A counterexample to a conjecture on facial unique-maximal colorings

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Abstract

A facial unique-maximum coloring of a plane graph is a proper vertex coloring by natural numbers where on each face α the maximal color appears exactly once on the vertices of α. Fabrici and Göring [4] proved that six colors are enough for any plane graph and conjectured that four colors suffice. This conjecture is a strengthening of the Four Color theorem. Wendland [6] later decreased the upper bound from six to five. In this note, we disprove the conjecture by giving an infinite family of counterexamples. Thus we conclude that facial unique-maximum chromatic number of the sphere is five.

Keywords: facial unique-maximum coloring, plane graph.

1 Introduction

We call a graph *planar* if it can be embedded in the plane without crossing edges and we call it *plane* if it is already embedded in this way. A *coloring* of a graph is an assignment of colors to vertices. A coloring is *proper* if adjacent vertices receive distinct colors. A proper coloring of a graph embedded on some surface, where colors are natural numbers and every face has a unique vertex colored with a maximal color, is called a *facial unique-maximum coloring*, or *FUM-coloring* for short. The minimum k such that a graph G has a FUM-coloring using the colors $\{1, 2, \ldots, k\}$ is called the *facial unique-maximum chromatic number* of G and is denoted $\chi_{\text{fum}}(G)$.

The cornerstone of graph colorings is the Four Color Theorem stating that every planar graph can be properly colored using at most four colors [2]. Fabrici and Göring [4] proposed the following strengthening of the Four Color Theorem.

Conjecture 1 (Fabrici and Göring)**.** *If G is a plane graph, then* $\chi_{\text{fum}}(G) \leq 4$.

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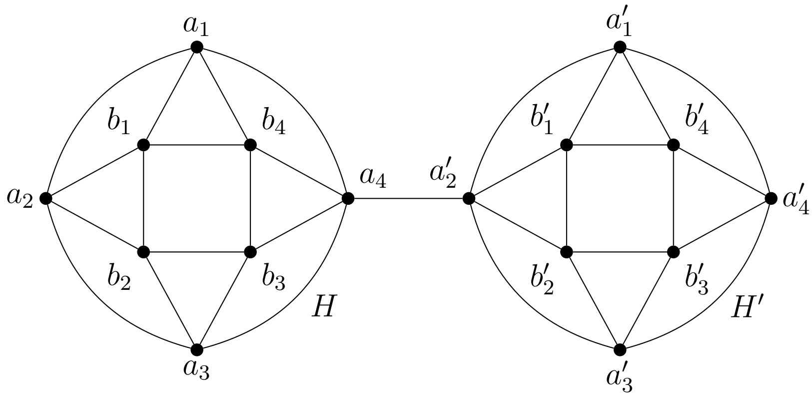

Figure 1: A counterexample to Conjecture 1.

When stating the conjecture, Fabrici and Göring [4] proved that $\chi_{\text{fum}}(G) \leq 6$ for every plane graph G. Promptly, this coloring was considered by others. Wendland [6] decreased the upper bound to 5 for all plane graphs. Andova, Lidický, Lužar, and Škrekovski [1] showed that 4 colors suffice for outerplanar graphs and for subcubic plane graphs. Wendland [6] also considered the list coloring version of the problem, where he was able to prove the upper bound 7 and conjectured that lists of size 5 are sufficient. Edge version of the problem was considered by Fabrici, Jendrol', and Vrbjarová [5]. For more results on facially constrained colorings, see a recent survey written by Czap and Jendrol' [3].

In this note we disprove Conjecture 1.

Proposition 1. *There exists a plane graph G with $\chi_{\text{fum}}(G) > 4$.*

Proof. Let G be the graph depicted in Figure 1. It consists of the induced graph H on the vertex set $\{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4\}$, H' (an isomorphic copy of H), and the edge $a_4a'_2$ connecting them. Suppose for contradiction that G has a FUM-coloring with the colors in $\{1, 2, 3, 4\}$. The color 4 is assigned to at most one vertex in the outer face of G, so by symmetry we may assume that a_1, a_2, a_3, and a_4 have colors in $\{1, 2, 3\}$. Next we proceed only with H to obtain the contradiction.

By symmetry, assume b_4 is the unique vertex in H that (possibly) has color 4. Without loss of generality, we assume a_1, b_1, and a_2 are colored by x, y, and z, respectively, where $\{x, y, z\} = \{1, 2, 3\}$. This forces b_2 to be colored with x, a_3 to be colored with y, and b_3 to be colored with z. Since a_4 is adjacent to vertices with colors x, y, and z, it must have color 4, a contradiction. □

The contradiction in Proposition 1 is produced from the property of H that every coloring of H by colors $\{1, 2, 3, 4\}$, where every interior face has a unique-maximum color, has a vertex in the outer face colored by 4. We can generalize the counterexample in Figure 1 by constructing an infinite family of graphs $\mathcal{H} = \{H_k\}_{k \geq 1}$ that can take the place of H. We construct a graph H_k on $6k + 2$ vertices by first embedding the cycle $b_1b_2 \cdots b_{3k+1}$ inside the cycle $a_1a_2 \cdots a_{3k+1}$. For $1 \leq i \leq 3k$, add edges a_ib_i and b_ia_{i+1}, then add the edges $a_{3k+1}b_{3k+1}$ and $b_{3k+1}a_1$. By this definition, the graph H is equivalent to H_1. See Figure 2(a) for an example of a generalization of the counterexample.

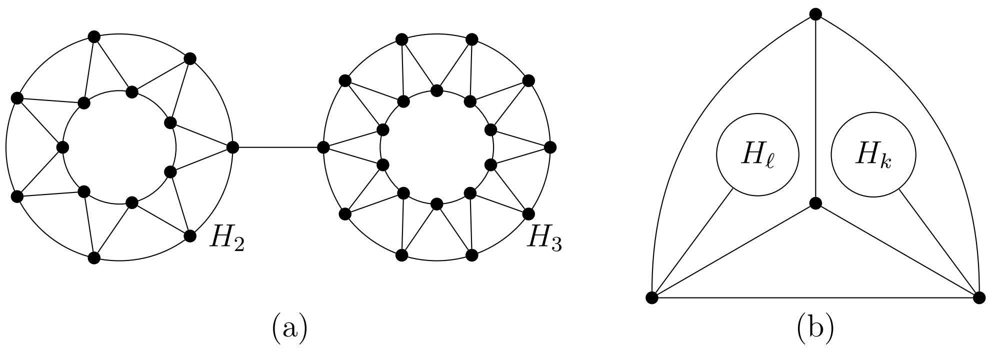

Figure 2: More counterexamples to Conjecture 1.

It is possible to construct more diverse counterexamples by embedding copies of members of $\mathcal{H}$ inside the faces of any 4-chromatic graph G and adding an edge from each copy to some vertex on the face it belongs to. It suffices to embed the graphs from $\mathcal{H}$ into a set of faces K such that in every 4-coloring of G, there is at least one face in K incident with a vertex of G colored by 4. An example of this with G being K_4 is given in Figure 2(b).

We now introduce a variation of Conjecture 1 with maximum degree and connectivity conditions added.

Conjecture 2. *If G is a connected plane graph with maximum degree 4, then $\chi_{\text{fum}}(G) \leq 4$.*

Notice that we constructed a counterexample of maximum degree five. Moreover, removing the edge a_4a_2' from the graph in Figure 1 gives a disconnected graph with maximum degree 4 that does not have a FUM-coloring with colors in $\{1, 2, 3, 4\}$. Recall that Andova et al. [1] showed that maximum degree 3 suffices.

For a surface Σ, we define the facial unique-maximum chromatic number of Σ,

$$\chi_{\text{fum}}(\Sigma) = \max_{G \hookrightarrow \Sigma} \chi_{\text{fum}}(G),$$

as the maximum of $\chi_{\text{fum}}(G)$ over all graphs G embedded into Σ. Our construction and the result of Wendland [6] implies that $\chi_{\text{fum}}(S_0) = 5$, where S_0 is the sphere. Our result motivates to study this invariant for graphs on other surfaces. It would be interesting to have a similar characterization to Heawood number for other surfaces of higher genus.

Problem 1. *Determine $\chi_{\text{fum}}(\Sigma)$ for surfaces Σ of higher genus.*

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