Hadamard diagonalizable graphs of order at most 36

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Abstract

If the Laplacian matrix of a graph has a full set of orthogonal eigenvectors with entries of the form ± 1, then the matrix formed by taking the columns as the eigenvectors form a Hadamard matrix and the graph is said to be Hadamard diagonalizable.

We determine all graphs which are Hadamard diagonalizable up through 36 vertices. This is done both via an efficient computation given a small Hadamard matrix combined with showing that if $n = 8k + 4$ then the only Hadamard diagonalizable graphs are K_n, $K_{n/2,n/2}$, $2K_{n/2}$, and nK_1.

1 Introduction

A Hadamard matrix is an $n \times n$ matrix H with entries in ± 1 with the property that $H^T H = nI$, or in other words the columns of H are orthogonal. These matrices have been extensively studied and it is known that a necessary condition for the existence of such a matrix is that $n = 1, 2$ or is a multiple of 4. A well-known and still open problem concerns the question of whether this is sufficient.

Conjecture 1.1. *Hadamard matrices exist for all order of the form $n = 4k$.*

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We will be interested in graphs which are *Hadamard diagonalizable* with respect to the Laplacian matrix of the graph. The Laplacian matrix is defined entrywise by

$$L_{uv} = \begin{cases} \deg(u) & \text{if } u = v, \\ -1 & \text{if } u \text{ adjacent to } v, \\ 0 & \text{otherwise.} \end{cases}$$

This corresponds to graphs for which there exists a full set of ± 1 orthogonal eigenvectors, e.g. there exists a collection of n eigenvectors which correspond with the columns of a Hadamard matrix. So if we let H denote the corresponding Hadamard matrix we have $\frac{1}{n}H^T LH = \Lambda$ where Λ is the diagonal matrix of the eigenvalues (e.g. there is a Hadamard matrix which diagonalizes the Laplacian).

Most graphs are not Hadamard diagonalizable. For example they must have order $n = 1$, 2 or $4k$ (so that the eigenvectors can make a Hadamard matrix); but this is not sufficient.

Proposition 1.2 ([1, 3])**.** *If G is Hadamard diagonalizable then the graph must be regular, and moreover all eigenvalues must be even integers.*

Proof that the graph is regular. The degrees of the graph correspond to the diagonal entries of the Laplacian matrix. Now let h_k denote the k-th column of H and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of L (the diagonal entries of Λ), then we have

$$L = \frac{1}{n} H \Lambda H^T = \frac{1}{n} \sum_{k=1}^{n} \lambda_k h_k h_k^T.$$

On the right hand side is a sum of matrices which have constant diagonal and so L has constant diagonal, showing the graph is regular. $\square$

We will also make use of the following; this follows immediately by noting that the eigenspaces for a graph and its complement are the same for the Laplacian matrix (the difference being the eigenvalues).

Proposition 1.3 ([1])**.** *A graph G is Hadamard diagonalizable if and only if G^c (the complement of G) is Hadamard diagonalizable.*

Previous research into Hadamard diagonalizable graphs has characterized Hadamard diagonalizable graphs up through order $n = 12$ [1] as well as all Hadamard diagonalizable graphs for the Sylvester construction for Hadamard matrices of order 2^k [3]. The goal of this current paper is to find all Hadamard diagonalizable graphs up through order $n = 36$. This will be done by using theory to show what is possible for graphs of order $n = 8k + 4$ (see Section 2); and then developing computational tools to search for all possible Hadamard diagonalizable graphs of small order (see Section 3). Information about the Hadamard diagonalizable graphs are given in Section 4. Concluding comments will be given in Section 5.

In Table 1 we summarize the number of Hadamard diagonalizable graphs as well as the number of inequivalent Hadamard matrices of the indicated order (two Hadamard matrices are equivalent if you can get from one to the other by some combination of the following operations: permuting rows, permuting columns, negating some subset of rows, negating some subset of columns).

Table 1: The order, number of inequivalent Hadamard matrices (H. matrices), and the number of Hadamard diagonalizable graphs (H. graphs)

Order	H. matrices	H. graphs
4	1	4
8	1	10
12	1	4
16	5	50
20	3	4
24	60	26
28	487	4
32	$13,710,027$	$10,196$
36	(unknown)	4

2 Hadamard diagonalizable graphs of order $n = 8k + 4$

We want to show that for order $n = 8k + 4$ there are at most four possible graphs which are Hadamard diagonalizable. We start with the following graph characterization property.

Lemma 2.1. *Suppose G is a connected graph on n vertices. Then G is a complete graph or a complete bipartite graph if and only if the following holds: for any four distinct vertices with u adjacent to v, v adjacent to w, w adjacent to x, it must be that x is adjacent to u.*

Proof. Assume that G satisfies the desired condition for all distinct vertices $u, v, w, x \in V(G)$. Suppose $u_1 \cdots u_k$ are the vertices in a cycle in G. If $k \geq 5$, then by assumption, $u_1u_2u_3u_4$ is a shorter cycle in G. So we have that the girth of G is either 3 or 4 or G is acyclic. If G is acyclic, then G contains no path on 3 edges, so G is a star making it a complete bipartite graph.

If the girth of G is 3, let U be a maximal clique. Then $|U| \geq 3$. Suppose $uv \in E(G)$ such that $u \in U$ and $v \in V(G) \setminus U$. By assumption, there exist two distinct vertices $x, y \in U$ such that $x, y \neq u$. Then $xyuv$ and $yxuv$ are paths of length 3, hence $vx, vy \in E(G)$. In particular, all vertices in U are adjacent to v, but this contradicts that U is a maximal clique. So it must be the case that no other vertices in G are connected to a vertex in U; and since G is connected we can conclude that G is a complete graph.

If the girth of G is 4, let U be a maximal induced complete bipartite subgraph of G with bipartition $U = U_1 \cup U_2$ such that $|U_1|, |U_2| \geq 2$. Suppose $uv \in E(G)$ where $u \in U$ and $v \in V(G) \setminus U$. Then without loss of generality, $u \in U_1$ and there exists a vertex $x \in U_2$. Then for every $w \in U_1$, $vuxw$ is a path of length 3 in G, hence $vw \in E(G)$. Since G is triangle-free, $vy \notin E(G)$ for all $y \in U_2$, but this contradicts that U is a maximal induced complete bipartite subgraph. So it must be the case that no other vertices in G are connected to a vertex in U; and since G is connected we can conclude that G is a complete bipartite graph.

The reverse implication holds by inspection. $\square$

We can now use this characterization of graphs to establish the possible Hadamard diagonalizable graphs of order $n = 8k + 4$.

Theorem 2.2. *Suppose H is an $n \times n$ Hadamard matrix with the first column consisting of all 1s. If $n = 8k+4$ and G is Hadamard diagonalizable, then $G \in \{K_n, K_{n/2,n/2}, nK_1, 2K_{n/2}\}$.*

Proof. Suppose for sake of contradiction that $n = 8k + 4$, G is Hadamard diagonalizable, and $G \notin \{K_n, K_{n/2,n/2}, nK_1, 2K_{n/2}\}$. By assumption, there exists a diagonal matrix Λ of eigenvalues with each eigenvalue even (see Proposition 1.2) and

$$L = \frac{1}{n} H \Lambda H^T = \frac{1}{n} \sum_{k=1}^{n} \lambda_k h_k h_k^T.$$

By Lemma 2.1, we have that G or G^c contains a path of length 3 whose endpoints are not adjacent (note that because G must be regular then we have that the only possible connected complete bipartite graph is $K_{n/2,n/2}$). Without loss of generality, we assume $uvwx$ is a path of length 3 in G. For any $i, j \in [n]$,

$$L_{ij} = \frac{1}{n} \sum_{k=1}^{n} \lambda_k (h_k)_i (h_k)_j.$$

Since $L_{uv} = L_{vw} = L_{wx} = -1$ and $L_{ux} = 0$, we have (rearranging)

$$\begin{aligned}
-3n &= n(L_{uv} + L_{vw} + L_{wx} + L_{ux}) \\
&= \sum_{k=1}^{n} \lambda_k ((h_k)_u (h_k)_v + (h_k)_v (h_k)_w + (h_k)_w (h_k)_x + (h_k)_u (h_k)_x) \\
&= \sum_{k=1}^{n} \lambda_k ((h_k)_u + (h_k)_w)((h_k)_v + (h_k)_x).
\end{aligned}$$

Since each λ_k is even and each $h_{ij} \in \{-1, 1\}$, it follows that each term in the sum is divisible by 8, meaning that 8 divides the right hand side. This impliese that n is a multiple of 8. But that contradicts the assumption that $n = 8k + 4$, concluding the proof. □

The preceding result shows that if a graph is Hadamard diagonalizable of order $n = 8k+4$ it must be one of the graphs mentioned. We now must argue that all four of these graphs are realizable.

Proposition 2.3. *If n is even and there exists a Hadamard matrix of order n, then the graphs $K_n, K_{n/2,n/2}, nK_1$, and $2K_{n/2}$ are Hadamard diagonalizable.*

Proof. Given that there exists a Hadamard matrix of order n we may assume that there is a Hadamard matrix H where h_1 is the all 1s vector and h_2 is 1 in entries $1, \ldots, n/2$ and -1 in entries $(n/2 + 1), \ldots, n$. It suffices to show how to write L as a linear combination of the projection matrices $h_k h_k^T$ for the graphs K_n and $2K_{n/2}$ (since this will have the Laplacian with the correct eigenvalues).

For $G = K_n$ we have

$$L = \sum_{k=1}^{n} h_k h_k^T - h_1 h_1^T,$$

since the sum becomes nI and the last term is $-J$.

For $G = 2K_{n/2}$ we have

$$L = \frac{1}{2}\sum_{k=1}^{n} h_k h_k^T - \frac{1}{2}h_1 h_1^T - \frac{1}{2}h_2 h_2^T,$$

since the sum becomes $\frac{n}{2}I$ and the last two terms combine to give $-\left(\begin{smallmatrix} J & O \\ O & J \end{smallmatrix}\right)$. $\square$

3 A procedure for finding all graphs diagonalizable by a given Hadamard matrix

In this section we will assume that we are working with Hadamard matrices where the first column and first row has all entries equal to 1. In particular, we have that our Hadamard matrices H will have the form

$$H = \left[\begin{array}{c|c} 1 & 1\cdots 1 \\ \hline 1 & \\ \vdots & \widehat{H} \\ 1 & \end{array}\right]$$

with $\widehat{H}$ a ± 1 matrix. Every Hadamard matrix is equivalent to a matrix of this form by negating combinations of rows and columns.

We will think of the columns of H as the eigenvectors of L and the first column of H, the all 1s vector, will correspond with eigenvalue 0. We will let Λ be the diagonal matrix with diagonal entries $(\lambda_1 = 0, \lambda_2, \ldots, \lambda_n)$.

Proposition 3.1. *The entries $L_{12}, \ldots, L_{1n}$ uniquely determine $\lambda_2, \ldots, \lambda_n$.*

Proof. Since we have

$$L = \frac{1}{n}H\Lambda H^T = \frac{1}{n}\sum_{k=1}^{n}\lambda_k h_k h_k^T, \quad \text{then} \quad L_{1j} = \frac{1}{n}\sum_{k=2}^{n}\lambda_k (h_k)_j.$$

Writing this in matrix form we have

$$\frac{1}{n}\begin{bmatrix} (h_2)_2 & (h_3)_2 & \cdots & (h_n)_2 \\ (h_2)_3 & (h_3)_3 & \cdots & (h_n)_3 \\ \vdots & \vdots & \ddots & \vdots \\ (h_2)_n & (h_3)_n & \cdots & (h_n)_n \end{bmatrix}\begin{bmatrix} \lambda_2 \\ \lambda_3 \\ \vdots \\ \lambda_n \end{bmatrix} = \frac{1}{n}\widehat{H}\begin{bmatrix} \lambda_2 \\ \lambda_3 \\ \vdots \\ \lambda_n \end{bmatrix} = \begin{bmatrix} L_{12} \\ L_{13} \\ \vdots \\ L_{1n} \end{bmatrix}.$$

The result now follows by noting $\left(\frac{1}{n}\widehat{H}\right)^{-1} = \widehat{H}^T - J$, showing that we can solve for the λ_i in terms of the off-diagonal entries in the first row. To see this we look at the rows of $\widehat{H}$ note that if we append 1s to the front we have rows of H then any two distinct rows in H are perpendicular. From this we can conclude that the dot product of two distinct rows in $\widehat{H}$ must be -1 (i.e. to compensate for the 1 appended to the front); the dot product of a row in $\widehat{H}$ with the all 1s vector must similarly be -1; finally, the dot product of a row with itself will be $n-1$.

Multiplying $(\frac{1}{n}\widehat{H})(\widehat{H}^T - J)$ is equivalent to looking at dot products of rows in $\frac{1}{n}\widehat{H}$ and rows in $\widehat{H} - J$. If the rows are the same, the result will be $\frac{1}{n}((n-1)-(-1)) = 1$; and if the rows are distinct the result will be $\frac{1}{n}((-1)-(-1)) = 0$. In particular, the result is the identity matrix, establishing the inverse. □

The preceding can be used to give a new proof that all Laplacian eigenvalues of a Hadamard diagonalizable graph are even integers

Proof that the eigenvalues are even integers. We have

$$\begin{bmatrix} \lambda_2 \\ \lambda_3 \\ \vdots \\ \lambda_n \end{bmatrix} = (\widehat{H}^T - J) \begin{bmatrix} L_{12} \\ L_{13} \\ \vdots \\ L_{1n} \end{bmatrix}.$$

Since the entries in $\widehat{H}^T - J$ are in $\{0, -2\}$ while the entries in the entries in $L_{12}, \ldots, L_{1n}$ are in $\{0, -1\}$, the result of multiplying will be a vector of eigenvalues which are even. □

Using Proposition 3.1, given our Hadamard matrix we can narrow our search space down to size 2^{n-1} by looking at all possible $\{-1, 0\}$ assignments to $L_{1,2}, \ldots, L_{1,n}$. Not every assignment will correspond to a graph as there might be other entries $L_{i,j} \notin \{-1, 0\}$; it is also possible to construct the same graph multiple ways (e.g. the same up to relabeling).

To further speed up the search we start by rewriting all of the off-diagonal entries of L as linear combinations of $L_{12}, \ldots, L_{1n}$. (This can be done since each entry is some linear combination of the $\lambda_2, \ldots, \lambda_n$ and then the proof of Proposition 3.1 show that each of the λ_i is a linear combination of $L_{12}, \ldots, L_{1n}$.)

To illustrate this we carry this procedure out for the Hadamard matrix `had.16.1` from Sloane [5] to produce an auxiliary matrix. For the 120 entries above the diagonal (by symmetry the entries below the diagonal will be equal) there were 27 different linear combinations. The auxiliary matrix is given in Table 2 where each row represents a linear combination and the columns correspond to the entries in the first row. For notational convenience we have labeled the rows and columns using hexadecimal and we have indicated which entries correspond to which linear combinations.

Looking at the auxiliary matrix in Table 2, the identity matrix of the first 15 rows is a reflection that the linear combination to produce an entry from the first row is trivial.

Let us view an assignment of $L_{12}, \ldots, L_{1n}$ as selecting some subset of the columns (so if the corresponding entry is -1 take the column; if it is 0 do not take the column). Then this will produce a Laplacian matrix for a graph *if and only if* the sum of the columns produce a 0-1 vector. Since this is the only case when the off-diagonal entries will be 0 and -1.

So to reduce the search space we explore all subsets of columns by deciding in a forest-like exploration of the space where at each stage we decide to either add or not add a particular column. After we add a particular column we then do a check if for each entry there is a possibility that some combination of the remaining columns can result in the value to be 0 or 1; if not then we prune the tree and don't explore any further on that branch.

Table 2: The auxiliary matrix for the Hadamard matrix `had.16.1` where the rows correspond with linear combinations in terms of the off-diagonal entries in the first row. At the bottom for each row we indicate which entries L_{ij} with i, j in hexadecimal to which the linear combination corresponds.

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\
0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Row	Entries	Row	Entries	Row	Entries
1	$01, 23, 45, 67, 89, AB, CD, EF$	10	$0A, 1B, 28, 39$	19	$27, 36, AF, BE$
2	$02, 13, 8A, 9B$	11	$0B, 1A, 29, 38$	20	$2C, 3D, 4A, 5B$
3	$03, 12, 8B, 9A$	12	$0C, 1D, 48, 59$	21	$2D, 3C, 4B, 5A$
4	$04, 15, 8C, 9D$	13	$0D, 1C, 49, 58$	22	$2E, 3F, 6A, 7B$
5	$05, 14, 8D, 9C$	14	$0E, 1F, 68, 79$	23	$2F, 3E, 6B, 7A$
6	$06, 17, 8E, 9F$	15	$0F, 1E, 69, 78$	24	$46, 57, CE, DF$
7	$07, 16, 8F, 9E$	16	$24, 35, AC, BD$	25	$47, 56, CF, DE$
8	$08, 19, 2A, 3B, 4C, 5D, 6E, 7F$	17	$25, 34, AD, BC$	26	$4E, 5F, 6C, 7D$
9	$09, 18, 2B, 3A, 4D, 5C, 6F, 7E$	18	$26, 37, AE, BF$	27	$4F, 5E, 6D, 7C$

For example, if we take the columns 2, 3, 11 then the last entry will be 1/2 and the remaining available entries in the remaining columns are 0. So no matter which combination of columns $12, 13, 14, 15$ we take we can never change that value from 1/2 and so there is no need to explore that part of the space. To get the most out of this it is useful to first presort the columns so that such conflicts will arise early.

If you get down to a leaf in the tree and the resulting combination of columns is a 0-1 vector, then we have found a Hadamard diagonalizable graph. To find the graph we find where the 1s are located and the corresponding entries to which they correspond. These corresponding entries match with the edges in the graph. As an example if we take the sum of the first three columns in Table 2 then this will produce a 1 in the rows of the resulting vector $1, 2, 3, 24, 25$. So this will be the graph on the vertex set with vertices $\{0, 1, \ldots, F\}$ and with edges

$$\underbrace{01, 23, 45, 67, 89, AB, CD, EF}_{\text{row 1}}, \underbrace{02, 13, 8A, 9B}_{\text{row 2}}, \underbrace{03, 12, 8B, 9A}_{\text{row 3}}, \underbrace{46, 57, CE, DF}_{\text{row 24}}, \underbrace{47, 56, CF, DE}_{\text{row 25}}$$

which becomes the graph $4K_4$ (cliques on the vertices $0, 1, 2, 3$; and $4, 5, 6, 7$; and $8, 9, A, B$; and C, D, E, F). As graphs are found they are then tested to see whether they have been seen before and we only keep those graphs which have not been seen before; this can be done, for example, by using canonical labeling methods.

The procedure outlined here was implemented in both `Python/SageMath` and `C++` with all computations done using integer computations. The only external call needed is to determine which graphs are discovered up to isomorphism. The program can be downloaded at `http://lidicky.name/pub/hadamard/`.

3.1 Equivalency of Hadamard matrices

We assumed that our Hadamard matrices have the form where the first row and column consists of all 1s. Given that for the Laplacian matrix one of the eigenvalues will be 0 with the all 1s eigenvalue this is a reasonable assumption to start with.

Note that a given Hadamard matrix is equivalent to possibly many such matrices where the first row and column consists of all 1s. To see this we can start with any Hadamard matrix and pick any column to be moved to the front. We can then multiply the rows by -1 as needed to make the first column consist of all 1s; and finally we multiply the columns by -1 as needed to make the first row consist of all 1s.

This raises the possibility that for any Hadamard matrix of order n we have potentially up to n different equivalent forms that need to be checked. Our calculations support the following conjecture.

Conjecture 3.2. *If H_1 and H_2 are equivalent Hadamard matrices which consist of 1 in the first row and first column, then G is Hadamard diagonalizable by H_1 if and only if G (up to some relabeling) is Hadamard diagonalizable by H_2.*

If true, this conjecture would significantly shorten the computational time.

4 Hadamard diagonalizable graphs of small order

For orders $4, 12, 20, 28, 36, \ldots$ it follows from Section 2 that as long as Hadamard matrices exist there are exactly four Hadamard diagonalizable graphs, namely $K_n, K_{n/2,n/2}, 2K_{n/2}, nK_1$.

For order 8 there is a unique Hadamard matrix and it is the Sylvester construction. So the Hadamard diagonalizable graphs have been determined [3] and consist of all Cayley graphs for $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

4.1 Order 16

As mentioned, there are 5 inequivalent Hadamard matrices of order 16, denote them by `had.16.`j for $j = 0, 1, 2, 3, 4$ [5], with `had.16.0` being the Sylvester matrix. Now denote the set of graphs diagonalizable by had.16.j by S_j. Then

- $|S_0| = 46$, $|S_1| = 50$, $|S_2| = 48$, $|S_3| = 10$, $|S_4| = 24$,
- $S_3 \subset S_4 \subset S_0$, $S_2 \subset S_1$, $S_0 \subset S_1$, and $|S_0 \backslash S_2| = 2$ (therefore $|S_2 \backslash S_0| = 4$, and $S_0 \cup S_2 = S_1$),
- S_0 consists of all the non-isomorphic cubelike graphs on 16 vertices (Cayley graphs in $\mathbb{Z}_2^4$).

In the following, $H_{n,n}$ denotes the crown graph: the graph obtained from $K_{n,n}$ by removing a perfect matching, $K_r(t)$ denotes the complete r-partite graph with t vertices in each part.

To describe the special graphs in the differences, we make use of one special operation on graphs of the same order:
Assume G_1 and G_2 are two graphs with the same vertex set. Then the graph with adjacency matrix $\begin{bmatrix} A(G_1) & A(G_2) \\ A(G_2) & A(G_1) \end{bmatrix}$ is denoted by $G_1 \ltimes G_2$. If G_1 and G_2 have disjoint edge sets, then $G_1 \ltimes G_2$ is a *double cover* of the graph whose adjacency matrix is $A(G_1) + A(G_2)$ [2]. For some visualization of this operation, see [3]: take $2G_1$, for each of its vertex v, connect v to all its neighbors' images in the other copy. The operation depends on the ordering of vertices of G_1 and G_2, so when using it to describe a graph, we also give the labelling of G_1 and G_2.

First we list the exact 10 graphs diagonalizable by all the 5 inequivalent Hadamard matrices, see Table 3, and the 24 graphs diagonalizable by all of them but `had.16.3`, see Table 4.

Table 3: Graphs diagonalizable by `had.16.3`

	G	G^c	ID
G_1	K_{16}^c	K_{16}	$(1)-(10)$
G_2	$K_{8,8}$	$2K_8$	$(2)-(8)$
G_3	$2K_{4,4}$	$(2K_4) \vee (2K_4)$	$(3)-(9)$
G_4	$4K_4$	$K_4 \wr K_4^c \cong K_4(4)$	$(4)-(6)$
G_5	$(K_2 \square K_4) \wr (K_2^c)$	$H_{4,4} \wr K_2$	$(5)-(7)$

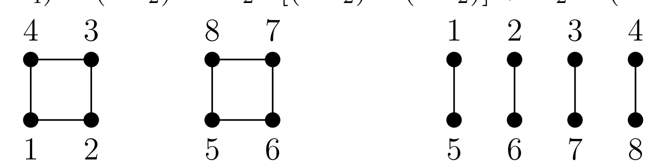
Table 4: Graphs diagonalizable by `had.16.4`

	G	G^c	ID
G_1	K_{16}^c	K_{16}	$(1)-(24)$
G_2	$8K_2$	$(8K_2)^c$	$(2)-(23)$
G_3	$K_{8,8}$	$2K_8$	$(3)-(18)$
G_4	$4C_4$	$K_4 \wr (2K_2)$	$(4)-(22)$
G_5	$2K_{4,4}$	$(2K_4) \vee (2K_4)$	$(5)-(21)$
G_6	$4K_4$	$K_4 \wr K_4^c \cong K_4(4)$	$(6)-(11)$
G_7	$H_{8,8}+1F$	$K_2\Box(4K_2)^c$	$(7)-(16)$
G_8	$(4K_2) \vee (4K_2)$	$2(4K_2)^c$	$(8)-(14)$
G_9	$(K_2\Box K_4) \wr (K_2^c)$	$H_{4,4} \wr K_2$	$(9)-(17)$
G_{10}	$(2C_4) \vee (2C_4)$	$2[(2K_2) \vee (2K_2)]$	$(10)-(12)$
G_{11}	$(K_2\Box K_4) \wr (K_2)$	$H_{4,4} \wr K_2^c$	$(13)-(20)$
G_{12}	$K_2\Box K_8$	$H_{8,8}$	$(15)-(19)$

As mentioned at the beginning of this section, there is only a pair of cubelike graphs that are not diagonalizable by `had.16.2` (that is, they are diagonalizable by `had.16.0`, but not `had.16.2`), they are given in Table 5, where

$$SS_1 = \{(0001),(0010),(1001),(1010),(1011),(1101),(1110),(1111)\}$$
$$SS_2 = \{(0011),(0100),(0101),(0110),(0111),(1000),(1100)\}.$$

Table 5: The only pair of cubelike graphs that are not diagonalizable by `had.16.2`

	G	G^c	ID
G_1	$\mathbb{Z}_2^4(S_1) \cong$ $(2C_4) \ltimes (4K_2)^c$	$\mathbb{Z}_2^4(S_2) \cong$ $K_2\Box[(2K_2) \vee (2K_2)] + K_2 \times (4K_2)$	$(12)-(35)$
	 $2C_4$	$4K_2$	

There are two pairs of graphs that are diagonalizable by had.16.2 but not by `had.16.0`, that is, they are diagonalizable by `had.16.2` but are not cubelike graphs. They are still Cayley graphs, but on the group $\mathbb{Z}_4^2$. The four graphs are listed in Table 6. Note that G_2, the Shrikhande graph is a subgraph of G_1. The two graphs in Table 5 (cubelike graphs) and Table 6 (not cubelike graphs), together with the other 44 cubelike graphs, gives all the 50 graphs on 16 vertices that are Hadamard diagonalizable.

Table 6: Graphs diagonalizable by had.16.3

	G	ID
G_1	$\mathbb{Z}_4^2(\{\pm(0,1),(0,2),\pm(1,0),\pm(1,1)\})$	(37)
G_1^c	$\mathbb{Z}_4^2(\{\pm(1,-1),\pm(1,2),\pm(2,1),(2,0),(2,2)\})$	(15)
G_2	$\mathbb{Z}_4^2(\{\pm(0,1),\pm(1,0),\pm(1,1)\})$, called Shrikhande graph	(31)
G_2^c	$\mathbb{Z}_4^2(\{\pm(1,-1),\pm(1,2),\pm(2,1),(0,2),(2,0),(2,2)\})$	(26)

4.2 Order 24

Now we give all the 26 graphs on 24 vertices that are Hadamard diagonalizable. Let Q_n denote the n-cube, $B_1 = \{\pm(0,1,0),(0,1,\pm1),(0,2,\pm1),(0,0,\pm1),\pm(1,1,0)\} \subseteq \mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_4$, $B_2 = \{(0,0,2),\pm(0,1,2),(1,0,0),\pm(1,0,1),(1,0,2),\pm(1,1,1),\pm(1,1,2),\pm(1,1,3)\} \subseteq \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$, $B_3 = \{(0,0,\pm1),(0,1,0),(0,1,\pm1),(0,1,\pm2),(0,1,3),(1,0,\pm1),(1,0,3)\} \subseteq \mathbb{Z}_2 \times \mathbb{Z}_2\times\mathbb{Z}_6$, $B_4 = \{(0,0,\pm2),(0,0,3),(1,0,\pm2),(1,1,0),(1,1,\pm1),(1,0,0),(1,1,\pm2),(1,1,0)\} \subseteq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$.

4.3 wreath/lexicographic product

The wreath product of two Hadamard diagonalizable graph is Hadamard diagonalizable.

Theorem 4.1. *Assume that the graph G_1 on m vertices is diagonalizable by a normalized Hadamard matrix H_1, the graph G_2 on n vertices is diagonalizable by a normalized Hadamard matrix H_2. Then the wreath product $G_1 \wr G_2$ of G_1 and G_2 is diagonalizable by the Hadamard matrix $H_1 \otimes H_2$*

Proof. Assume that H_1 diagonalizes $A(G_1)$ to Λ_1, and H_2 diagonalizes $A(G_2)$ to Λ_2. Since for any graph G, $L(G)$ is diagonalizable by a Hadamard matrix H if and only $A(G)$ is diagonalizable by H (Hadamard diagonalizable graphs are regular), we show that $A(G_1 \wr G_2) = A(G_1 \wr G_2) = I_m \otimes A(G_2) + A(G_1) \otimes J_n$ is diagonalizable by the normalized Hadamard matrix $H_1 \otimes H_2$ instead. Now for any normalized Hadamard matrix H of size n, $H^{-1}J_nH = ne_1e_1^T = nE_{1,1}$, we have

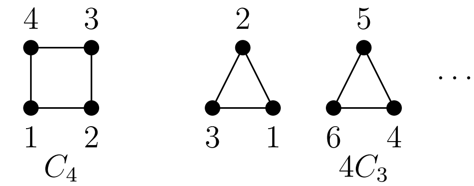
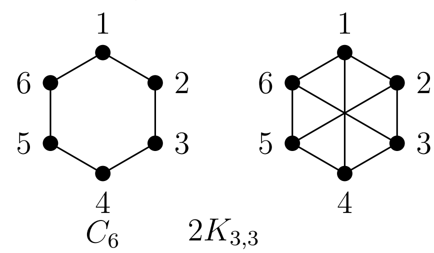
$$\begin{aligned}
&(H_1 \otimes H_2)^{-1}A(G_1 \wr G_2)H_1 \otimes H_2) \\
&= (H_1 \otimes H_2)^{-1}(I_m \otimes A(G_2) + A(G_1) \otimes I_n)H_1 \otimes H_2) \\
&= H_1^{-1}I_mH_1 \otimes H_2^{-1}A(G_2)H_2 + H_1^{-1}A(G_1)H_1 \otimes H_2^{-1}AJ_nH_2 \\
&= I_m \otimes \Lambda_2 + n\Lambda_1 \otimes E_{1,1},
\end{aligned}$$

which is a diagonal matrix. ∎

4.4 Order 32

The calculation for order 32 was performed by a program written in C++ by Lidický. The program uses `nauty` [4] for graph isomorphism testing. To speed up the calculation, it utilized `parallel` [6]. The calculation was performed on a server maintained by the Department of Applied Mathematics at Charles University in Prague. The source codes and outputs can be

Table 7: Graphs diagonalizable by `had.24`

	G	G^c	ID
G_1	K_{24}^c	K_{24}	$(1)-(26)$
G_2	$12K_2$	$(12K_2)^c$	$(2)-(25)$
G_3	$K_{12,12}$	$2K_{12}$	$(3)-(20)$
G_4	$2K_{6,6}$	$(2K_6)\vee(2K_6)$	$(4)-(23)$
G_5	$2(K_{12}-6K_2)$	$(6K_2)\vee(6K_2)$	$(5)-(15)$
G_6	$\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_4(B_1)\cong$ $(C_4\wr K_3)\odot(4C_3)$ 	$\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_4(B_2)\cong$ $(C_4\wr K_3^c)\odot(4C_3)^c+K_2\square(C_4\wr K_3^c)$	$(6)-(18)$
G_7	$(K_2\square K_4)\wr K_3^c$	$Q_3\wr K_3$	$(7)-(19)$
G_8	$K_2\square K_{6,6}$	$(2K_6)\vee(2K_6)-12K_2$	$(8)-(22)$
G_9	$(K_{6,6}-6K_2)\vee(K_{6,6}-6K_2)$	$2(K_2\square K_6)$	$(9)-(14)$
G_{10}	$K_4\wr K_6^c$	$4K_6$	$(10)-(11)$
G_{11}	$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_6(B_3)\cong$ $(C_6\vee C_6)\odot(2K_{3,3})$ 	$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_6(B_4)\cong$ $(2(K_2\square K_3))\odot K_4(3)$	$(12)-(16)$
G_{12}	$2(K_{6,6}-6K_2)$	$(K_2\square K_6)\vee(K_2\square K_6)$	$(13)-(24)$
G_{13}	$H_{12,12}$	$K_2\square K_{12}$	$(17)-(21)$

downloaded from `http://lidicky.name/pub/hadamard/`. We also provide an example how to load the Hadamard graphs to `SAGE` and explore their properties. The calculation took 179,736,390 seconds of CPU time, which was about 2 months of real time due to parallel processing. If Conjecture 3.2 was true, the calculation would take 2 days.

5 Conclusion

The obstacles with moving forward with larger Hadamard matrices consists both in terms of the size of the computations for any individual Hadamard matrix combined with a lack of the classification of all Hadamard matrices of order 36 or above.

We can run the computation on some known Hadamard matrices and we summarize the computation results in Table 8.

Table 8: Some Hadamard matrices from Sloane [5] and the number of graphs for which that matrix Hadamard diagonalizes the graph.

Hadamard matrix	Number of H. graphs
`had.40.tpal`	26
`had.40.ttoncheviv`	26
`had.40.twill`	26
`had.48.pal`	4
`had.56.tpal2`	26
`had.56.twll`	26

For the three Hadamard matrices of order 40 the 26 graphs are the same; similarly for the two Hadamard matrices of order 56. The data, combined with what we know for order 24 suggests the following.

Conjecture 5.1. *For $n = 24+16k$ there are exactly 26 distinct graphs which are Hadamard diagonalizable for some Hadamard matrix of order n.*

A proof of this might follow along the lines of that carried out for $n = 4+8k$; on the other hand it might be false and so to disprove it then computations should be run for additional Hadamard matrices of order 40 or 56 to find additional graphs.

The Hadamard matrix of order 48 from Sloane [5] has few Hadamard diagonalizable graphs. When we reran the computation using the Hadamard matrix generated by `SAGE` there were 762 distinct Hadamard diagonalizable graphs. Given the lack of classification for Hadamard matrices of order 48 it is not clear how to determine all Hadamard diagonalizable graphs of order 48.

One of the motivations for exploring Hadamard diagonalizable graphs is that these graphs correspond to graphs which exhibit perfect state transfer in quantum walks. A more thorough investigation of the Hadamard diagonalizable graphs of order 32 could yield further insight into the phenomenon on perfect state transfer.

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