

Fine structure of 4-critical triangle-free graphs II. Planar triangle-free graphs with two precolored 4-cycles

Zdeněk Dvořák* Bernard Lidický†

December 27, 2016

Abstract

We study 3-coloring properties of triangle-free planar graphs G with two precolored 4-cycles C_1 and C_2 that are far apart. We prove that either every precoloring of $C_1 \cup C_2$ extends to a 3-coloring of G , or G contains one of two special substructures which uniquely determine which 3-colorings of $C_1 \cup C_2$ extend. As a corollary, we prove that there exists a constant $D > 0$ such that if H is a planar triangle-free graph and $S \subseteq V(H)$ consists of vertices at pairwise distances at least D , then every precoloring of S extends to a 3-coloring of H . This gives a positive answer to a conjecture of Dvořák, Král' and Thomas, and implies an exponential lower bound on the number of 3-colorings of triangle-free planar graphs of bounded maximum degree.

1 Introduction

The interest in the 3-coloring properties of planar graphs was started by a celebrated theorem of Grötzsch [9], who proved that every planar triangle-free graph is 3-colorable. This result was later generalized and strengthened in many different ways. The one relevant to the topic of this paper concerns graphs embedded in surfaces. While the direct analogue of Grötzsch's theorem is false for any surface other than the sphere, 3-colorability of triangle-free graphs embedded in a fixed surface is nowadays quite well understood.

A first step in this direction was taken by Thomassen [11], who proved that every graph of girth at least five embedded in the projective plane or torus is 3-colorable. Thomas and Walls [10] extended this result to graphs embedded in the Klein bottle. More generally, Thomassen [12] proved that for any fixed

*Computer Science Institute (CSI) of Charles University, Malostranské náměstí 25, 118 00 Prague, Czech Republic. E-mail: rakdver@iuuk.mff.cuni.cz. Supported by project 14-19503S (Graph coloring and structure) of Czech Science Foundation.

†Iowa State University, Ames IA, USA. E-mail: lidicky@iastate.edu. Supported by NSF grants DMS-1266016 and DMS-1600390.

surface Σ , there are only finitely many 4-critical graphs of girth at least 5 that can be drawn in Σ (a graph G is 4-critical if it is not 3-colorable, but every proper subgraph of G is 3-colorable). In other words, there exists a constant c_Σ such that if a graph of girth at least 5 drawn in Σ is not 3-colorable, then it contains a subgraph with at most c_Σ vertices that is not 3-colorable. Thomassen's bound on c_Σ is double exponential in the genus of Σ . This was improved by Dvořák, Král' and Thomas [2], who gave a bound on c_Σ linear in the genus of Σ .

In follow-up papers [3, 5], they also described structure of triangle-free graphs on surfaces with respect to 3-coloring. Essentially, they show that such a graph (without non-facial contractible 4-cycles, whose interiors can be cleaned out without affecting 3-colorability) can be cut into a bounded number of pieces, each of them satisfying one of the following: every 3-coloring of the boundary of the piece H extends to a 3-coloring of H ; or a 3-coloring of the boundary of the piece H extends to a 3-coloring of H if and only if it satisfies a specific condition ("winding number constraint"); or the piece is homeomorphic to the cylinder and either all the faces of the piece have length 4, or both boundary cycles of the piece have length 4 (the *cylinder* is the sphere with two holes). While the first two possibilities for the pieces of the structure give all the information about their 3-colorings, in the last subcase the information is much more limited.

The aim of this series of papers is to fix this shortcoming. Let us remark that we cannot eliminate this subcase entirely—based on the construction of Thomas and Walls [10], one can find 4-critical triangle-free graphs G_1, G_2, \dots drawn in a fixed surface Σ , such that for any $i \geq 1$, G_i contains two non-contractible 4-faces bounding a subset Π of Σ homeomorphic to the cylinder, such that at least i 5-faces of G are contained in Π . Such a class of graphs cannot be described using only the first two subcases of the structure theorem.

The main problem studied in this paper concerns describing subgraphs of 4-critical triangle-free graphs embedded in the cylinder with rings of length 4. The exact description of such subgraphs under the additional assumption that all (≤ 4)-cycles are non-contractible was given by Dvořák and Lidický [6]. Even this special case is rather involved (in addition to one infinite class of such graphs mentioned in the previous paragraph, there are more than 40 exceptional graphs) and extending it to the general triangle-free case would be difficult. However, in the applications it is mostly sufficient to deal with the case that the boundary 4-cycles of the cylinder are far apart, and this is the case considered in this paper. With this restriction, it turns out that there are only two infinite classes of critical graphs.

Before stating the result precisely, let us mention one application of this characterization.

Lemma 1.1. *There exists a constant $D \geq 0$ with the following property. Let G be a plane triangle-free graph, let C be a 4-cycle bounding a face of G and let v be a vertex of G . Let ψ be a 3-coloring of $C + v$. If the distance between C and v is at least D , then ψ extends to a 3-coloring of G .*

This confirms Conjecture 1.5 of Dvořák et al. [4], who also proved that this implies the following more interesting result (stated as Conjecture 1.4 in [4]).

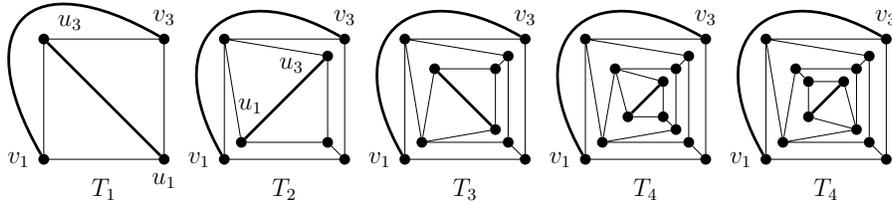


Figure 1: Some Thomas-Walls graphs. Notice that T_4 has two embeddings.

Corollary 1.2. *There exists a constant $D \geq 0$ with the following property. Let G be a planar triangle-free graph, let S be a set of vertices of G and let $\psi : S \rightarrow \{1, 2, 3\}$ be an arbitrary function. If the distance between every two vertices of S is at least D , then ψ extends to a 3-coloring of G .*

Thomassen [13] conjectured that all triangle-free planar graphs have exponentially many 3-colorings. If G is an n -vertex graph of maximum degree at most Δ , then there exists a set $S \subseteq V(G)$ of size at least n/Δ^D such that the distance between every two vertices of S is at least D . Hence, Corollary 1.2 implies that this conjecture holds for triangle-free planar graphs of bounded maximum degree.

Corollary 1.3. *Let D be the constant of Corollary 1.2. If G is an n -vertex planar triangle-free graph of maximum degree at most Δ , then G has at least $(3^{1/\Delta^D})^n$ distinct 3-colorings.*

Let us now introduce a few definitions. The *disk* is the sphere with a hole. In this paper, we generally consider graphs embedded in the sphere, the disk, or the cylinder. Furthermore, we always assume that each face of the embedding contains at most one hole, and if a face contains a hole, then the face is bounded by a cycle, which we call a *ring*. Note that the rings do not have to be disjoint. Let C be the union of the rings of such a graph G (C is empty when G is embedded in the sphere without holes). We say that G is *critical* if $G \neq C$ and for every proper subgraph G' of G such that $C \subseteq G'$, there exists a 3-coloring of C that extends to a 3-coloring of G' , but not to a 3-coloring of G ; that is, removing any edge or vertex not belonging to C affects the set of precolorings of C that extend to a 3-coloring of the graph.

We construct a sequence of graphs T_1, T_2, \dots , which we call *Thomas-Walls graphs* (Thomas and Walls [10] proved that they are exactly the 4-critical graphs that can be drawn in the Klein bottle without contractible cycles of length at most 4). Let T_1 be equal to K_4 . For $n \geq 1$, let uv be any edge of T_n that belongs to two triangles and let T_{n+1} be obtained from $T_n - uv$ by adding vertices x, y and z and edges ux, xy, xz, vy, vz and yz . The first few graphs of this sequence are drawn in Figure 1. For $n \geq 2$, note that T_n contains unique 4-cycles $C_1 = u_1u_2u_3u_4$ and $C_2 = v_1v_2v_3v_4$ such that $u_1u_3, v_1v_3 \in E(T_n)$. Let $T'_n = T_n - \{u_1u_3, v_1v_3\}$. We also define T'_1 to be a 4-cycle. We call the graphs

T'_1, T'_2, \dots reduced Thomas-Walls graphs, and we say that u_1u_3 and v_1v_3 are their *interface pairs*. Note that T'_n has an embedding in the cylinder with rings C_1 and C_2 .

A *patch* is a graph drawn in the disk with ring C of length 6, such that C has no chords and every face of the patch other than the one bounded by C has length 4. Let G be a graph embedded in the sphere, possibly with holes. Let G' be any graph which can be obtained from G as follows. Let S be an independent set in G such that every vertex of S has degree 3. For each vertex $v \in S$ with neighbors x, y and z , remove v , add new vertices a, b and c and a 6-cycle $C = xaybzc$, and draw any patch with ring C in the disk bounded by C . We say that any such graph G' is obtained from G by *patching*. This operation was introduced by Borodin et al. [1] in the context of describing planar 4-critical graphs with exactly 4 triangles.

Consider a reduced Thomas-Walls graph $G = T'_n$ for some $n \geq 1$, with interface pairs u_1u_3 and v_1v_3 . A *patched Thomas-Walls graph* is any graph obtained from such a graph G by patching, and u_1u_3 and v_1v_3 are its interface pairs (note that u_1, u_3, v_1 , and v_3 have degree two in G , and thus they are not affected by patching).

Let G be a graph embedded in the cylinder, with the rings $C_i = x_iy_iz_iw_i$ of length 4, for $i = 1, 2$. Let y'_i be either a new vertex or y_i , and let w'_i be either a new vertex or w_i . Let G' be obtained from G by adding 4-cycles $x_iy'_iz_iw'_i$ forming the new rings. We say that G' is obtained by *framing on pairs x_1z_1 and x_2z_2* .

Let G be a graph embedded in the cylinder with rings C_1 and C_2 of length 3, such that every other face of G has length 4. We say that such a graph G is a *(3,3)-quadrangulation*. Let G' be obtained from G by subdividing at most one edge in each of C_1 and C_2 . We say that such a graph G' is a *near (3,3)-quadrangulation*.

We say that a graph G embedded in the cylinder is *tame* if G contains no contractible triangles, and all triangles of G are pairwise vertex-disjoint. The main result of this paper is the following.

Theorem 1.4. *There exists a constant $D \geq 0$ such that the following holds. Let G be a tame graph embedded in the cylinder with rings C_1 and C_2 of length at most 4. If the distance between C_1 and C_2 is at least D and G is critical, then either*

- *G is obtained from a patched Thomas-Walls graph by framing on its interface pairs, or*
- *G is a near (3,3)-quadrangulation.*

This clearly implies the previously mentioned result on precolored 4-cycle and vertex.

Proof of Lemma 1.1. Suppose for a contradiction that there exists a precoloring ψ of $C + v$ that does not extend to a 3-coloring of G . Let G' be obtained from G by adding three new vertices u_1, u_2 and u_3 and edges vu_1, u_1u_2, u_2u_3 and

u_3v , with the 4-cycle $C' = vu_1u_2u_3$ drawn so that it forms a face. Drill a hole inside C' , turning it into a ring. Let ψ' be any 3-coloring of $C \cup C'$ that extends ψ , and note that ψ' does not extend to a 3-coloring of G' . Hence, G' has a critical subgraph G'' . By Theorem 1.4, either G'' is obtained from a patched Thomas-Walls graph by framing on its interface pairs, or G'' is a near $(3, 3)$ -quadrangulation. However, this is a contradiction, since in either of the cases, each ring contains at most two vertices of degree two, while C' contains three vertices of degree two. \square

Let us remark that which 3-colorings of the rings of a framed patched Thomas-Walls graph or a near $(3, 3)$ -quadrangulation extend to a 3-coloring of the whole graph was determined in the first paper of this series (Dvořák and Lidický [7], Lemmas 2.5 and 2.9).

In Section 2, we prove a lemma based on an idea of [4] that allows us to introduce new non-contractible (≤ 4) -cycles. This enables us to apply the result of the first paper of the series [7], where we considered graphs in cylinder containing many non-contractible (≤ 4) -cycles, and thus prove Theorem 1.4 (Section 3).

2 Cutting a cylinder

Let us start by introducing several previous results. The following is a special case of the main result of Dvořák et al. [3].

Theorem 2.1 ([3]). *For every integer $k \geq 0$, there exists a constant β with the following property. Let G be a graph embedded either in the disk or the cylinder such that the sum of the lengths of the rings is at most k . Suppose that G contains no contractible triangles, and that every non-contractible (≤ 4) -cycle in G is equal to one of the rings. Let F be the set of faces of G . If G is critical, then*

$$\sum_{f \in F} (|f| - 4) \leq \beta. \quad (1)$$

Notice that (1) gives an upper bound β on the number of (≥ 5) -faces as well as upper bound 5β on number of vertices incident to (≥ 5) -faces in G .

Furthermore, we need a result of Gimbel and Thomassen [8], which essentially states that in a plane triangle-free graph, every precoloring of a facial cycle C of length at most 6 extends to a 3-coloring, unless $|C| = 6$ and G contains a subgraph such that every face other than the one bounded by C has length 4.

Theorem 2.2. *Let G be a triangle-free graph drawn in the disk with the ring C of length at most 6. If G is critical, then $|C| = 6$ and all other faces of G have length 4. Furthermore, a precoloring ψ of $C = v_1v_2v_3v_4v_5v_6$ does not extend to a 3-coloring of G if and only if either*

- C has a chord $v_i v_{i+3}$ for some $i \in \{1, 2, 3\}$ and $\psi(v_i) = \psi(v_{i+3})$, or

- C is an induced cycle and $\psi(v_1) = \psi(v_4)$, $\psi(v_2) = \psi(v_5)$, and $\psi(v_3) = \psi(v_6)$.

Let us give an observation about critical graphs that is often useful.

Lemma 2.3. *Let G be a graph drawn in the sphere with holes so that every triangle is non-contractible. If G is critical, then*

- every vertex $v \in V(G)$ that does not belong to the rings has degree at least three,
- every contractible (≤ 5)-cycle in G bounds a face, and
- if K is a closed walk of length 6 in G forming the boundary of an open disk Λ , then either Λ is a face of G , or all faces of G contained in Λ have length 4.

Proof. Let C be the union of the rings of G . Consider any vertex $v \in V(G) \setminus V(C)$. Suppose for a contradiction that v has degree at most 2. Let ψ be any 3-coloring of C that extends to a 3-coloring φ of $G - v$. Then ψ also extends to a 3-coloring of G , by giving the vertices of $V(G) \setminus \{v\}$ the same color as in the coloring φ and by choosing a color of v distinct from the colors of its neighbors. This contradicts the assumption that G is critical.

Next, suppose for a contradiction that K is a contractible (≤ 5)-cycle in G that does not bound a face. Let G_2 be the subgraph of G drawn in the closed disk bounded by K , and let $G_1 = G - (V(G_2) \setminus V(K))$. Since G is critical, there exists a 3-coloring ψ of C that extends to a 3-coloring φ of G_1 , but does not extend to a 3-coloring of G . Hence, the restriction of φ to K does not extend to a 3-coloring of G_2 . However, this contradicts Theorem 2.2.

The last conclusion of the lemma follows by a similar argument as the one for (≤ 5)-cycle. \square

Let G_1 and G_2 be two graphs embedded in the cylinder, with the same rings. We say that G_1 *dominates* G_2 if every precoloring ψ of the rings that extends to a 3-coloring of G_1 also extends to a 3-coloring of G_2 . For two subgraphs H_1 and H_2 of a graph G , let $d(H_1, H_2)$ denote the length of the shortest path in G with one end in H_1 and the other end in H_2 . We now prove the key result of this section.

Lemma 2.4. *There exists an integer $d_0 \geq 3$ with the following property. Let G be a critical graph embedded in the cylinder with rings C_1 and C_2 of length at most 4, such that the distance between C_1 and C_2 is $d \geq d_0$, G contains no contractible triangles, and every non-contractible (≤ 4)-cycle of G is equal to one of the rings. There exists a tame graph G' in the cylinder with the same rings, such that*

- G' dominates G ,
- the distance between C_1 and C_2 in G' is at least $d - 2$,

- G' contains a non-contractible (≤ 4)-cycle distinct from C_1 and C_2 ,
- there exists a vertex $z \in V(G')$ such that the distance between $C_1 \cup C_2$ and z is at least three, and z is contained in all non-contractible (≤ 4)-cycles of G' distinct from the rings, and
- if H' is a near $(3, 3)$ -quadrangulation with rings C_1 and C_2 and $H' \subseteq G'$, then G is a near $(3, 3)$ -quadrangulation with the same 5-faces as H' .

Proof. Let β be the constant of Theorem 2.1. Let $d_0 = (5\beta+1)(5(\beta+4)+2)+12$. We prove the claim by induction on the number of vertices of G ; hence, assume that the claim holds for all graphs with fewer than $|V(G)|$ vertices.

Let $f = v_1v_2v_3v_4$ be a 4-face of G at distance at least 3 from $C_1 \cup C_2$ such that no vertex of f is incident with a face of length greater than 4. Let G_1 be the graph obtained from G by identifying v_1 with v_3 to a new vertex z_1 , and let G_2 be the graph obtained from G by identifying v_2 with v_4 to a new vertex z_2 . If G_1 contained a contractible triangle z_1xy , then G would contain a contractible 5-cycle $v_1v_2v_3xy$. Since G is critical, Lemma 2.3 implies that $v_1v_2v_3xy$ bounds a face, contrary to the assumption that all faces incident with v_2 have length 4. We conclude that every triangle in G_1 is non-contractible. By the assumption on the distance between f and $C_1 \cup C_2$, the triangles of G_1 that contain z_1 are disjoint with $C_1 \cup C_2$.

Let us first discuss the case that the distance between C_1 and C_2 in G_1 is at least d . Observe that every 3-coloring of G_1 extends to a 3-coloring of G (by giving v_1 and v_3 the color of z_1), and thus G_1 dominates G . Let G'_1 be a maximal critical subgraph of G_1 , and note that G'_1 also dominates G . If G'_1 is tame, then let $G''_1 = G'_1$. Otherwise, let T_1 and T_2 be triangles of G'_1 containing z_1 , chosen so that the part Σ of the cylinder between T_1 and T_2 is maximal. Let G''_1 be the graph obtained from G'_1 by removing all vertices and edges contained in the interior of Σ and by identifying the corresponding vertices of T_1 and T_2 .

In the latter case, Theorem 2.2 implies that every 3-coloring of G''_1 extends to a 3-coloring of G'_1 , and thus G''_1 also dominates G . Note that G''_1 is tame and all non-contractible (≤ 4)-cycles in G''_1 distinct from the rings contain z_1 .

Let us consider the situation that a near $(3, 3)$ -quadrangulation H' with rings C_1 and C_2 is a subgraph of G''_1 . Consider any face h of H' . Then h either corresponds to a face in G , or h is incident with z_1 and corresponds to a contractible $(|h| + 2)$ -cycle K in G containing the path $v_1v_2v_3$. Suppose the latter. Since the distance between z_1 and $C_1 \cup C_2$ in H' is at least three, it follows that h has length 4. Since v_2 is not incident with a 6-face in G , all the faces of G contained in the closed disk bounded by K have length 4 by Lemma 2.3. Since v_1 is not incident with a 6-face in G , the same argument shows that if $G''_1 \neq G'_1$, then all faces of G contained in the disk bounded by the closed walk corresponding to $T_1 \cup T_2$ have length 4. Note that one of these cases accounts for all faces of G , and thus G is a near $(3, 3)$ -quadrangulation with the same 5-faces as H' .

If G''_1 contains a non-contractible (≤ 4)-cycle distinct from the rings, then the distance between the rings of G''_1 is at least $d - 2$ (due to the possible

identification of T_1 with T_2 during the construction of G_1'') and we can set $G' = G_1''$ in the conclusions of Lemma 2.4. Otherwise, $G_1'' = G_1'$, and thus G_1'' is critical and the distance between the rings of G_1'' is at least d . By the induction hypothesis, there exists a graph G' dominating G_1' (and thus also G) that satisfies the conclusions of Lemma 2.4.

Therefore, we can assume that the distance between C_1 and C_2 in G_1 is less than d , and by symmetry, the distance between C_1 and C_2 in G_2 is also less than d . Thus, without loss of generality, there exist paths P_1 and P_2 joining v_1 and v_2 , respectively, to C_1 , and paths P_3 and P_4 joining v_3 and v_4 , respectively, to C_2 , such that $|P_1| + |P_3| \leq d - 1$ and $|P_2| + |P_4| \leq d - 1$. Since the distance between C_1 and C_2 in G is at least d , we have $|P_1| + |P_4| \geq d - 1$ and $|P_2| + |P_3| \geq d - 1$. Summing the inequalities, we have $2d - 2 \leq |P_1| + |P_2| + |P_3| + |P_4| \leq 2d - 2$, and thus all the inequalities hold with equality. Consequently, $|P_1| = |P_2|$ and $|P_3| = |P_4|$.

Therefore, we can assume that for every 4-face f of G at distance at least 3 from $C_1 \cup C_2$ that shares no vertex with a (≥ 5)-face, there exists a labelling $v_1 v_2 v_3 v_4$ of the vertices of f such that $d(C_1, v_1) = d(C_1, v_2)$ and $d(C_1, v_3) = d(C_1, v_4) = d(C_1, v_1) + 1$. For an integer $a \geq 0$, let A_a denote the set of vertices of G at distance exactly a from C_1 . Suppose that $5 \leq a \leq d_0 - 5$ and that all vertices of G in $A_{a-1} \cup A_a \cup A_{a+1}$ are only incident with 4-faces. Observe that if two vertices of A_a are incident with the same face, then they are adjacent, and that C_1 and C_2 are in different components of $G - A_a$. Therefore, there exists a non-contractible cycle in G with all vertices in A_a ; let Q_a denote the shortest such cycle (in particular, Q_a is induced).

Let $Q_a = v_1 \dots v_k$. For $1 \leq i \leq k$, let P_i denote a path of length a from v_i to C_1 . Observe that we can choose the paths so that for any $i, j \in \{1, \dots, k\}$, if P_i and P_j intersect, then $P_i \cap P_j$ is a path with an end in C_1 . Let $P_{k+1} = P_1$ and $v_{k+1} = v_1$. For $1 \leq i \leq k$, if P_i and P_{i+1} intersect, then let K_i be the cycle contained in $P_i \cup P_{i+1} \cup \{v_i v_{i+1}\}$. Observe that since C_1 has at most 4 edges, the cycle K_i exists for at least $k - 4$ values of i . On the other hand, K_i has odd length, and thus an odd face of G is contained in the open disk bounded by K_i . Therefore, the cycle K_i exists for at most β values of i . We conclude that $k \leq \beta + 4$.

By Theorem 2.1, at most 5β vertices of G are incident with a face of length greater than 4. By the choice of d_0 , there exists an integer $a \geq 5$ such that all vertices of G at distance at least $a - 1$ and at most $a + 5(\beta + 4) + 1 \leq d_0 - 4$ from C_1 are incident only with 4-faces. Since $|Q_a| \leq \beta + 4$, there exists an integer b such that $a \leq b \leq a + 5(\beta + 4) - 5$ and $|Q_b| \leq |Q_j|$ for all $b \leq j \leq b + 5$.

Let $Q_b = v_1 \dots v_k$ and consider any $i \in \{1, \dots, k\}$. Since no 4-face has three vertices at distance b from C_1 , there exists an edge $v_i u_i \in E(G)$ such that Q_b separates u_i from C_1 . If there existed another such edge incident with v_i , then G would contain a 4-face $u_i v_i u_i' w$ with $d(u_i, C_1) = d(u_i', C_1) = b + 1$, which is a contradiction. It follows that there exists exactly one such vertex u_i for each $i = 1, \dots, k$. Since all faces incident with Q_b have length 4, $u_1 u_2 \dots u_k$ is a non-contractible closed walk in G . By the choice of Q_b , $u_1 u_2 \dots u_k$ is a cycle, and we can choose it as Q_{b+1} .

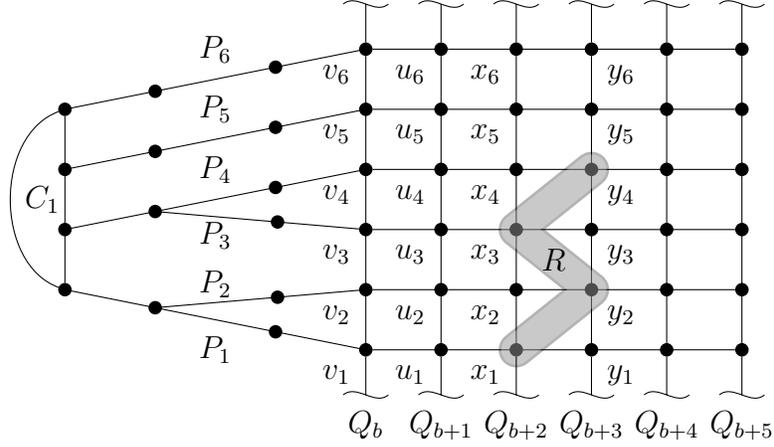


Figure 2: Situation at the end of Lemma 2.4 for $k = 6$.

The same argument shows that we can choose Q_{b+2}, \dots, Q_{b+5} so that the subgraph of G drawn between Q_b and Q_{b+5} is a $6 \times k$ cylindrical grid, see Figure 2. Let $Q_{b+2} = x_1x_2 \dots x_k$ and $Q_{b+3} = y_1y_2 \dots y_k$. If k is even, then let $R = \{x_1, y_2, x_3, y_4, \dots, x_{k-3}\}$. If k is odd, then let $R = \{x_1, y_2, x_3, \dots, x_{k-2}\}$. Let G' be the graph obtained from G by identifying the vertices of R to a single vertex r . Observe that G' dominates G and that r is contained in a new non-contractible (≤ 4)-cycle in G' . Furthermore, since we only contracted 4-faces inside a quadrangulated cylinder, observe that if G' contains a near (3, 3)-quadrangulation with rings C_1 and C_2 as a subgraph, then G is a near (3, 3)-quadrangulation as well. Hence, G' satisfies the conclusions of Lemma 2.4. \square

We need an iterated version of this lemma. Let G be a tame graph embedded in the cylinder with rings of length at most 4. We say that G is a *chain* of graphs G_1, \dots, G_n , if there exist non-contractible (≤ 4)-cycles C_0, \dots, C_n in G such that

- the cycles are pairwise vertex-disjoint except that for $(i, j) \in \{(0, 1), (n, n-1)\}$, C_i can intersect C_j if C_i is a 4-cycle and C_j is a triangle,
- for $0 \leq i < j < k \leq n$ the cycle C_j separates C_i from C_k ,
- the cycles C_0 and C_n are the rings of G ,
- every triangle of G is equal to one of C_0, \dots, C_n , and
- for $1 \leq i \leq n$, the subgraph of G drawn between C_{i-1} and C_i is isomorphic to G_i .

We say that C_0, \dots, C_n are the *cutting cycles* of the chain.

Lemma 2.5. *For every integer $c > 0$, there exists an integer $d_1 \geq 0$ with the following property. Let G be a tame graph embedded in the cylinder with rings C_1 and C_2 of length at most 4, such that the distance between C_1 and C_2 is $d \geq d_1$. If G is not a chain of at least c graphs, then there exists a tame graph G' in the cylinder with the same rings, such that*

- G' dominates G ,
- the distance between C_1 and C_2 in G' is at least $d - 6c$,
- G' is a chain of c graphs G_1, \dots, G_c ,
- every non-contractible (≤ 4)-cycle in G' intersects a ring of one of G_1, \dots, G_c , and
- if G' contains a near $(3, 3)$ -quadrangulation with rings C_1 and C_2 as a subgraph, then G contains a near $(3, 3)$ -quadrangulation with rings C_1 and C_2 as a subgraph as well.

Proof. Let d_0 be the constant from Lemma 2.4, and let $d_1 = c(d_0 + 7) + 6c$.

Let $\mathcal{C} = K_0, K_1, \dots, K_t$ be a sequence of cutting cycles of a chain in G with $t < c$, chosen so that every non-contractible (≤ 4)-cycle in G intersects one of the cutting cycles. We prove that the conclusions of the lemma hold if the distance between C_1 and C_2 is at least $c(d_0 + 7) + 6(c - t)$. The proof is by induction on decreasing t ; hence, we assume that the claim holds for all chains of length greater than t .

Since the distance between K_0 and K_t is greater than $c(d_0 + 7)$, there exists i such that the distance between K_i and K_{i+1} is at least $d_0 + 4$. Let K'_i and K'_{i+1} be non-contractible (≤ 4)-cycles in G intersecting K_i and K_{i+1} , respectively, such that the subgraph F of G drawn between K'_i and K'_{i+1} is minimal. Let F_0 be a maximal critical subgraph of F . Note that the distance between the rings of F_0 is at least d_0 , and that F_0 contains no non-contractible (≤ 4)-cycles distinct from the rings by the choice of \mathcal{C} . Let F' be the graph obtained from F_0 by applying Lemma 2.4. Let Q be the graph obtained from G by replacing the subgraph drawn between K'_i and K'_{i+1} by F' .

If Q contains a near $(3, 3)$ -quadrangulation H with rings C_1 and C_2 as a subgraph, then $H \cap F'$ is also a near $(3, 3)$ -quadrangulation, and thus F_0 is a near $(3, 3)$ -quadrangulation with the same 5-faces as $H \cap F'$ by the conclusions of Lemma 2.4; and thus the subgraph of G obtained from H by replacing $H \cap F'$ by F_0 is a near $(3, 3)$ -quadrangulation.

Since the distance between K'_i and K'_{i+1} in F' is smaller than the distance in F by at most two, it follows that the distance between C_1 and C_2 in Q is smaller than the distance in G by at most 6. Let \mathcal{C}' be the sequence obtained from \mathcal{C} by adding one of the (≤ 4)-cycles of F' distinct from its rings. Observe that every non-contractible (≤ 4)-cycle in Q intersects one of the cycles of \mathcal{C}' .

If $t = c - 1$, then Lemma 2.5 holds with $G' = Q$. Otherwise, Lemma 2.5 follows by the induction hypothesis for Q and \mathcal{C}' . \square

3 Graphs in cylinder with rings far apart

In [7], we proved the following.

Theorem 3.1 ([7], Theorem 1.1). *There exists a constant $c \geq 0$ such that the following holds. Let G be a tame graph embedded in the cylinder with the rings C_1 and C_2 of length at most 4. If G is a chain of at least c graphs, then one of the following holds:*

- every precoloring of $C_1 \cup C_2$ extends to a 3-coloring of G , or
- G contains a subgraph obtained from a patched Thomas-Walls graph by framing on its interface pairs, with rings C_1 and C_2 , or
- G contains a near $(3, 3)$ -quadrangulation with rings C_1 and C_2 as a subgraph.

It is now straightforward to prove our main result by combining Theorem 3.1 and Lemma 2.5.

Proof of Theorem 1.4. Let c be the constant of Theorem 3.1, and let d_1 be the constant of Lemma 2.5 with this c . Let $D = \max(d_1, 15c)$.

Since G is critical, there exists a precoloring of $C_1 \cup C_2$ that does not extend to a 3-coloring of G . If G is a chain of at least c graphs, then the conclusions of Theorem 1.4 follow from Theorem 3.1, since all contractible cycles of length at most 5 in G bound faces by Lemma 2.3.

Hence, assume that G is not a chain of at least c graphs, and let G' be the graph obtained by Lemma 2.5. Since G' dominates G , there exists a 3-coloring of $C_1 \cup C_2$ that does not extend to a 3-coloring of G' . The graph G' is a chain of graphs G_1, \dots, G_c , such that every non-contractible (≤ 4)-cycle in G' intersects one of their rings. By Theorem 3.1, there exists a subgraph H of G' with rings C_1 and C_2 such that either H is obtained from a patched Thomas-Walls graph by framing on its interface pairs, or H is a near $(3, 3)$ -quadrangulation. In the latter case, the last condition in the conclusions of Lemma 2.5 implies that G is a near $(3, 3)$ -quadrangulation.

Hence, assume the former. However, since the distance between C_1 and C_2 in G' is at least $9c$, there exists $i \in \{1, \dots, c\}$ such that the distance between the rings of G_i is at least 5. But since $H \subseteq G$, it follows that G_i contains a non-contractible 4-cycle disjoint from its rings, which is a contradiction. \square

References

- [1] BORODIN, O. V., DVOŘÁK, Z., KOSTOCHKA, A., LIDICKÝ, B., AND YANCEY, M. Planar 4-critical graphs with four triangles. *ArXiv e-prints 1306.1477* (June 2013).
- [2] DVOŘÁK, Z., KRÁL', D., AND THOMAS, R. Three-coloring triangle-free graphs on surfaces III. Graphs of girth five. *ArXiv e-prints 1402.4710* (Feb. 2014).

- [3] DVOŘÁK, Z., KRÁL', D., AND THOMAS, R. Three-coloring triangle-free graphs on surfaces IV. Bounding face sizes of 4-critical graphs. *ArXiv e-prints 1404.6356v2* (Jan. 2015).
- [4] DVOŘÁK, Z., KRÁL', D., AND THOMAS, R. Three-coloring triangle-free graphs on surfaces V. Coloring planar graphs with distant anomalies. *ArXiv e-prints 0911.0885v2* (Jan. 2015).
- [5] DVOŘÁK, Z., KRÁL', D., AND THOMAS, R. Three-coloring triangle-free graphs on surfaces VI. 3-colorability of quadrangulations. *ArXiv e-prints 1509.01013* (Sept. 2015).
- [6] DVOŘÁK, Z., AND LIDICKÝ, B. 4-critical graphs on surfaces without contractible (≤ 4)-cycles. *SIAM Journal on Discrete Mathematics 28* (2014).
- [7] DVOŘÁK, Z., AND LIDICKÝ, B. Fine structure of 4-critical triangle-free graphs I. Planar graphs with two triangles and 3-colorability of chains. *ArXiv e-prints 1505.07294* (May 2015).
- [8] GIMBEL, J., AND THOMASSEN, C. Coloring graphs with fixed genus and girth. *Trans. Amer. Math. Soc. 349* (1997), 4555–4564.
- [9] GRÖTZSCH, H. Ein Dreifarbensatz für Dreikreisfreie Netze auf der Kugel. *Math.-Natur. Reihe 8* (1959), 109–120.
- [10] THOMAS, R., AND WALLS, B. Three-coloring Klein bottle graphs of girth five. *J. Combin. Theory, Ser. B 92* (2004), 115–135.
- [11] THOMASSEN, C. Grötzsch's 3-color theorem and its counterparts for the torus and the projective plane. *J. Combin. Theory, Ser. B 62* (1994), 268–279.
- [12] THOMASSEN, C. The chromatic number of a graph of girth 5 on a fixed surface. *J. Combin. Theory, Ser. B 87* (2003), 38–71.
- [13] THOMASSEN, C. Many 3-colorings of triangle-free planar graphs. *J. Combin. Theory, Ser. B 97* (2007), 334–349.