.

Note that the distance between C_0 and C_n in G' is at least 6. Suppose that not every precoloring of $C_0 \cup C_n$ extends to a 3-coloring of G. Then G' is dominated by its maximal critical subgraph G'', and by Theorem 3.6, G'' is a reduced Thomas-Walls graph in that the distance between C_0 and C_n is at least 6. By Lemma 2.7, G is dominated by the reduced Thomas-Walls graph T_4'. □

We call graphs $Xq_1, \ldots, Xq_7$ *almost quadrangulations*. We need the following claim.

Lemma 3.9. *Let G be a tame chain of n basic graphs, such that each of them is an almost quadrangulation. If $n \geq 4$, then every precoloring of the rings of G extends to a 3-coloring of G.*

Proof. Let C_0, $\ldots$, C_n be the cutting cycles of the chain, and let ψ be a 3-coloring of $C_0 \cup C_n$. The graph G is dominated by a quadrangulation of the cylinder with rings at distance at least 4 from each other. Hence, if $|C_0| = 4$, then ψ extends to a 3-coloring of G by Lemma 2.10. If $|C_0| = 3$, then G is a chain of the copies of Xq_7. Observe that ψ extends to a 3-coloring φ of the subgraph of G drawn between C_0 and C_1 such that the winding number of φ on C_1 is the same as the winding number of ψ on C_n. By Lemma 2.10, it follows that ψ extends to a 3-coloring of G. □

By combining these results, we obtain the following.

Lemma 3.10. *Let G be a tame graph embedded in the cylinder with rings of length at most 4. If G is a chain of graphs, at least 131 of which are not quadrangulated, then either every precoloring of the rings of G extends to a 3-coloring of G, or both rings of G have length 4 and G is dominated by the reduced Thomas-Walls graph T_4'.*

Proof. Let $\mathcal{C} = C_0, C_1, \ldots, C_n$ be the sequence of cutting cycles of the chain G, and for $i = 1, \ldots, n$, let G_i be the subgraph of G drawn between G_{i-1} and G_i.

Without loss of generality, G_i is critical, as otherwise we can replace G_i by its maximal critical subgraph. Furthermore, by Lemma 3.5, we can assume that G_i is basic. If G_i is a quadrangulation, then G_i is one of Q_1, ..., Q_5; in this case, we identify the vertices of the rings of G_i as indicated by dotted lines in Figure 3.

Hence, assume that none of G_1, ..., G_n is a quadrangulation. By the assumptions of the lemma, we have $n \geq 131$.

Suppose that there exists $i \in \{1, \ldots, n-3\}$ such that G_i, ..., G_{i+3} are almost quadrangulations. By Theorem 1.3, any 3-coloring ψ of $C_0 \cup C_n$ extends to a 3-coloring of $G_1 \cup \ldots \cup G_{i-1} \cup G_{i+4} \cup \ldots \cup G_n$, and by Lemma 3.9, ψ extends to a 3-coloring of G. Therefore, we can assume that no four consecutive graphs in the chain G''' are almost quadrangulations.

Extend all almost quadrangulations in G to quadrangulations Q_1, ..., Q_5 and identify the vertices of their rings as indicated by dotted lines in Figure 3. The resulting graph G' is a chain of at least 32 basic graphs dominating G, such that none of the graphs in the chain is a quadrangulation or almost quadrangulation. Also, if J_1 appears in the chain (necessarily as the first or the last element), identify the opposite vertices of its 4-face not contained in the ring of G', thus turning it into I_1. Similarly, we turn each of the graphs in the chain that is isomorphic to Tr_1' or Tr_2' into Tr_1 or Tr_2 by adding an edge. The claim of Lemma 3.10 then follows by Lemma 3.8. □

Actually, we only need the following consequence of Lemmas 2.7 and 3.10

Corollary 3.11. *Let G be a tame graph embedded in the cylinder with rings C_1 and C_2 of length at most 4. Suppose that G is a chain of graphs, at least 131 of which are not quadrangulated. If $|C_1| = 3$ or $|C_2| = 3$, then every precoloring of $C_1 \cup C_2$ extends to a 3-coloring of G. If $|C_1| = |C_2| = 4$, then for every 3-coloring ψ_1 of C_1, there exists $v \in V(C_2)$ such that for every v-diagonal 3-coloring ψ_2 of C_2, the precoloring $\psi_1 \cup \psi_2$ extends to a 3-coloring of G.*

Let us remark that using computer, we can signficantly improve the bound of Corollary 3.11 as follows.

Lemma 3.12. *Let G be a tame graph embedded in the cylinder with rings C_1 and C_2 of length at most 4. Suppose that G is a chain of $n \geq 7$ graphs, at least 3 of which are not quadrangulated. If $|C_1| = 3$ or $|C_2| = 3$, then every precoloring of $C_1 \cup C_2$ extends to a 3-coloring of G. If $|C_1| = |C_2| = 4$, then for every 3-coloring ψ_1 of C_1, there exists $v \in V(C_2)$ such that for every v-diagonal 3-coloring ψ_2 of C_2, the precoloring $\psi_1 \cup \psi_2$ extends to a 3-coloring of G.*

Proof. By Lemma 3.5, we can assume that all the graphs in the chain are basic. Furthermore, without loss of generality, they are critical. We consider the chains consisting of copies of the graphs depicted in Figure 3, the graph I_4, and the graphs from [8] whose rings are disjoint and not separated by another triangle or disjoint 4-cycle. By computer enumeration, we verify that the claim holds for $n = 7$.

For longer chains, we prove the claim by induction on n. By symmetry, we can assume that if $|C_1| = 3$, then $|C_2| = 3$. If any of the graphs in the chain is isomorphic to a quadrangulation Q_1, ..., or Q_5, then identify the vertices of their rings as indicated by dotted lines in Figure 3, and apply induction to the resulting graph. Otherwise, let K be the first cutting cycle of the chain distinct from C_1. We extend the 3-coloring of C_1 to the subgraph of G drawn between C_1 and K by Theorem 1.3, and then the claim follows by the induction hypothesis for the subgraph of G drawn between K and C_2. $\square$

Let us mention that Lemma 3.12 is not true if we demand just two non-quadrangulations or if the chain has length at most 6.

4 Disk with two triangles

In this section, we consider the case of the graphs embedded in the disk with a ring of length 4 and with exactly two triangles, and we aim to prove Theorem 1.4.

Borodin et al. [3] described all non-3-colorable planar graphs with exactly 4 triangles. The following is a special case of their main result relevant to us.

Theorem 4.1 (Borodin et al. [3]). *If G is a 4-critical planar graph with exactly four triangles and there exists an edge $e \in E(G)$ intersecting two of them, then $G - e$ is a patched Havel-Thomas-Walls graph with the interface pair e.*

As a corollary, we have the following special case of Theorem 1.4.

Corollary 4.2. *Let G be a graph embedded in the disk with at most two triangles and with ring $C = v_1 v_2 v_3 v_4$. If G is critical and has no 3-coloring that is v_1-diagonal on C, then G is obtained from a patched Havel-Thomas-Walls graph by framing on its interface pair $v_1 v_3$.*

Proof. Since G is critical, Theorem 1.3 implies that any non-facial 4-cycle in G separates the hole of the disk from both triangles of G.

Let C' be a non-facial 4-cycle in G containing v_1 and v_3, such that the subgraph of G drawn between C' and C is maximal. Let $G' = G - (V(C) \setminus V(C'))$. Since the 4-cycles in $C \cup C'$ distinct from C and C' do not separate the hole from the triangles, they bound faces, and thus G is obtained from G' by framing on $v_1 v_3$.

Let $G'' = G' + v_1 v_3$. Note that G'' is not 3-colorable, and by the choice of C', it contains exactly four triangles. By Theorem 1.2, G'' contains a 4-critical subgraph H with all four triangles. By Theorem 4.1, $H - v_1 v_3$ is a patched Havel-Thomas-Walls graph with the interface pair $v_1 v_3$. Since G is critical, Lemma 2.5 implies that every non-facial (≤ 5)-cycle of G separates the hole of the disk from at least one triangle of G. However, all faces $H - v_1 v_3$ have length at most 5, and thus $G'' = H$.

It follows that G is obtained from the patched Havel-Thomas-Walls graph G' by framing on its interface pair $v_1 v_3$. $\square$

Finally, we need a result on the density of 4-critical graphs by Kostochka and Yancey [14].

Theorem 4.3. *For every $n \geq 4$, every 4-critical graph with n vertices has at least $\frac{5n-2}{3}$ edges.*

We can now prove the main theorem of this section.

Proof of Theorem 1.4. We prove the claim by the induction on the number of vertices of G; hence, we assume that the claim holds for all graphs with less than $|V(G)|$ vertices.

Let $C = v_1 v_2 v_3 v_4$ be the ring of G. Let ψ_1, ψ_2 and ψ_3 be a v_1-diagonal, a v_2-diagonal, and a bichromatic 3-coloring of C, respectively. If either ψ_1 or ψ_2 does not extend to a 3-coloring of G, then the claim follows from Corollary 4.2. Hence, we can assume that ψ_1 and ψ_2 extend to 3-colorings φ_1 and φ_2 of G, respectively. Since G is critical, ψ_3 does not extend to a 3-coloring of G.

Note that if say v_1 had degree 2, then we could recolor v_1 in the coloring φ_1 and obtain a 3-coloring of G whose restriction to C is bichromatic, which is a contradiction. Similarly, we conclude that every vertex of C has degree at least three. Also, since $\varphi_1(v_2) = \varphi_1(v_4)$, the graph $G_2^\star$ obtained from G by identifying v_2 with v_4 is 3-colorable. Symmetrically, the graph $G_1^\star$ obtained from G by identifying v_1 with v_3 is 3-colorable.

Suppose for a contradiction that G has no non-ring 4-faces. Let n, m and s denote the number of vertices, edges and faces of G, respectively. Then $2m \geq 5(s-3) + 4 + 2 \cdot 3 = 5s - 5$. By Euler's formula, we have $s = m + 2 - n$, and thus

$$
\begin{aligned}
2m &\geq 5(m + 2 - n) - 5 = 5m - 5n + 5 \\
5n - 5 &\geq 3m
\end{aligned}
$$

and $m \leq \frac{5n-5}{3}$. Let G' be the graph (not embedded in the disk) obtained from G by identifying v_1 with v_3 to a vertex z_1 and v_2 with v_4 to a vertex z_2 and by suppressing parallel edges. Note that G' is not 3-colorable, since ψ_3 does not extend to a 3-coloring of G. Let G'' be a 4-critical subgraph of G'. Since $G_1^\star$ and $G_2^\star$ are 3-colorable, we have $z_1, z_2 \in V(G'')$. For every $v \in V(G') \setminus \{z_1, z_2\}$, note that ψ_3 extends to a 3-coloring of $G - v$ by the criticality of G, and thus $v \in V(G'')$. Thus, $V(G'') = V(G')$. Note that $|E(G')| = m - 3$ and $|V(G')| = n - 2$ since C is replaced by the edge $z_1 z_2$. Thus

$$
|E(G'')| \leq |E(G')| = m - 3 \leq \frac{5n - 14}{3} = \frac{5|V(G'')| - 4}{3}.
$$

This contradicts Theorem 4.3.

It follows that G contains a 4-face $x_1 x_2 x_3 x_4$. Since all vertices of C have degree at least 3, we can assume that $x_1, x_2 \notin V(C)$. If $x_3, x_4 \in V(C)$, say $x_3 = v_3$ and $x_4 = v_4$, and x_1 is adjacent to v_1 and x_2 is adjacent to v_2, then G is a tent. Hence, by symmetry, we can assume that either $x_3 \notin V(C)$, or $x_3 = v_i$ for some $i \in \{3, 4\}$ and x_1 is not adjacent to v_{i-2}. Let G_0 be the graph

obtained from G by identifying x_1 with x_3 to a new vertex x. Note that C is the ring of G_0 and C is an induced cycle. Observe that every 3-coloring of G_0 corresponds to a 3-coloring of G, obtained by giving x_1 and x_3 the color of x. Consequently, ψ_3 does not extend to a 3-coloring of G_0.

Consider any triangle xyz in G_0 created by the identification; i.e., $K = x_1 x_2 x_3 yz$ is a 5-cycle in G. Since G is critical, x_2 has degree at least three, and thus K does not bound a face. By Lemma 2.5, K separates a triangle of G from the hole of the disk. If K separated both triangles, then Theorems 1.3 and 1.2 would imply that ψ_3 extends to a 3-coloring of G_0, which is a contradiction. Consequently, K separates exactly one of the triangles of G from the hole. Let G_1 be a maximal critical subgraph of G_0, and note that by Theorem 1.3, G_1 contains exactly two triangles.

By the induction hypothesis, G_1 is either a tent or obtained from a patched Havel-Thomas-Walls graph by framing on its interface pair.

Let us first discuss the case that G_1 is a tent. Then G_1 contains two vertex-disjoint triangles, each of them sharing an edge with C. At least one of the triangles does not contain x, say a triangle $v_1 v_2 z_1$. Hence, $v_1 v_2 z_1$ is a triangle in G as well. By Theorem 2.1 applied to the disk bounded by the 5-cycle $K = v_1 z_1 v_2 v_3 v_4$ and the 3-coloring of K that extends ψ_3, we conclude that G also contains a triangle containing the edge $v_3 v_4$ and all other faces of G have length 4. Therefore, G is a tent.

Hence, it remains to consider the case that G_1 is obtained from a patched Havel-Thomas-Walls graph by framing on its interface pair, say $v_1 v_3$. Since ψ_3 does not extend to a 3-coloring of G_1, Lemma 2.8 implies that C is strong in G_1, and thus G_1 is a patched Havel-Thomas-Walls graph.

As the next case, suppose that G_1 is not obtained by patching from the graph depicted in Figure 2(b). Then, since C is a strong ring G_1 and since C is an induced cycle in G_1, it follows that G_1 contains vertices w_1, w_2, y_1 and y_2 and facial 5-cycles $K_1 = v_2 v_1 v_4 w_1 y_1$ and $K_2 = v_2 v_3 v_4 w_2 y_2$, where possibly $w_1 = w_2$. Furthermore, if $w_1 \neq w_2$, then G_1 also contains a 6-cycle $y_1 w_1 v_4 w_2 y_2 z$ with quadrangulated interior. If $w_1 = w_2$, let us define $z = w_1$. Let $K = z y_1 v_2 y_2$ and let $G_{1,K}$ be the subgraph of G_1 drawn in the closed disk bounded by K; note that $G_{1,K}$ is obtained from a patched Havel-Thomas-Walls graph by framing on its interface pair $v_2 z$. See Figure 5.

Both v_1 and v_3 have degree two in G_1. Since v_1 and v_3 have degree at least three in G and every non-facial (≤ 5)-cycle in G separates the hole from at least one of the triangles of G, it follows that neither K_1 nor K_2 is a cycle in G. In particular, x is one of v_4, v_2 or w_1 (in the case that $w_1 = w_2$). Furthermore, K corresponds to a 4-cycle K' in G, and the subgraph $G_{K'}$ of G drawn in the closed disk bounded by K' is isomorphic to $G_{1,K}$. By Lemma 2.8, any y_1-diagonal 3-coloring of K' extends to a 3-coloring of $G_{K'}$. Let us distinguish two subcases.

- If $x = v_2$, then G contains cycles $K_1' = v_2 v_1 v_4 w_1 y_1 x_1 x_2$ and $K_2' = v_2 v_3 v_4 w_2 y_2 x_1 x_2$ (recall that $x_2 \notin V(C)$, and that v_1, v_3 have degree at least three in C and non-facial (≤ 5)-cycles in G separate the hole from
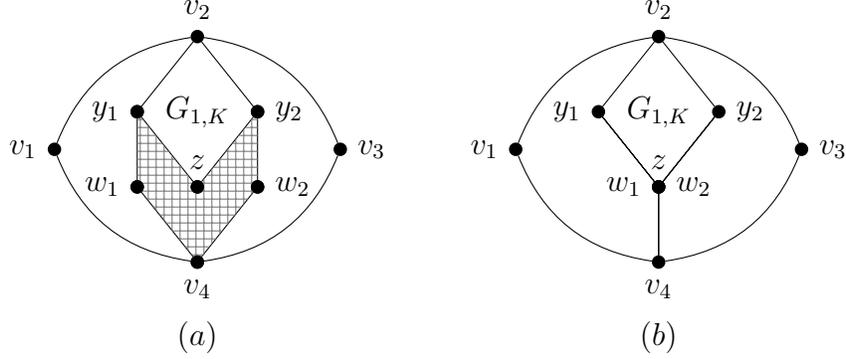
Figure 5: Graph G_1 in proof of Theorem 1.4.

at least one of the triangles, and thus $x_2 \notin \{v_1, v_3, y_1, y_2\}$). Let $G_{K_1'}$ and $G_{K_2'}$ denote the subgraphs of G drawn in the closed disks bounded by K_1' and K_2', respectively.

Since x_2 has degree at least three in G and every non-facial (≤ 5)-cycle in G separates a triangle from the hole, either $v_1 y_1$ or $v_3 y_2$ is not an edge. By symmetry, we can assume the former. Let φ be a 3-coloring defined by $\varphi(v_1) = \varphi(v_3) = \varphi(x_2) = \varphi(y_1) = 1$, $\varphi(v_2) = \varphi(v_4) = \varphi(y_2) = 2$ and $\varphi(x_1) = \varphi(w_1) = \varphi(w_2) = \varphi(z) = 3$. Since φ is y_1-diagonal on K', it extends to a 3-coloring of $G_{K'}$. If $w_1 \neq w_2$, then by Lemma 2.4, φ extends to the subgraph of G drawn in the closed disk bounded by $v_4 w_1 y_1 z y_2 w_2$.

Suppose that φ does not extend to a 3-coloring of $G_{K_1'}$. By Theorem 3.2, $G_{K_1'}$ contains a 5-face whose intersection with K_1' is a path containing v_1, v_2, x_1, and y_1. However, this is not possible, since y_1 is not adjacent to v_1. Hence, φ extends to a 3-coloring of $G_{K_1'}$.

Suppose that φ does not extend to a 3-coloring of $G_{K_2'}$. By Theorem 3.2, $G_{K_2'}$ contains a 5-face whose intersection with K_2' is a path containing w_2, y_2, v_2, and v_3. However, this is not possible, since v_3 has degree at least three in G. Hence, φ extends to a 3-coloring of $G_{K_2'}$.

We conclude that φ extends to a 3-coloring of G. This is a contradiction, since φ is bichromatic on C.

- If $x \neq v_2$, then G contains cycles $K_1' = v_2 v_1 v_4 a b w_1 y_1$ and $K_2' = v_2 v_3 v_4 a b w_2 y_2$, with $\{a, b\} = \{x_1, x_2\}$. Let $G_{K_1'}$ and $G_{K_2'}$ denote the subgraphs of G drawn in the closed disks bounded by K_1' and K_2', respectively.

  Let φ be a 3-coloring defined by $\varphi(v_1) = \varphi(v_3) = \varphi(a) = \varphi(y_1) = 1$, $\varphi(v_2) = \varphi(v_4) = \varphi(w_1) = \varphi(w_2) = \varphi(z) = 2$ and $\varphi(y_2) = \varphi(b) = 3$. Since φ is y_1-diagonal on K', it extends to a 3-coloring of $G_{K'}$. If $w_1 \neq w_2$, then by Lemma 2.4, φ extends to the subgraph of G drawn in the closed disk bounded by $b w_1 y_1 z y_2 w_2$.

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Suppose that φ does not extend to a 3-coloring of $G_{K_1'}$. By Theorem 3.2, $G_{K_1'}$ contains a 5-face whose intersection with K_1' is a path containing v_1, v_2, v_4, and y_1. However, this is not possible, since v_1 has degree at least three in G.

Suppose that φ does not extend to a 3-coloring of $G_{K_2'}$. By Theorem 3.2, $G_{K_2'}$ contains a 5-face whose intersection with K_2' is a path containing v_3, v_4, w_2 and y_2. However, this is not possible, since v_3 has degree at least three in G.

We conclude that φ extends to a 3-coloring of G. This is a contradiction, since φ is bichromatic on C.

Finally, let us consider the case that G_1 is obtained by patching from the graph depicted in Figure 2(b). Since C is strong, G_1 contains 5-faces $v_2v_1v_4w_1y_1$ and $v_2v_3v_4w_2y_2$. Let H denote the subgraph of G_1 drawn in the closed disk bounded by the 6-cycle $K = v_2y_1w_1v_4w_2y_2$, and observe that a precoloring φ of K extends to a 3-coloring of H, unless $\{\varphi(y_1), \varphi(w_1)\} = \{\varphi(y_2), \varphi(w_2)\}$. Since both v_1 and v_3 have degree at least three in G, we conclude that $x \in \{v_2, v_4\}$, say $x = x_3 = v_2$, and G contains a 6-cycle $K' = x_1y_1w_1v_4w_2y_2$ such that the subgraph drawn in the closed disk bounded by K' is isomorphic to H.

Because x_2 has degree at least three in G and non-facial (≤ 5)-cycles in G separate the hole from at least one of the triangles, either v_1y_1 or v_3y_2 is not an edge; assume the latter. Let φ be a 3-coloring defined by $\varphi(v_2) = \varphi(v_4) = \varphi(y_1) = 1$, $\varphi(v_1) = \varphi(v_3) = \varphi(y_2) = \varphi(x_2) = 2$ and $\varphi(x_1) = \varphi(w_1) = \varphi(w_2) = 3$. Note that φ extends to a 3-coloring of H. We consider the subgraphs $G_{K_1'}$ and $G_{K_2'}$ of G drawn inside the 7-cycles $v_2x_2x_1y_1w_1v_4v_1$ and $v_2x_2x_1y_2w_2v_4v_3$, respectively.

Suppose that φ does not extend to a 3-coloring of $G_{K_1'}$. By Theorem 3.2, $G_{K_1'}$ contains a 5-face whose intersection with K_1' is a path containing v_1, v_2, y_1, and w_1. However, this is not possible, since v_1 has degree at least three in G.

Suppose that φ does not extend to a 3-coloring of $G_{K_2'}$. By Theorem 3.2, $G_{K_2'}$ contains a 5-face whose intersection with K_2' is a path containing v_2, v_3, x_1 and y_2. However, this is not possible, since y_2 is not adjacent to v_3.

Therefore, φ extends to a 3-coloring of G. This is a contradiction, since the restriction of φ to C is bichromatic. $\square$

Theorem 1.4 enables us to give some information about critical graphs embedded in the cylinder with rings of length 4.

Corollary 4.4. *Let G be a critical tame graph embedded in the cylinder with rings $C_1 = u_1u_2u_3u_4$ and $C_2 = v_1v_2v_3v_4$. Let ψ be a 3-coloring of C_1. If no 3-coloring of G that extends ψ is v_1-diagonal on C_2, then G is obtained from a patched Thomas-Walls graph by framing on its interface pairs, one of which is v_1v_3.*

Proof. Let C_2' be a non-contractible 4-cycle in G containing v_1 and v_3 such that the subgraph G_2 of G drawn between C_2' and C_2 is maximal. Let G_1 be the

subgraph of G drawn between C_1 and C_2'. Note that all faces of $C_2' \cup C_2$ have length 4. Since G is critical, Lemma 2.5 implies that $G_2 = C_2' \cup C_2$, and thus G is obtained from G_1 by framing on $v_1 v_3$.

Let $G_1' = G_1 + v_1 v_3$. Note that ψ does not extend to a 3-coloring of G_1'. By the choice of C_2', the edge $v_1 v_3$ belongs to exactly two triangles $v_1 v_2' v_3$ and $v_1 v_4' v_3$ in G_1'. If G_1' contains a triangle T distinct from $v_1 v_2' v_3$ and $v_1 v_4' v_3$, then T separates the hole bounded by C_1 from $v_1 v_2' v_3$ and $v_1 v_4' v_3$, and ψ extends to a 3-coloring of G_1' by Theorems 1.3 and 1.2. This is a contradiction, and thus G_1' contains exactly two triangles. Since the two triangles of G_1' share an edge, the examination of the outcomes of Theorem 1.4 shows that G_1' contains a subgraph H' that is obtained from a patched Thomas-Walls graph by framing on its interface pair in C_1, $v_1 v_3$ is an interface pair of H', and the rings of H' are C_1 and C_2'.

Let $H = H' \cup C_2$. To prove Corollary 4.4, it suffices to show that $G = H$. This is the case, since $H \subseteq G$, every face of H has length at most 5, and every contractible (≤ 5)-cycle in G bounds a face by Lemma 2.5. $\qquad\square$

We can now strengthen the conclusions of Lemma 3.10.

Lemma 4.5. *Let G be a tame graph embedded in the cylinder with rings of length at most 4. If G is a chain of graphs, at least 264 of which are not quadrangulated, then either every precoloring of the rings of G extends to a 3-coloring of G, or G contains a subgraph obtained from a patched Thomas-Walls graph by framing on its interface pairs, with the same rings as G.*

Proof. Let C_1 and C_2 be the rings of G. There exists a non-contractible ($\leq$ 4)-cycle K such that for $i \in \{1, 2\}$, if G_i denotes the subgraph of G drawn between C_i and K, then G_i is a chain of graphs, at least 132 of which are not quadrangulated.

Suppose that there exists a 3-coloring ψ of $C_1 \cup C_2$ that does not extend to a 3-coloring of G. By Corollary 3.11, for $i \in \{1, 2\}$ there exists a vertex $v_i \in V(K)$ such that for any v_i-diagonal 3-coloring ψ' of K, the coloring $(\psi \restriction V(C_i)) \cup \psi'$ extends to a 3-coloring of G_i. If v_1 is either equal or non-adjacent to v_2, then we can choose a 3-coloring ψ' of K that is both v_1-diagonal and v_2-diagonal, and extend $\psi \cup \psi'$ to both to G_1 and G_2, which is a contradiction.

Therefore, assume that v_1 and v_2 are adjacent, $K = v_1 v_2 v_3 v_4$. Consider any 3-coloring ψ' of $C_1 \cup C_2 \cup K$ that extends ψ, such that ψ' is v_2-diagonal on K. It follows that ψ' extends to a 3-coloring of G_2, and thus it does not extend to a 3-coloring of G_1. By Corollary 4.4, G_1 contains a subgraph H_1 obtained from a patched Thomas-Walls graph by framing on its interface pairs, $v_2 v_4$ is an interface pair of H_1, and the rings of H_1 are C_1 and K. By symmetry, G_2 contains a subgraph H_2 obtained from a patched Thomas-Walls graph by framing on its interface pairs, $v_1 v_3$ is an interface pair of H_2, and the rings of H_2 are C_2 and K.

Note that all faces of $H_1 \cup H_2$ have length at most 5, and since ψ does not extend to a 3-coloring of G, Lemma 2.4 implies that ψ does not extend to a 3-coloring of $H_1 \cup H_2$. If K is weak in both H_1 and H_2, then we can extend ψ to a

3-coloring ψ' of $C_1 \cup C_2 \cup K$ that is bichromatic on K, and further extend ψ' to a 3-coloring of H_1 and H_2 by Lemma 2.7, which is a contradiction. Hence, K is weak in at most one of H_1 and H_2. Let H be the subgraph of $H_1 \cup H_2$ obtained by removing all vertices of degree two not belonging to $C_1 \cup C_2$. Observe that H is obtained from a patched Thomas-Walls graph by framing on its interface pairs, as required by the conclusion of the lemma. $\qquad\square$

Let us remark that by using Lemma 3.12 instead of Corollary 3.11, the assumption of Lemma 4.5 could be relaxed to "If G is a chain of at least 14 graphs, at least 10 of which are not quadrangulated".

5 Colorings of quadrangulations

Next, we explore the graphs containing a long chain of quadrangulations, which complements Lemma 4.5. We need the following fact, which follows from Lemmas 4 and 5 of [7].

Lemma 5.1. *Let G be a graph embedded in the cylinder with a ring $C = v_1 v_2 v_3 v_4$ and a ring T of length 3, such that T is the only triangle in G, G has exactly one face f of length 5, and all non-ring faces of G other than f have length 4. Let ψ be a 3-coloring of C, and let $w \in \{-1, 1\}$. If ψ does not extend to a 3-coloring of G with winding number w on T, then either T shares an edge with C, or there exists a path $v_i x y v_{i+2}$ in G for some $i \in \{1, 2\}$ such that f is drawn inside the contractible 5-cycle of $C + v_i x y v_{i+2}$, and $\psi(v_i) \neq \psi(v_{i+2})$.*

Let G be a graph embedded in the cylinder with rings C and T such that $|T| = 3$. Let ψ be a 3-coloring of C. We say that ψ *forces the winding number of* T if there exists $w \in \{-1, 1\}$ such that for every 3-coloring φ of G that extends ψ, the winding number of φ on T is w.

Lemma 5.2. *Let G be a critical graph embedded in the cylinder with rings C of length at most 4 and T of length 3, such that all triangles in G are non-contractible. If there exists a 3-coloring ψ of C that forces the winding number of T, then G is a near $3, 3$-quadrangulation and for some $w \in \{-1, 1\}$, ψ on C causes winding number w.*

Proof. We proceed by induction, assuming that the claim holds for all graphs with less than $|V(G)|$ vertices. If C and T share at least two vertices, or if $|C| = 3$ and $|V(C) \cap V(T)| = 1$, then the claim follows from Lemma 2.4. Hence, assume that C intersects T in at most one vertex, and if $|C| = 3$, then C and T are vertex-disjoint.

If G contains a triangle T' distinct from C and T, then let G_1 be the subgraph of G drawn between C and T', and let G_2 be the subgraph of G drawn between T' and T. By Theorem 1.3, ψ extends to a 3-coloring φ of G_1. Note that $\varphi \upharpoonright V(T')$ must force the winding number of T in G_2. By the induction hypothesis, G_2 is a $3, 3$-quadrangulation. By Lemma 2.9, the winding number of any 3-coloring of G_2 on T is equal to its winding number on T', and we conclude that ψ forces

the winding number of T' in G_1. The claims of Lemma 5.2 then follow by the induction hypothesis applied to G_1. Therefore, we can assume that G contains no triangle distinct from the rings, and in particular G is tame.

Suppose that G contains at most one non-ring face f of length other than 4, and if G has such a face f, that $|f| = 5$. Since G has even number of odd faces, note that f exists if and only if $|C| = 4$. If $|C| = 3$, then it follows that G is a $3, 3$-quadrangulation, and ψ automatically causes winding number. If $|C| = 4$, then G satisfies the assumptions of Lemma 5.1. Since T does not share an edge with C, it follows that say $C = v_1 v_2 v_3 v_4$ and G contains a path $v_1 x y v_3$ such that f is contained inside the contractible 5-cycle $v_3 v_4 v_1 x y$, and $\psi(v_1) \neq \psi(v_3)$. Since G is critical, Lemma 2.5 implies that $f = v_3 v_4 v_1 x y$, and thus G is a near $3, 3$-quadrangulation and ψ on C causes winding number w.

Hence, assume that

G contains either at least two faces of length at least 5, or a face of length at least 6.

$$(8)$$

If G contains no 4-face, then G is one of the critical graphs determined in [8]. We depict those without separating triangles in Figure 6. A straightforward case analysis shows that ψ does not force the the winding number of T in any of these graphs.

Therefore, we can assume that G contains a 4-face $f = u_1 u_2 u_3 u_4$. Suppose first that three vertices of f, say u_1, u_2, and u_3, either all belong to C, or all belong to T. Since G is tame, this is only possible if $u_1, u_2, u_3 \in V(C)$ and $|C| = 4$. Let $C = u_1 u_2 u_3 v_4$. Let $G' = G - u_2$, and let ψ' be the 3-coloring of $C' = u_1 u_4 u_3 v_4$ given by $\psi'(x) = \psi(x)$ for $x \in \{u_1, u_3, v_4\}$ and $\psi'(u_4) = \psi(v_4)$. Note that ψ' forces the winding number of T in G', and by the induction hypothesis, G' is a near $3, 3$-quadrangulation. Since f is the only face of G that does not belong to G', this contradicts (8). It follows that we can assume that $|V(f) \cap V(C)| \leq 2$ and $|V(f) \cap V(T)| \leq 2$ for every 4-face f of G.

In particular, we can by symmetry assume that $|\{u_1, u_3\} \cap V(C)| \leq 1$ and $|\{u_1, u_3\} \cap V(T)| \leq 1$; hence, u_1 and u_3 are not both contained in the same triangle, and thus they are non-adjacent. Let G_1 be obtained from G by identifying u_1 with u_3, and let G_2 be a maximal critical subgraph of G_1.

Suppose for a contradiction that G_2 contains a contractible triangle, and thus G contains a contractible 5-cycle K with $u_1 u_2 u_3 \subset K$. By Lemma 2.5, K bounds a face in G, and thus u_2 has degree two. Since G is critical, we conclude that u_2 is incident with C or T. However, then u_2 and its neighbors u_1 and u_3 all belong to C or all belong to T, which is a contradiction.

Hence, every triangle in G_2 is non-contractible. Note that ψ forces the winding number of T in G_2, since every precoloring of $C \cup T$ that extends to a 3-coloring of G_2 also extends to a 3-coloring of G. By the induction hypothesis, G_2 is a near $3, 3$-quadrangulation. Each (≤ 5)-face K in G_2 corresponds either to a $|K|$-face in G, or to a contractible $(|K| + 2)$-cycle K' in G containing either the path $u_1 u_2 u_3$ or the path $u_1 u_4 u_3$. Since neither u_2 nor u_4 has degree 2 in G, in the latter case K' does not bound a face, and by Theorems 2.2 and 3.2, all the faces contained in the disk bounded by K' have length 4 except for one of
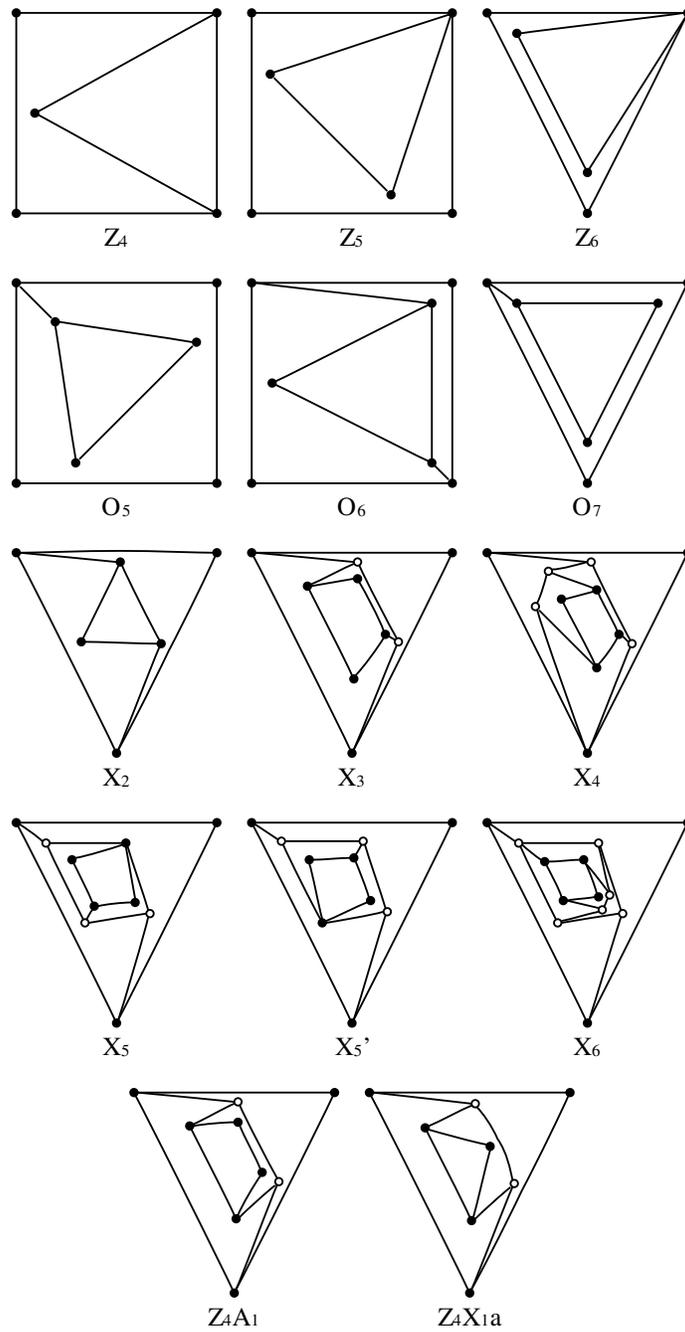
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Figure 6: Tame critical graphs with rings of length 3 and at most 4, no contractible 4-cycles, and no non-ring triangles.

length $|K|$. We conclude that all faces of G distinct from C and T have length four, except possibly for one of length 5. This contradicts (8). $\qquad\square$

As a corollary, we obtain the following.

Lemma 5.3. *Let G be a tame graph embedded in the cylinder with rings C_1 and C_2 of length at most 4. Suppose that G contains non-contractible (≤ 4)-cycles K_1 and K_2 at distance at least 4 from each other, such that all faces of G drawn between K_1 and K_2 have length 4. Then either every precoloring of $C_1 \cup C_2$ extends to a 3-coloring of G, or G contains a near $3,3$-quadrangulation with rings C_1 and C_2 as a subgraph.*

Proof. Without loss of generality, K_1 separates C_1 from K_2. For $i \in \{1, 2\}$, let G_i be the subgraph of G drawn between C_i and K_i. Let G_0 be the subgraph of G drawn between K_1 and K_2.

Suppose that there exists a precoloring ψ of $C_1 \cup C_2$ that does not extend to a 3-coloring of G. By Theorem 1.3, ψ extends to a 3-coloring φ of $G_1 \cup G_2$. Since ψ does not extend to a 3-coloring of G, $\varphi \restriction V(K_1 \cup K_2)$ does not extend to a 3-coloring of G_0. By Lemma 2.10, it follows that $|K_1| = |K_2| = 3$ and φ has opposite winding numbers on K_1 and K_2. Furthermore, ψ forces the winding number of K_1 and K_2, and thus by Lemma 5.2, for $i \in \{1, 2\}$, G_i contains a near $3,3$-quadrangulation H_i with rings C_i and K_i as a subgraph. Then, $H_1 \cup G_0 \cup H_2$ is a near $3,3$-quadrangulation with rings $C_1 \cup C_2$. $\qquad\square$

6 Chains in cylinder

We can now prove the main result of the paper.

Proof of Theorem 1.1. Let $c_1 = 264$ be the constant of Lemma 4.5. Let $c = 4c_1 = 1056$. If at least c_1 of the graphs forming the chain G are not quadrangulations, then by Lemma 4.5, either every precoloring of $C_1 \cup C_2$ extends to a 3-coloring of G, or G contains a subgraph H obtained from a patched Thomas-Walls graph by framing on its interface pairs, and the rings of H are C_1 and C_2.

On the other hand, if all but at most $c_1 - 1$ graphs in the chain forming G are quadrangulations, then there exist four consecutive graphs in the chain that are quadrangulations, and thus by Lemma 5.3, either every precoloring of $C_1 \cup C_2$ extends to a 3-coloring of G, or G contains a near $3,3$-quadrangulation with rings C_1 and C_2 as a subgraph. $\qquad\square$

Using the bound from Lemma 3.12, the constant c of Theorem 1.1 can be improved to 40. However, even this improved bound is still likely to be far from the best possible.

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