Fine structure of 4-critical triangle-free graphs III. General surfaces

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Abstract

Dvořák, Král' and Thomas [4, 6] gave a description of the structure of triangle-free graphs on surfaces with respect to 3-coloring. Their description however contains two substructures (both related to graphs embedded in plane with two precolored cycles) whose coloring properties are not entirely determined. In this paper, we fill these gaps.

1 Introduction

The interest in the 3-coloring properties of planar graphs was started by a celebrated theorem of Grötzsch [10], who proved that every planar triangle-free graph is 3-colorable. This result was later generalized and strengthened in many different ways. The one relevant to the topic of this paper concerns graphs embedded in surfaces. While the direct analogue of Grötzsch's theorem is false for any surface other than the sphere, 3-colorability of triangle-free graphs embedded in a fixed surface is nowadays quite well understood. Building upon several previous results ([3, 9, 11, 12, 13, 14, 15, 16]), Dvořák et al. [4, 6] gave a description of the structure of triangle-free graphs on surfaces with respect to 3-coloring. To state the description precisely, we need to introduce a number of definitions.

A *surface* is a compact connected 2-manifold with (possibly null) boundary. Each component of the boundary is homeomorphic to a circle, and we call it a *cuff*. Consider a graph G embedded in the surface Σ; when useful, we identify G with the topological space consisting of the points corresponding to the vertices of G and the simple curves corresponding to the edges of G. A *face* f of G is a maximal connected subset of $\Sigma - G$. A face f is a *closed 2-cell* if it is homeomorphic to an open disk and its boundary forms a cycle C in G; the

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length of f is defined as $|E(C)|$. A graph H is a *quadrangulation* of a surface Σ if all faces of H are closed 2-cells and have length 4 (in particular, the boundary of Σ is formed by a set of pairwise vertex-disjoint cycles in H, called the *boundary cycles* of H). A vertex of G contained in the boundary of Σ is called a *boundary vertex.*

Let G be a graph, let ψ be a proper coloring of G by colors $\{1,2,3\}$ and let $Q = v_1v_2\ldots v_kv_1$ be a directed closed walk in G. One can view ψ as mapping Q to a closed walk in a triangle T with vertices 1, 2, and 3, and the *winding number* of ψ on Q is then the number of times this walk goes around T in a fixed orientation. More precisely, for $uv \in E(G)$, let $\delta_\psi(u,v) = 1$ if $\psi(v) - \psi(u) \in \{1,-2\}$, and $\delta_\psi(u,v) = -1$ otherwise. For a walk $W = u_1u_2\ldots u_m$, let $\delta_\psi(W) = \sum_{i=1}^{m-1} \delta_\psi(u_i,u_{i+1})$. The winding number $\omega_\psi(Q)$ of ψ on Q is defined as $\delta_\psi(Q)/3$.

Suppose that G is embedded in an orientable surface Σ so that every face of G is closed 2-cell. Let $\mathcal{C}$ be the set consisting of all facial and boundary cycles of G. For each of the cycles in $\mathcal{C}$, choose an orientation of its edges with outdegree one so that every edge $e \in G$ is oriented in opposite directions in the two cycles of $\mathcal{C}$ containing e. We call such orientations *consistent.* Note that there are exactly two consistent orientations opposite to each other. There is a well known constraint on winding numbers in embedded graphs.

Observation 1.1. *Let G be a graph embedded in an orientable surface Σ so that every face of G is closed 2-cell. Let $\mathcal{C}$ be the set consisting of all facial and boundary cycles of G. Let $Q_1, \ldots, Q_m$ be the cycles of $\mathcal{C}$ viewed as closed walks in a consistent orientation. If ψ is a 3-coloring of G, then*

$$\sum_{i=1}^{m} \omega_\psi(Q_i) = 0.$$

As the winding number of any 3-coloring of a 4-cycle is 0, we obtain the following constraint on 3-colorings of quadrangulations.

Corollary 1.2. *Let G be a quadrangulation of an orientable surface Σ. Let $B_1, \ldots, B_k$ be the boundary cycles of G in a consistent orientation. If ψ is a 3-coloring of G, then*

$$\sum_{i=1}^{k} \omega_\psi(B_i) = 0.$$

For non-orientable surfaces, the situation is a bit more complicated. Suppose that G is a quadrangulation of a surface, and let us fix directed closed walks $B_1, \ldots, B_k$ tracing the cuffs of G. For each facial cycle, choose an orientation arbitrarily. Let D denote the directed graph with vertex set $V(G)$ and uv being an edge of D if and only if uv is an edge of G oriented towards v in both cycles of $\mathcal{C}$ that contain it. Let $p(G,B_1,\ldots,B_k) = 2|E(D)| \bmod 4$. Note that $p(G,B_1,\ldots,B_k)$ is independent on the choice of the orientations of the 4-faces, since reversing an orientation of a 4-face with d edges belonging to D changes $2|E(D)|$ by $2(4-2d) \equiv 0 \pmod 4$.

Consider the sum of winding numbers of a 3-coloring ψ of G on cycles in $\mathcal{C}$. As before, the contributions of all edges of G that do not belong to D cancel out, and since G is a quadrangulation, the winding number on any non-boundary cycle in $\mathcal{C}$ is 0. Hence,

$$\sum_{i=1}^{k} \delta_\psi(B_i) = 2 \sum_{uv \in E(D)} \delta_\psi(u, v).$$

Since $\delta_\psi(u, v) = \pm 1$ for every $uv \in E(D)$,

$$2 \sum_{uv \in E(D)} \delta_\psi(u, v) \equiv 2|E(D)| \equiv -2|E(D)| \equiv -p(G, B_1, \ldots, B_k) \pmod{4},$$

regardless of the 3-coloring ψ. Furthermore,

$$\sum_{i=1}^{k} \delta_\psi(B_i) = 3 \sum_{i=1}^{k} \omega_\psi(B_i) \equiv - \sum_{i=1}^{k} \omega_\psi(B_i) \pmod{4}.$$

Therefore, we get the following necessary condition for the existence of a 3-coloring.

Observation 1.3. *Let G be a quadrangulation of a surface Σ. Let B_1, ..., B_k be the boundary cycles of G. If ψ is a 3-coloring of G, then*

$$\sum_{i=1}^{k} \omega_\psi(B_i) \equiv p(G, B_1, \ldots, B_k) \pmod{4}.$$

If a 3-coloring ψ of the boundary cycles satisfies the condition of Observation 1.3, we say that ψ is *parity-compliant.*

We say that a coloring ψ of the boundary cycles of a quadrangulation of a surface Σ *satisfies the winding number constraint* if either

- Σ is orientable and the sum of winding numbers of ψ on the boundary cycles of G in their consistent orientation is 0, or
- Σ is non-orientable and ψ is parity-compliant.

The structure theorem of Dvořák, Král' and Thomas [6] now can be stated as follows.

Theorem 1.4 ([6]). *For every surface Σ and integer $k \geq 0$, there exists a constant β_0 with the following property. Let G be a triangle-free graph embedded in Σ so that every cuff of Σ traces a cycle in G and so that the sum of the lengths of the boundary cycles of G is at most k. Suppose that every contractible 4-cycle in G bounds a face. Then G has a subgraph H with at most β_0 vertices, such that H contains all the boundary cycles and each face h of H satisfies one of the following (where $G[h]$ is the subgraph of G drawn in the closure of h).*

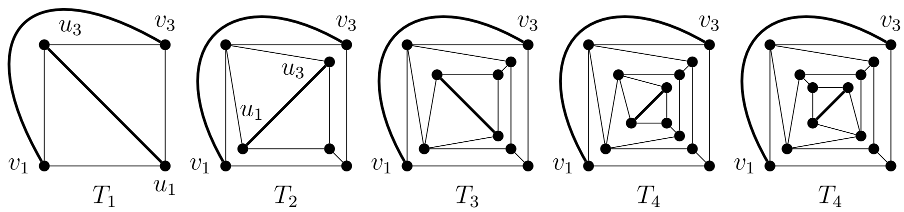

Figure 1: Some Thomas-Walls graphs (with two different drawings of T_4).

(a) Every precoloring of the boundary of h extends to a 3-coloring of $G[h]$, or

(b) $G[h]$ is a quadrangulation and every precoloring of the boundary of h which satisfies the winding number constraint extends to a 3-coloring of $G[h]$, or

(c) h is an open cylinder and $G[h]$ is its quadrangulation, or

(d) h is an open cylinder and both boundary cycles of h have length exactly 4.

Given the subgraph H as in the theorem, we can decide whether a given precoloring of the boundary cycles of G extends to a 3-coloring of G, by first trying all possible (constantly many) 3-colorings of H that extend the precoloring, and then testing whether they extend to the subgraphs $G[h]$ for each face h of H. In the cases (a) and (b), the theorem also shows how to test the existence of the extension to $G[h]$. In the cases (c) and (d), much fewer details are given (although the structure is still sufficiently restrictive to enable many applications of the theorem). The goal of this paper is to fill in this gap. Before stating our result, let us first give a few more definitions.

We construct a sequence of graphs T_1, T_2, ..., which we call *Thomas-Walls graphs* (Thomas and Walls [14] proved that they are exactly the 4-critical graphs that can be drawn on Klein bottle without contractible cycles of length at most 4). Let T_1 be equal to K_4. For $n \geq 1$, let uv be any edge of T_n that belongs to two triangles and let T_{n+1} be obtained from $T_n - uv$ by adding vertices x, y and z and edges ux, xy, xz, vy, vz and yz. The first few graphs of this sequence are drawn in Figure 1. The sequence is unique but embeddings in the plane are not combinatorially unique. For $n \geq 2$, note that T_n contains unique 4-cycles $C_1 = u_1u_2u_3u_4$ and $C_2 = v_1v_2v_3v_4$ such that $u_1u_3, v_1v_3 \in E(G)$. Let $T'_n = T_n - \{u_1u_3, v_1v_3\}$. We call the graphs T'_1, T'_2, ... *reduced Thomas-Walls graphs*, and we say that u_1u_3 and v_1v_3 are their *interface pairs*. Note that for $n \geq 3$, the 4-cycles C_1 and C_2 are vertex-disjoint, and thus T'_n has an embedding in the cylinder with boundary cycles C_1 and C_2 (as we mentioned before, this embedding is not unique for $n \geq 4$). We always use only such embeddings with boundary 4-cycles when drawing T_i in the cylinder.

A *patch* is a graph drawn in the disk with a cycle C of length 6 tracing the cuff, such that C has no chords and every face of the patch other than the one bounded by C has length 4. Let G be a graph embedded in a surface. Let G' be any graph which can be obtained from G as follows. Let S be an independent set in G such that every vertex of S has degree 3. For each vertex $v \in S$ with neighbors x, y and z, remove v, add new vertices a, b and c (drawn very close to the original position of v) and 6-cycle $C = xaybzc$, and draw any patch with the boundary cycle C in the disk bounded by C. We say that any such graph G' is obtained from G by *patching*. This operation was introduced by Borodin et al. [2] in the context of describing planar 4-critical graphs with exactly 4 triangles.

Consider a reduced Thomas-Walls graph $G = T'_n$ for some $n \geq 1$, with interface pairs u_1u_3 and v_1v_3. A *patched Thomas-Walls graph* is any graph obtained from such a graph G by patching, and u_1u_3 and v_1v_3 are its interface pairs.

Next, let us construct another class of graphs, *forced extension quadrangulations*. Let G be a graph embedded in the cylinder whose holes are contained in distinct faces of G bounded by (not necessarily disjoint) cycles C_1 and C_2 of the same length k (let us remark that these cycles are disjoint from the cuffs). We say that G is a forced extension quadrangulation if all other faces of G have length 4, every non-contractible cycle in G has length at least k, and there is a sequence $K_1, \ldots, K_n$ of non-contractible k-cycles in G such that $C_1 = K_1$, $C_2 = K_n$ and K_i intersects K_{i+1} in at least one vertex for $1 \leq i \leq n-1$. Note that if C_1 and C_2 are vertex-disjoint, then G can be drawn as a quadrangulation of the cylinder with boundary cycles C_1 and C_2. If the distance between C_1 and C_2 is at least $4k$, we say that G is a *wide forced extension quadrangulation.*

Our main result now can be stated as follows.

Theorem 1.5. *For every surface Σ and integer $k \geq 0$, there exists a constant β with the following property. Let G be a triangle-free graph embedded in Σ so that every cuff of Σ traces a cycle in G and so that the sum of the lengths of the boundary cycles of G is at most k. Suppose that every contractible (≤ 5)-cycle in G bounds a face. Then G has a subgraph H with at most β vertices, such that H contains all the boundary cycles and each face h of H satisfies one of the following (where $G[h]$ is the subgraph of G drawn in the closure of h).*

(a) Every precoloring of the boundary of h extends to a 3-coloring of $G[h]$, or

(b) $G[h]$ is a quadrangulation and every precoloring of the boundary of h which satisfies the winding number constraint extends to a 3-coloring of $G[h]$, or

(c) h is an open cylinder and $G[h]$ is its wide forced extension quadrangulation with boundary cycles whose length is divisible by 3, or

(d) h is an open cylinder and $G[h]$ is a patched Thomas-Walls graph.

Let us remark that Aksenov [1] showed that in a triangle-free planar graph G', every precoloring of a (≤ 5)-cycle extends to a 3-coloring of G'. Hence, given an embedded graph, removing vertices and edges contained inside a contractible (≤ 5)-cycle does not affect the 3-coloring properties of the graph. Therefore, the assumption of Theorem 1.5 that G contains no non-facial contractible (≤ 5)-cycles does not affect its generality.

Which precolorings of the boundary cycles of a patched Thomas-Walls graph F extend to a 3-coloring of F was determined in the first paper of this series, see Lemma 2.7 of [7]. In Section 2, we describe which precolorings of the boundary cycles of a wide forced extension quadrangulation F extend to a 3-coloring of F, thus making Theorem 1.5 fully explicit about the 3-coloring properties of G. In Section 3, we prove Theorem 1.5 for quadrangulations of the cylinder. Finally, Section 4 is devoted to the proof of Theorem 1.5.

2 Forced extension quadrangulations

A function $\theta : V(G) \to V(H)$ is a *homomorphism* if $\theta(u)\theta(v) \in E(H)$ for every edge $uv \in E(G)$. If C' and C are cycles of the same length, then a homomorphism $\theta : V(C') \to V(C)$ is *cyclic* if it is an isomorphism. If $\theta' : V(C') \to V(C)$ is another cyclic homomorphism, we say that θ' and θ are *offset by* 2 if the distance between $\theta(v)$ and $\theta'(v)$ in C is 2 for every $v \in V(C')$. If $X \subseteq V(G)$ and $\theta : V(G) \to V(H)$, then $\theta \restriction X$ denotes the restriction of a θ to X.

Lemma 2.1. *Let G be a graph embedded in the cylinder so that the holes of the cylinder are contained in distinct faces bounded by cycles C_1 and C_2 of the same length k, such that all other faces of G have length 4. Suppose that every non-contractible cycle in G has length at least k. Let C be a k-cycle. Then there exists a homomorphism $\theta : V(G) \to V(C)$ such that θ restricted to C_1 as well as C_2 is cyclic. Furthermore, if G is not a forced extension quadrangulation, then there exists another such homomorphism θ' such that $\theta' \restriction V(C_1) = \theta \restriction V(C_1)$, and $\theta' \restriction V(C_2)$ and $\theta \restriction V(C_2)$ are offset by 2.*

Proof. We prove the lemma by induction on the number of vertices of G.

Suppose that G contains a non-contractible k-cycle K distinct from C_1 and C_2. For $i \in \{1, 2\}$, let G_i be the subgraph of G drawn between C_i and K. Let $\theta_i : V(G_i) \to V(C)$ be the homomorphism obtained by the induction hypothesis, chosen so that $\theta_1 \restriction V(K) = \theta_2 \restriction V(K)$. Then, $\theta_1 \cup \theta_2$ gives a homomorphism from G to C as required. If G is not a forced extension quadrangulation, then by symmetry, we can assume that G_2 is not a forced extension quadrangulation, and thus there exists a homomorphism $\theta_2' : V(G_2) \to V(C)$ such that $\theta_2 \restriction V(K) = \theta_2' \restriction V(K)$, and $\theta_2 \restriction V(C_2)$ and $\theta_2' \restriction V(C_2)$ are offset by 2. Hence, $\theta_1 \cup \theta_2'$ is the other homomorphism required by the claim of Lemma 2.1.

Therefore, we can assume that every non-contractible cycle distinct from C_1 and C_2 has length greater than k. Note that each such cycle has the same parity as C_1, and thus it has length at least $k + 2$.

Suppose that G contains a contractible cycle $K = z_1z_2z_3z_4$ of length 4 which does not bound a face. Let G' be obtained from G by removing the interior of K. By the induction hypothesis, G' has a homomorphism θ to C. Since G is a quadrangulation, the subgraph of G induced by K and its interior is bipartite with parts B_1 and B_2 such that $z_1, z_3 \in B_1$ and $z_2, z_4 \in B_2$. The homomorphism θ can be extended to G by for $i \in \{1, 2\}$ assigning $\theta(v) = \theta(z_i)$ for all $v \in B_i \setminus \{z_{i+2}\}$. Observe that G' is a forced extension quadrangulation if and only if G is a forced extension quadrangulation. If G' is not a forced extension quadrangulation then it has another homomorphism θ' as described in the statement of the lemma, and this homomorphism also extends to G. Hence, we can assume that every contractible 4-cycle in G bounds a face.

Clearly, the claim of the lemma holds if $G = C_1 = C_2$. Hence, we have $C_1 \neq C_2$. Suppose now that C_1 and C_2 intersect. Then G is a forced extension quadrangulation. Furthermore, there exists a 4-face $z_1z_2z_3z_4$ such that $z_1 \in V(C_1) \cap V(C_2)$. We apply induction to the graph obtained from G by identifying z_2 with z_4 (this graph has no parallel edges, since every contractible 4-cycle in G bounds a face), and extend the resulting homomorphism to G by setting $\theta(z_2) = \theta(z_4)$.

Suppose now that C_1 and C_2 are vertex-disjoint. Since every non-contractible cycle distinct from C_1 and C_2 has length greater than k, every vertex of C_1 has degree at least three. If all vertices of C_1 have degree exactly three, then let $C_1 = v_1v_2 \ldots v_k$ and for $1 \leq i \leq k$, let u_i be the neighbor of v_i not belonging to C_1. Observe that $u_1u_2 \ldots u_k$ is a k-cycle, which must be equal to C_2. If $C = x_1x_2 \ldots x_k$, we can define the homomorphisms θ and θ' by setting $\theta(v_i) = \theta'(v_i) = x_i$, $\theta(u_i) = x_{i+1}$ and $\theta'(u_i) = x_{i-1}$ for $1 \leq i \leq k$, where $x_0 = x_k$ and $x_{k+1} = x_1$.

Therefore, we can assume that a vertex z_1 of C_1 has degree at least 4, and thus G has a 4-face $z_1z_2z_3z_4$ with $z_2, z_4 \notin V(C_1)$. Since all non-contractible cycles in G distinct from C_1 and C_2 have length at least $k+2$, at most one of z_2 and z_4 belongs to C_2, and the graph G' obtained from G by identifying z_2 with z_4 contains no non-contractible cycle of length less than k. Furthermore, since every contractible 4-cycle in G bounds a face, the graph G' has no parallel edges.

Suppose that G' contains a non-contractible k-cycle $K \neq C_1$ that intersects C_1 in a vertex z. Then G contains paths P_1 from z_2 to z and P_2 from z_4 to z such that $|P_1| + |P_2| = k$ and K is obtained from $P_1 \cup P_2$ by identifying z_2 with z_4. Let P'_1 and P'_2 be the subpaths of C_1 between z_1 and z, chosen so that both closed walks $K_1 = P'_1 \cup P_1 \cup \{z_1z_2\}$ and $K_2 = P'_2 \cup P_2 \cup \{z_1z_4\}$ are contractible. Note that $|P'_1| + |P'_2| = k$, and thus by symmetry, we can assume that $|P_1| \leq |P'_1|$. Since K_1 is contractible and all faces of G other than the ones bounded by C_1 and C_2 have length 4, it follows that $|K_1|$ is even, and thus P_1 and P'_1 have opposite parity. Hence, $|P_1| \leq |P'_1| - 1$, and $Q = P_1 \cup P'_2 \cup \{z_1z_2\}$ is a non-contractible closed walk of length at most k in G. Since G contains no non-contractible cycles of length less than k, it follows that Q is a cycle. However, Q is distinct from C_1 and C_2, which is a contradiction.

We conclude that G' contains no such k-cycle K, and thus G' is not a forced extension quadrangulation. By the induction hypothesis, there exist homomorphisms θ and θ' from G' to C that are offset by 2, and they can be extended to homomorphisms from G to C satisfying the conditions of Lemma 2.1 by setting $\theta(z_2) = \theta(z_4)$ and $\theta'(z_2) = \theta'(z_4)$. □

Next, let us recall a result from [5].

Lemma 2.2 ([5], Corollary 4.7)**.** *Let G be a quadrangulation of the cylinder with boundary cycles C_1 and C_2 of the same length k, such that every non-contractible cycle in G has length at least k, and the distance between C_1 and C_2 is at least $4k$. Let ψ be a precoloring of $C_1 \cup C_2$ satisfying the winding number constraint. If ψ does not extend to a 3-coloring of G, then k is divisible by 3 and ψ has winding number $\pm k/3$ on C_1.*

This makes it easy to analyze the colorings of wide forced extension quadrangulations. Note that the existence of the homomorphism θ in the following statement is guaranteed by Lemma 2.1.

Lemma 2.3. *Let G be a wide forced extension quadrangulation of a cylinder with boundary cycles of length k, and let $\theta : V(G) \to V(C)$ be a homomorphism to a k-cycle C such that the restrictions of θ to both boundary cycles of G are cyclic. A precoloring ψ of the boundary cycles of G extends to a 3-coloring of G if and only if*

- *ψ satisfies the winding number constraint, and additionally,*
- *if k is divisible by 3 and ψ has winding number $\pm k/3$ on the boundary cycles, then $\psi(u) = \psi(v)$ for all vertices u and v in the boundary cycles such that $\theta(u) = \theta(v)$.*

Proof. Let G be a wide forced extension quadrangulation with boundary cycles C_1 and C_2 of length k, and let $K_1, \ldots, K_n$ be the non-contractible k-cycles such that $K_1 = C_1$, $K_n = C_2$ and K_i intersects K_{i+1} for $1 \le i \le n-1$.

Let ψ be any precoloring of $C_1 \cup C_2$. If ψ does not satisfy the winding number constraint, then it does not extend to a 3-coloring of G, as we observed in the introduction (Corollary 1.2). Suppose that ψ satisfies the winding number constraint. If the winding number of ψ on C_1 is not $\pm k/3$, then ψ extends to a 3-coloring of G by Lemma 2.2.

Finally, consider the case that the winding number of ψ on C_1 is $\pm k/3$, and in particular, k is divisible by 3. Let $C = v_1 v_2 \ldots v_k$. Note that θ is cyclic on cycles $K_1, \ldots, K_n$: for each edge $e = uv$ of G with $\theta(u) = v_a$ and $\theta(v) = v_b$, orient e so that the head of e is v if and only if $b \equiv a + 1 \pmod{k}$, and for each walk Q in G, let $p(Q)$ denote the difference between the number of edges of Q oriented forwards and backwards. Since $k \neq 4$, we have $p(Q) = 0$ for each 4-cycle Q, and since all faces of G other than the ones bounded by C_1 and C_2 have length 4, we conclude that $p(C_1) = p(K)$ for every non-contractible cycle K. It follows that $p(K_1) = \ldots = p(K_n) = k$, and thus θ is cyclic on these cycles.

For $1 \le i \le n$, let $K_i = v_{i,1} v_{i,2} \ldots v_{i,k}$, where the labels are chosen so that $\theta(v_{i,j}) = v_j$ for $1 \le j \le k$. Consider any 3-coloring φ of G, and any $i \in \{1, \ldots, n-1\}$. Suppose that φ has winding number $\pm k/3$ on K_i; without loss of generality, we have $\varphi(v_{i,j}) = j \bmod 3$ for $1 \le j \le k$. By the winding number constraint, φ has the same winding number on K_{i+1}, and thus there exists $\alpha \in \{0, 1, 2\}$ such that $\varphi(v_{i+1}, j) = (j + \alpha) \bmod 3$ for $1 \le j \le k$. However,

K_i and K_{i+1} share a vertex, and thus $\alpha = 0$. We conclude that if φ has winding number $\pm k/3$ on C_1, then $\varphi(u) = \varphi(v)$ for any $u, v \in V(K_1) \cup \ldots \cup V(K_n)$ such that $\theta(u) = \theta(v)$.

Hence, in the case that k is divisible by 3 and ψ has winding number $\pm k/3$ on the boundary cycles, if ψ extends to a 3-coloring of G, then $\psi(u) = \psi(v)$ for all vertices u and v in the boundary cycles such that $\theta(u) = \theta(v)$. Conversely, if ψ satisfies this condition on the boundary cycles, then we can extend ψ to a 3-coloring φ of G by setting $\varphi(u) = \psi(u')$ for every $u \in V(G)$, where u' is the vertex of C_1 such that $\theta(u) = \theta(u')$. □

3 The structure theorem for quadrangulations of the cylinder

Lemma 2.1 of Dvořák, Král' and Thomas [6] gives the following result on precoloring extension in quadrangulations of the disk.

Lemma 3.1. *Let G be a quadrangulation of the disk with boundary cycle C. Let ψ be a 3-coloring of C with winding number 0. If ψ does not extend to a 3-coloring of G, then G contains a path P intersecting C exactly in its endpoints so that both cycles in $C \cup P$ distinct from C are strictly shorter than C.*

Iterating this lemma, we get the following result

Corollary 3.2. *Let $k \geq 4$ be an integer. For every quadrangulation G of the disk with the boundary cycle C of length k, there exists a subgraph $H \subseteq G$ with at most $2^{k/2}$ vertices, such that H contains the boundary cycle and for every face h of H, every precoloring of the boundary of h which satisfies the winding number constraint extends to a 3-coloring of $G[h]$.*

Proof. We prove the claim by induction on k. If $k = 4$, then every precoloring of C extends to a 3-coloring of G by the result of Aksenov [1], and we can set $H = C$. Hence, assume that $k \geq 6$. If G contains no path P intersecting C exactly in its endpoints such that both cycles in $C \cup P$ distinct from C are strictly shorter than C, then by Lemma 3.1 we can again set $H = C$.

Hence, suppose that G contains such a path P. Let C_1 and C_2 be the cycles of $C \cup P$ distinct from C; we have $|C_1|, |C_2| \leq k - 2$. For $i \in \{1, 2\}$, let G_i be the subgraph of G drawn in the closed disk bounded by C_i, and let H_i be its subgraph obtained by the induction hypothesis. We can set $H = H_1 \cup H_2$; note that $|V(H)| \leq |V(H_1)| + |V(H_2)| \leq 2 \cdot 2^{(k-2)/2} = 2^{k/2}$. □

We need another result of [5].

Lemma 3.3 ([5], Lemma 4.5)**.** *Let G be a quadrangulation of the cylinder with boundary cycles C_1 and C_2, such that every non-contractible cycle in G distinct from C_1 and C_2 has length greater than $\max(|C_1|, |C_2|)$. If the distance between C_1 and C_2 is at least $|C_1| + |C_2|$, then a precoloring of $C_1 \cup C_2$ extends to a 3-coloring of G if and only if it satisfies the winding number constraint.*

We also need a variation of Lemma 3.3.

Lemma 3.4. *Let G be a quadrangulation of the cylinder with boundary cycles C_1 and C_2 of the same length k, such that every non-contractible cycle in G has length at least k. Suppose that G contains a non-contractible k-cycle K, and for $i \in \{1,2\}$, let G_i be the subgraph of G drawn between K and C_i. Assume that neither G_1 nor G_2 is a forced extension quadrangulation. If the distance between C_1 and C_2 is at least $4k$, then a precoloring of $C_1 \cup C_2$ extends to a 3-coloring of G if and only if it satisfies the winding number constraint.*

Proof. As we have shown in the introduction (Corollary 1.2), the winding number constraint is a necessary condition for the precoloring to extend. Let ψ be a precoloring of $C_1 \cup C_2$ that satisfies the winding number constraint. If ψ does not have winding number $\pm k/3$ on the boundary cycles, then the claim follows from Lemma 2.2. Hence, assume that k is divisible by 3 and ψ has winding number $\pm k/3$ on the boundary cycles.

Let C be a k-cycle, $C = x_1x_2 \dots x_k$. Let θ_1 and θ'_1 be the homomorphisms from G_1 to C obtained by Lemma 2.1. Let $C_1 = v_1 \dots v_k$ and $K = w_1 \dots w_k$, with the labels chosen so that $\theta_1(w_i) = \theta'_1(w_i) = \theta_1(v_i) = x_i$ and $\theta'_1(v_i) = x_{i+2}$ for $1 \le i \le k$, where $x_{k+1} = x_1$ and $x_{k+2} = x_2$. Let θ_2 and θ'_2 be the homomorphisms from G_2 to C obtained by Lemma 2.1. We can choose the homomorphisms and the labels of $C_2 = z_1z_2 \dots z_k$ so that $\theta_2(w_i) = \theta'_2(w_i) = \theta_2(z_i) = x_i$ and $\theta'_2(z_i) = x_{i+2}$ for $1 \le i \le k$.

Let $\theta^\star_1 = \theta_1 \cup \theta_2$, $\theta^\star_2 = \theta'_1 \cup \theta_2$ and $\theta^\star_3 = \theta_1 \cup \theta'_2$. For $i \in \{1,2,3\}$, let φ_i be the 3-coloring of G defined by setting $\varphi_i(y) = \psi(v)$ for every $y \in V(G)$ and $v \in V(C_1)$ such that $\theta^\star_i(y) = \theta^\star_i(v)$. Observe that $\varphi_1 \restriction V(C_1) = \varphi_2 \restriction V(C_1) = \varphi_3 \restriction V(C_1) = \psi \restriction V(C_1)$ and that $\varphi_1 \restriction V(C_2) \neq \varphi_2 \restriction V(C_2) \neq \varphi_3 \restriction V(C_2) \neq \varphi_1 \restriction V(C_2)$.

Since ψ has winding number $\pm k/3$, there are only three colorings of C_2 with the same winding number as ψ has on C_1. Consequently, one of φ_1, φ_2 and φ_3 extends ψ. □

We can now prove Theorem 1.5 for quadrangulations of the cylinder, which exposes the importance of the forced extension quadrangulations.

Lemma 3.5. *For every $k \ge 3$, there exists a constant β_2 with the following property. Let G be a quandrangulation of the cylinder, such that both boundary cycles of G have length at most k. Then G has a subgraph H with at most β_2 vertices, such that H contains all the boundary cycles and each face h of H satisfies one of the following (where $G[h]$ is the subgraph of G drawn in the closure of h):*

(a) every precoloring of the boundary of h which satisfies the winding number constraint extends to a 3-coloring of $G[h]$, or

(b) h is an open cylinder and $G[h]$ is its wide forced extension quadrangulation with boundary cycles whose length is divisible by 3.

Proof. Let C_1 and C_2 be the boundary cycles of G, where $|C_1| \leq |C_2| = k$. If K is a shortest non-contractible cycle of G distinct from C_1 and C_2, let $\ell(G) = \min(|K|, k+1)$. We now proceed by induction on k, and subject to that on decreasing $\ell(G)$; i.e., we assume that Lemma 3.5 holds for all quadrangulations G' with boundary cycles of length less than k, and for those with boundary cycles of length at most k and with $\ell(G') > \ell(G)$.

We will exhibit a sequence $K_1, \ldots, K_n$ of non-contractible ($\leq k$)-cycles with $n \leq 5$ such that $C_1 = K_1$, $C_2 = K_n$, for $1 \leq i < j < m \leq n$, the cycle K_j separates K_i from K_m, and such that for $1 \leq i \leq n-1$, the subgraph G_i of G drawn between K_i and K_{i+1} satisfies one of the following properties:

(i) $\max(|K_i|, |K_{i+1}|) < k$, or

(ii) $\ell(G_i) > \ell(G)$, or

(iii) $\ell(G_i) = k+1$, or

(iv) $|K_i| = |K_{i+1}| = \ell(G_i) = k$ and G_i contains a non-contractible k-cycle K'_i such that neither the subgraph of G drawn between K_i and K'_i nor the subgraph drawn between K'_i and K_{i+1} is a forced extension quadrangulation, or

(v) G_i is a forced extension quadrangulation.

Before we construct such a sequence, we show how its existence implies Lemma 3.5. If the distance between K_i and K_{i+1} is less than $4k$, then let G'_i be the graph obtained from G_i by cutting the cylinder along a shortest path P_i between K_i and K_{i+1}. Note that G'_i is embedded in the disk; let H'_i be the subgraph of G'_i obtained by Corollary 3.2, and let $H_i \subseteq G_i$ be obtained from H'_i by merging back the vertices of the path P_i.

Otherwise, in cases (i) and (ii), let H_i be the subgraph of G_i obtained by the induction hypothesis. In the cases (iii) and (iv), note that by Lemmas 3.3 and 3.4, every precoloring of $K_i \cup K_{i+1}$ which satisfies the winding number constraint extends to a 3-coloring of G_i, and set $H_i = K_i \cup K_{i+1}$. In the case (v), since G_i is a wide forced extension quadrangulation, Lemma 2.3 implies that either every precoloring of $K_i \cup K_{i+1}$ which satisfies the winding number constraint extends to a 3-coloring of G_i, or $|K_i|$ is divisible by 3; we set $H_i = K_i \cup K_{i+1}$.

Observe that every face of $H = \bigcup_{i=1}^{n-1} H_i$ satisfies either the condition (a) or (b) of the statement of the lemma. Hence, it suffices to construct the sequence $K_1, \ldots, K_n$ with the described properties.

If $\ell(G) = k+1$, then we set $n = 2$, $K_1 = C_1$ and $K_2 = C_2$, and note that G satisfies (iii). If $\ell(G) < k$, then let $n = 4$, let $K_1 = C_1$, $K_4 = C_2$ and let K_2 and K_3 be non-contractible $\ell(G)$-cycles chosen so that G_1 and G_3 are minimal (with possibly $K_2 = K_3$). Consequently, G_1 and G_3 satisfy (ii) and G_2 satisfies (i).

Hence, assume that $\ell(G) = k$. We set $K_1 = C_1$ and let K_2 be a non-contractible k-cycle chosen so that G_1 is minimal (where possibly $K_2 = C_1$ if $|C_1| = k$), so that G_1 satisfies (ii). If the subgraph of G drawn between K_2 and

C_2 satisfies (iii), (iv) or (v), then we set $n = 3$ and $K_3 = C_2$. Otherwise, let $n = 5$, $K_5 = C_2$, and let K_3 and K_4 be chosen as non-contractible k-cycles such that G_2 and G_4 are forced extension quadrangulations and they are maximal with this property. Consequently, every non-contractible k-cycle in G_3 distinct from K_3 and K_4 is disjoint from $K_3 \cup K_4$, and since the subgraph of G drawn between K_2 and C_2 does not satisfy (iv), we conclude that K_3 and K_4 are the only non-contractible k-cycles in G_3, and thus G_3 satisfies (iii). □

We can now prove Theorem 1.5 in the special case of graphs embedded in the disk.

Theorem 3.6. *For every integer $k \geq 4$, there exists a constant β_3 with the following property. Let G be a triangle-free graph embedded in the disk whose cuff traces a cycle C in G of length at most k. Suppose that every 4-cycle in G bounds a face. Then G has a subgraph H with at most β_3 vertices, such that $C \subseteq H$ and each face h of H satisfies one of the following (where $G[h]$ is the subgraph of G drawn in the closure of h).*

(a) Every precoloring of the boundary of h extends to a 3-coloring of $G[h]$, or

(b) $G[h]$ is a quadrangulation and every precoloring of the boundary of h which satisfies the winding number constraint extends to a 3-coloring of $G[h]$, or

(c) h is an open cylinder and $G[h]$ is its wide forced extension quadrangulation with boundary cycles whose length is divisible by 3.

Proof. Let H_0 be the subgraph of G obtained by Theorem 1.4. If a face h of H_0 satisfies (a) or (b), then let F_h be the subgraph of G consisting of the boundary walks of h. If a face h of H_0 satisfies (c), then let F_h be the subgraph of $G[h]$ obtained by Lemma 3.5. Note that no face of H_0 satisfies (d), since G does not contain non-facial 4-cycles. Observe that every face of the graph $H = \bigcup_h F_h$ satisfies one of the conditions (a), (b), or (c) of Theorem 3.6. □

4 The structure theorem

In order to prove the structure theorem, we use a result that we derived in the second part of the series [8]. We need a few more definitions.

Let G be a graph embedded in the cylinder, with the boundary cycles $C_i = x_i y_i z_i w_i$ of length 4, for $i = 1, 2$. Let y'_i be either a new vertex or y_i, and let w'_i be either a new vertex or w_i. Let G' be obtained from G by adding 4-cycles $x_i y'_i z_i w'_i$ forming the boundary cycles. We say that G' is obtained by *framing on pairs $x_1 z_1$ and $x_2 z_2$.*

Let G be a graph embedded in the cylinder with boundary cycles C_1 and C_2 of length 3, such that every face of G has length 4. We say that such a graph G is a $3,3$-*quadrangulation*. Let G' be obtained from G by subdividing at most one edge in each of C_1 and C_2. We say that such a graph G' is a *near* $3,3$-*quadrangulation.*

If G is a graph embedded in a surface and C is the union of the boundary cycles of G, we say that G is *critical* if $G \neq C$ and for every $G' \subsetneq G$ such that $C \subseteq G'$, there exists a 3-coloring of C that extends to a 3-coloring of G', but does not extend to a 3-coloring of G.

Theorem 4.1 (Dvořák and Lidický [8]). *There exists a constant $D \geq 0$ such that the following holds. Let G be a triangle-free graph embedded in the cylinder with boundary cycles C_1 and C_2 of length 4. If the distance between C_1 and C_2 is at least D and G is critical, then either*

- *G is obtained from a patched Thomas-Walls graph by framing on its interface pairs, or*
- *G is a near $3,3$-quadrangulation.*

Finally, we are ready to combine all the results.

Proof of Theorem 1.5. Let D be the constant of Theorem 4.1. Let H_0 be the subgraph of G obtained by Theorem 1.4. If a face h of H_0 satisfies (a) or (b), then let F_h be the subgraph of G consisting of the boundary walks of h. If a face h of H_0 satisfies (c), then let F_h be the subgraph of $G[h]$ obtained by Lemma 3.5.

Suppose that h is a face of H_0 satisfying (d). Let C_1 and C_2 be the boundary cycles of $G[h]$. If the distance between C_1 and C_2 is less than D, then cut the cylinder in that $G[h]$ is embedded along the shortest path P between C_1 and C_2, let F'_h be the graph obtained by applying Theorem 3.6 to the resulting graph embedded in the disk, and let F_h be obtained from F'_h by gluing back the two paths created by cutting.

Assume now that the distance between C_1 and C_2 in $G[h]$ is at least D. If every 3-coloring of $C_1 \cup C_2$ extends to a 3-coloring of $G[h]$, then let $F_h = C_1 \cup C_2$. Otherwise, let F' be a maximal critical subgraph of $G[h]$, and apply Theorem 4.1 to F'.

- If F' is obtained from a patched Thomas-Walls graph by framing on its interface pairs, then note that all faces of G' have length at most 5, and thus by the assumptions of Theorem 1.5, we have $F' = G[h]$. Let F_h consist of the union of the 4-cycles containing an interface pair of F' (hence, $|V(F_h)| \leq 12$).
- If F' is a near $3,3$-quadrangulation, then similarly to the previous case, we have $F' = G[h]$. Let C'_1 and C'_2 be the 5-cycles in F' such that for $i = 1,2$, $C_i \cap C'_i$ is a path of length two and the symmetric difference of C_i and C'_i bounds a 5-face. Let F_h consist of $C_1 \cup C'_1 \cup C_2 \cup C'_2$ and the subgraph of the quadrangulation between C'_1 and C'_2 obtained by Lemma 3.5.

Observe that every face of the graph $H = \bigcup_h F_h$ satisfies one of the conditions (a), (b), (c) or (d) of Theorem 1.5. □

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