Injective choosability of subcubic planar graphs with girth 6

Boris Brimkov [*] Jennifer Edmond [†] Robert Lazar [‡] Bernard Lidický [§]
Kacy Messerschmidt [¶] Shanise Walker [‖]

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Abstract

An injective coloring of a graph G is an assignment of colors to the vertices of G so that any two vertices with a common neighbor have distinct colors. A graph G is injectively k-choosable if for any list assignment L, where $|L(v)| \geq k$ for all $v \in V(G)$, G has an injective L-coloring. Injective colorings have applications in the theory of error-correcting codes and are closely related to other notions of colorability. In this paper, we show that subcubic planar graphs with girth at least 6 are injectively 5-choosable. This strengthens a result of Lužar, Škrekovski, and Tancer that subcubic planar graphs with girth at least 7 are injectively 5-colorable. Our result also improves several other results in particular cases.

1 Introduction

A *proper coloring* of a graph G is an assignment of colors to the vertices of G so that any neighboring vertices receive distinct colors. The *chromatic number* of G, $\chi(G)$, is the minimum number of colors needed for a proper coloring of G. An *injective coloring* of a graph G is an assignment of colors to the vertices of G so that any two vertices with a common neighbor receive distinct colors. The *injective chromatic number*, $\chi_i(G)$, is the minimum number of colors needed for an injective coloring of G. An injective coloring of G is not necessarily a proper coloring of G. Define the *neighboring graph* $G^{(2)}$ by $V(G^{(2)}) = V(G)$ and $E(G^{(2)}) = \{uv : u \text{ and } v \text{ have a common neighbor in } G\}$. Note that $\chi_i(G) = \chi(G^{(2)})$.

Injective colorings were first introduced by Hahn, Kratochvíl, Širáň, and Sotteau [13], where the authors showed injective colorings can be used in coding theory, by relating the injective chromatic number of the hypercube to the theory of error-correcting codes. The authors showed that for a graph G with maximum degree Δ, $\chi_i(G) \leq \Delta(\Delta-1)+1$. They also showed that computing the injective chromatic number is NP-complete and gave bounds and structural results for the injective chromatic numbers of graphs with special properties. It is easy to see that $\Delta(G) \leq \chi_i(G) \leq |V(G)|$.

[*]Department of Computational and Applied Mathematics, Rice University, USA (boris.brimkov@rice.edu).
[†]Mathematics Department, Syracuse University, USA (jledmond@syr.edu).
[‡]Department of Mathematics, Iowa State University, USA (rllazar@iastate.edu).
[§]Department of Mathematics, Iowa State University, USA (lidicky@iastate.edu).
[¶]Department of Mathematics, Iowa State University, USA (kacymess@iastate.edu).
[‖]Department of Mathematics, Iowa State University, USA (shanise1@iastate.edu).

For each $v \in V(G)$, let $L(v)$ be a set of colors assigned to v. Then $L = \{L(v)|v \in V(G)\}$ is a *list assignment* of G. Given a list assignment L of G, an injective coloring φ of G is called an *injective L-coloring* of G if $\varphi(v) \in L(v)$ for every $v \in V(G)$. A graph is injectively *k-choosable* if for any list assignment L, where $|L(v)| \geq k$ for all $v \in V(G)$, G has an injective L-coloring. The *injective choosability number* of G, denoted $\chi_i^\ell(G)$, is the minimum k needed such that G is injectively k-choosable. It is clear that $\chi_i(G) \leq \chi_i^\ell(G)$.

Graphs with low injective chromatic numbers have been studied extensively. A number of authors have studied the injective chromatic number of graphs G in relation to their maximum degree, $\Delta(G)$, or their maximum average degree, $\mathrm{mad(G)} = \max_{\mathrm{H} \subseteq \mathrm{G}}\{2|\mathrm{E(H)}|/|\mathrm{V(H)}|\}$, for instance [3, 4, 5, 16]. As $\mathrm{mad(G)} < \frac{2\mathrm{g(G)}}{\mathrm{g(G)}-2}$ for all planar graphs, we can compute a bound for the girth of G, $g(G)$, given mad(G). Table 1 consists of results for the injective chromatic number and the injective choosability number of graphs which depend on planarity, the maximum degree, the maximum average degree, and the girth.

For a planar graph G with girth at least 6 and any maximum degree Δ, the best known result about the injective chromatic number is $\chi_i(G) \leq \Delta + 3$ [11]. In this paper, we improve this result for the case $\Delta = 3$. Moreover, we improve the result of Lužar, Škrekovski, and Tancer [16] by decreasing the girth condition and by changing to injective list coloring. We also improve the other two highlighted bounds in Table 1 in special cases.

Theorem 1. *Every planar graph G with $\Delta(G) \leq 3$ and $g(G) \geq 6$ is injectively 5-choosable.*

This theorem is a step towards the conjecture of Chen, Hahn, Raspaud, and Wang [5] that all planar subcubic graphs are injectively 5-colorable. In order to prove Theorem 1 we prove a slightly stronger result in Theorem 2. Let G be a graph and let L be a list assignment. A *precolored path* in G is a path P_k on k vertices where $|L(v)| = 1$ for all $v \in V(P_k)$ and there is at most one vertex $v \in V(P_k)$ with a neighbor in $G - P_k$. Moreover, $\deg_{P_k}(v)$ is maximal among the other vertices in P_k and v has at most one neighbor in $G - P_k$. Vertices with lists of size one are called *precolored*. The set of all precolored vertices $\mathcal{P}$ in G is *proper* if the lists of precolored vertices give a proper coloring of $G^{(2)}$ when restricted to $\mathcal{P}$. That is, the precolored vertices do not conflict among themselves.

Theorem 2. *Let G be a plane graph with $\Delta(G) \leq 3$ and $g(G) \geq 6$. Let $\mathcal{P} \subseteq V(G)$. Let L be a list assignment of G such that $|L(v)| \geq 5$ for $v \in V(G) \setminus \mathcal{P}$ and $|L(v)| = 1$ for $v \in \mathcal{P}$. If the precolored vertices are proper, are all in the same face, form at most two precolored paths, each of which is on at most three vertices, then G is injectively L-colorable.*

2 Preliminaries

The following notation shall be used in the sequel. A *k-vertex* is a vertex of degree k. We denote the degree of a vertex v by $\deg(v)$. We denote the set of neighbors of v by $N(v)$ and $N(v) \cup \{v\}$ by $N[v]$. If we want to stress that the degree or neighborhood is in a particular graph G, we use subscript G, e.g. $\deg_G(v)$. A *cut edge* or *bridge* is an edge which, when removed, increases the number of components in G. Given $S \subset V$, the induced subgraph $G[S]$ is the subgraph of G whose vertex set is S and whose edge set consists of all edges of G which have both ends in S.

The length of a face f, denoted by $\ell(f)$, is the length of a closed walk around the boundary of f. This is the same as the number of edges incident to f plus the number of cut edges incident to f. A face of length ℓ is called an ℓ-face. A graph G is *planar* if it is possible to draw G in the

Bounds	Planar	$\Delta(G)$	mad(G)	$g(G)$	Authors
$\chi_i(G) \leq \Delta + 1$	Yes	≥ 18		≥ 6	Borodin and Ivanova [3]
$\chi_i(G) \leq \Delta + 3$	Yes			≥ 6	Dong and Lin [11]
$\chi_i(G) \leq \Delta + 3$	No		$< \frac{14}{5}$	$\geq 7^*$	Doyon, Hahn, and Raspaud [12]
$\chi_i(G) \leq \Delta + 4$	No		< 3	$\geq 6^*$	Doyon, Hahn, and Raspaud [12]
$\chi_i(G) \leq \Delta + 8$	No		$< \frac{10}{3}$	$\geq 5^*$	Doyon, Hahn, and Raspaud [12]
$\chi_i(G) \leq 5$	Yes	≤ 3		≥ 7	Lužar, Škrekovski, and Tancer [16]
$\chi_i^\ell(G) \leq \Delta + 1$	No		$< \frac{5}{2}$	$\geq 10^*$	Cranston, Kim and Yu [9]
$\chi_i^\ell(G) \leq \Delta + 1$	Yes	≥ 4		≥ 9	Cranston, Kim and Yu [9]
$\chi_i^\ell(G) = \Delta$	Yes	≥ 4		≥ 13	Cranston, Kim and Yu [9]
$\chi_i^\ell(G) = \Delta$	No		$< \frac{42}{19}$	$\geq 21^*$	Cranston, Kim and Yu [9]
$\chi_i^\ell(G) \leq 5$	No	≥ 3	$< \frac{36}{13}$	$\geq 8^*$	Cranston, Kim and Yu [10]
$\chi_i^\ell(G) \leq \Delta + 2$	No	≥ 4	$< \frac{14}{5}$	$\geq 7^*$	Cranston, Kim and Yu [10]
$\chi_i^\ell(G) \leq \Delta + 1$	Yes	≥ 24		≥ 6	Borodin and Ivanova [2]
$\chi_i^\ell(G) \leq \Delta + 2$	Yes	≥ 12		≥ 6	Li and Xu [14]
$\chi_i^\ell(G) \leq \Delta + 2$	Yes	≥ 8		≥ 6	Bu and Lu [6]
$\chi_i^\ell(G) \leq \Delta + 3$	Yes			≥ 6	Chen and Wu [7]
$\chi_i^\ell(G) \leq \Delta + 4$	Yes	≥ 30		≥ 5	Li and Xu [14]
$\chi_i^\ell(G) \leq \Delta + 5$	Yes	≥ 18		≥ 5	Li and Xu [14]
$\chi_i^\ell(G) \leq \Delta + 6$	Yes	≥ 14		≥ 5	Li and Xu [14]
$\chi_i^\ell(G) \leq 5$	Yes	≤ 3		≥ 6	This paper

Table 1: Known results on the injective chromatic number and injective list chromatic number. A 'Yes' in the 'Planar' column indicates that the result holds only for planar graphs, and a 'No' indicates that the result holds for both planar and non-planar graphs. A * in the '$g(G)$' column indicates that the girth was obtained using the bound $\mathrm{mad(G)} < \frac{2\mathrm{g(G)}}{\mathrm{g(G)}-2}$. Results in [9, 10] are state for injective coloring. However, the same proofs work also for injective list coloring.

plane without edge crossings; G is *plane* if it is drawn in the plane without edge crossings. The set of faces of a plane graph G will be denoted by $F(G)$. We say that a 2-vertex v is *nearby* f if v is adjacent to a vertex which is incident to f but v is not incident to f itself. The set of all 2-vertices incident to a face f will be denoted as $I(f)$:

$$I(f) = \{v \in V(G) : \deg(v) = 2, v \text{ is incident to } f\}.$$

The set of all 2-vertices nearby to a face f will be denoted as $N(f)$:

$$N(f) = \{v \in V(G) : \deg(v) = 2, v \text{ is nearby } f\}.$$

2.1 Overview of Method

In order to prove Theorem 2, we use the discharging method. We start with a minimum counterexample and assign initial charges to all vertices and faces of G. By Euler's formula, the sum of charges of all vertices and faces of G is negative. Next, we apply rules that move charges between faces and vertices while preserving the sum of the charges. By the minimality of G, certain subgraphs cannot appear in G. We call these subgraphs reducible configurations. By using the fact that G does not contain any *reducible configurations*, we show that the final charge of every face and every vertex of G is nonnegative, contradicting that the sum of all charges is negative. For further details and examples of the discharging method, see [8].

3 Proof of Theorem 2

In this section, we prove Theorem 2. Let G be a minimum counterexample. The minimality of G is defined by first minimizing the number of connected components of G, then, subject to that, minimizing the number of non-precolored vertices, and finally, subject to the first two conditions, maximizing the number of precolored vertices. Recall that G is a plane graph with $\Delta(G) \leq 3$ and $g(G) \geq 6$. Let L be a list assignment for G such that the precolored vertices are proper and form at most two precolored paths, each on at most three vertices in the same face of G, and such that there is no injective L-coloring of G. Denote the set of all precolored vertices by $\mathcal{P}$. Note that there are at most six vertices in $\mathcal{P}$.

By the minimality of G, we obtain that G is connected. In addition, each subgraph of G with fewer non-precolored vertices is injectively L-colorable. Moreover, if G' is a connected graph obtained from G by adding precolored vertices which still satisfies the assumptions of Theorem 2, then G' is injectively L-colorable.

3.1 Preliminary observations

We first make some preliminary observations about the structure of G.

Claim 3. *Every precolored path is on three vertices, every vertex of $\mathcal{P}$ has degree one or three in G. and there are at least two vertices that are not precolored.*

Proof. If P is a precolored path on less than 3 vertices, then we can add a new precolored vertex to one end of P, contradicting the maximality of $|\mathcal{P}|$.

Suppose for contradiction there exists $v \in \mathcal{P}$ with $\deg_G(v) = 2$. Since each precolored path P has three vertices and by the assumptions of Theorem 2, only the middle vertex of P can be adjacent to a vertex in $G - P$, v is a middle vertex of a precolored path. However, $\deg_P(v) = 2$. Since G is connected, we get $P = G$. Since $\mathcal{P}$ is proper, there exists an injective L-coloring of G, which is a contradiction.

Suppose v is the only non-precolored vertex. Then $\deg(v) \leq 2$ and G is a tree. Therefore, v has at most four neighbors in $G^{(2)}$ and G is injectively L-colorable, a contradiction. □

Claim 4. *If v_1 and v_2 are two distinct 2-vertices, then the distance between them is at least four.*

Proof. Let v_1 and v_2 be distinct 2-vertices that are endpoints of a path Z of length ℓ, where $\ell \leq 3$. Let G' be obtained from G by removing the vertices of Z. By the minimality of G, there exists an

Shape	List Size	2-vertex	With External Edge
●	1	⦿	●- - -
○	2	◎	○- - -
△	3	(△)	△- - -
□	4	(□)	□- - -
⬠	5	(⬠)	⬠- - -

Table 2: Key of list sizes.

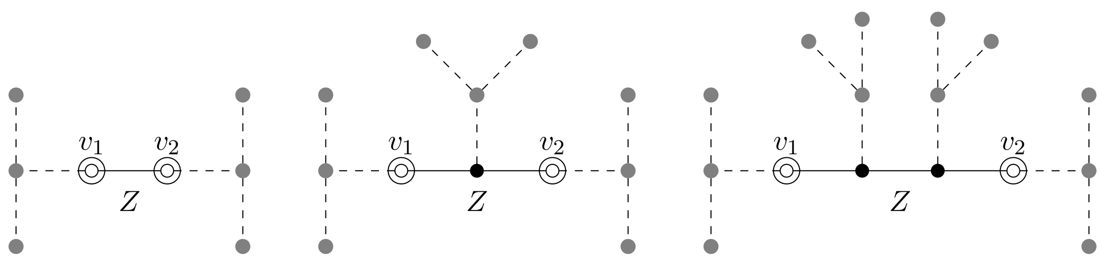


Figure 1: Path Z connecting two vertices of distance at most three in Claim 4. Dashed edges and gray vertices correspond to (possible) edges and vertices of G outside of Z.

injective L-coloring φ of G'. Observe that the subgraph of $G^{(2)}$ induced by $V(G')$ is the same as $G'^{(2)}$.

Let L_Z be a list assignment for Z defined in the following way:

$$L_Z(v) = L(v) \setminus \{\varphi(u) : u \in N_{G^{(2)}}(v) \cap V(G')\}.$$

See Figure 1 for possible cases based on ℓ, and refer to Table 2 for shape meanings in Figure 1 and subsequent figures. In all three cases, $|L_Z(v_i)| \geq 2$ for $i \in \{1, 2\}$ and all the remaining vertices of Z have at least one color available. Hence there exists an injective L_Z-coloring ρ of Z.

Observe that ρ and φ form an injective L-coloring of G, which is a contradiction. □

Claim 5. *If $e = uv$ is a bridge in G, then u or v is in $\mathcal{P}$.*

Proof. Suppose for contradiction that $e = uv$ is a bridge and neither u nor v is precolored. Denote the two connected components of $G - e$ by X_u and X_v where $u \in V(X_u)$ and $v \in V(X_v)$. Moreover, if possible, pick e such that X_v does not contain any precolored vertices.

First we show that each of u and v have a neighbor in $\mathcal{P}$. To show this, assume that either X_v does not contain any precolored vertices, or if both X_u and X_v contain precolored vertices, that v is not adjacent to any of them.

By the minimality of G, there exists an injective L-coloring φ of X_u. Let $X'_v = G[X_v \cup N[u]]$. We create a list assignment L' for X'_v, where $L'(y) = L(y)$ if $y \in V(X_v)$ and $L'(y) = \{\varphi(y)\}$ if $y \in V(X_u)$.

Observe that X'_v with L' is a plane graph with at most two precolored paths, one on the vertices $N[u] \cap X_u$ and possibly another one in X_v. Moreover, the set of precolored vertices in L' is proper since v is not adjacent to precolored vertices in X_v. Finally, if there are two precolored paths in X'_v, they must both be part of the outer face F since the two precolored paths in G are in F. Hence e is also in F.

By the minimality of G, X'_v has an injective L'-coloring ρ. Since ρ and φ agree on u and its neighbors in X_u, it is possible to combine φ and ρ into an injective L-coloring of G, which is a contradiction.

Hence we conclude that each u and v have a neighbor in $\mathcal{P}$. By Claim 4, u cannot be a 2-vertex. Then u is a 3-vertex and has a non-precolored neighbor w distinct from v. Since uv is a bridge, uw is also a bridge. Since only two vertices in $G - \mathcal{P}$ have neighbors in $\mathcal{P}$, we get a contradiction with our choice of e since vw is a bridge and the connected component of $G - vw$ containing w has no precolored vertices. □

Claim 6. *If a vertex v has two precolored neighbors, then v is also precolored.*

Proof. Suppose for contradiction that v is not precolored. If v is a vertex of degree two, the connectivity of G implies that v is the only non-precolored vertex, contradicting Claim 3. If v is a vertex of degree three, then consider its non-precolored neighbor w. The edge vw has no endpoint precolored and it is a bridge, contradicting Claim 5. □

Claim 7. *If v is a vertex of degree one in G, then v is precolored.*

Proof. Suppose for contradiction that v is a vertex of degree one that is not precolored. By the minimality of G, there exists an injective L-coloring φ of $G - v$. Since v has at most 4 neighbors in $G^{(2)}$ and $|L(v)| \geq 5$, it is possible to extend φ to an injective L-coloring of G, which is a contradiction to the minimality of G. □

Claim 8. *Every face of $G - \mathcal{P}$ is bounded by a cycle.*

Proof. Claims 5, 6, and 7 imply that $G - \mathcal{P}$ is bridgeless and every face f of $G - \mathcal{P}$ is bounded by a cycle of length $\ell(f)$. □

Claim 9. *Let e_1 and e_2 be edges with distinct endpoints such that $G - \{e_1, e_2\}$ is disconnected but neither e_1 nor e_2 is a bridge. Then each connected component of $G - \{e_1, e_2\}$ contains precolored vertices. In particular, both e_1 and e_2 are in the outer face.*

Proof. Let e_1 and e_2 be edges with distinct endpoints such that $G - \{e_1, e_2\}$ is disconnected but neither e_1 nor e_2 is a bridge.

Since neither e_1 nor e_2 is a bridge, $G - \{e_1, e_2\}$ contains exactly two connected components X and Y. Suppose for contradiction that Y does not contain any precolored vertices.

By the minimality of G, there is an injective L-coloring φ of X. Let u_i be a vertex of e_i in $V(X)$ for $i \in \{1, 2\}$. Notice that neither u_1 nor u_2 is precolored since precolored vertices are incident only to bridges.

Next we build a graph Y' and a list assignment L'. We start with $Y' = Y$ and define $L'(v) = L(v)$ for all $v \in V(Y')$. Then for $i \in \{1, 2\}$ we add u_i and e_i to Y' and define $L'(u_i) = \varphi(u_i)$. Then for every $x \in N_X(u_i)$ we add to Y' a new vertex u_i^x adjacent to u_i and define $L'(u_i^x) = \varphi(x)$; see Figure 2.

Notice that if u_1 and u_2 have a common neighbor x, there are two vertices u_1^x and u_2^x corresponding to x in Y'. Similarly, if u_1 and u_2 are adjacent in G, we have edges $u_1 u_1^{u_2}$ and $u_2 u_2^{u_1}$

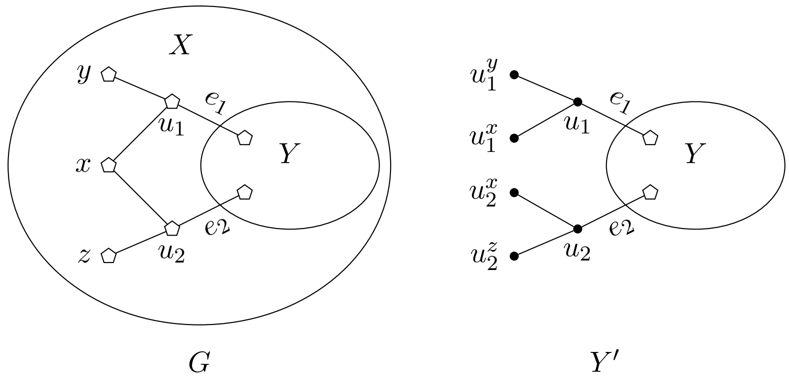


Figure 2: Dealing with edges e_1 and e_2 that form a cut in Claim 9.

instead of the edge u_1u_2 in Y'. We do this to keep the assumption that precolored vertices form two paths. Since e_1 and e_2 do not share vertices, the precolored vertices in Y' with L' are proper.

By the minimality of G, there is an injective L'-coloring ρ of Y'. Since φ and ρ agree on u_1, u_2 and their neighbors in X, it is possible to combine φ and ρ and obtain an injective L-coloring of G, which is a contradiction. □

3.2 Reducible Configurations

A *configuration* is a pair (H, md), where H is a plane graph and md is a mapping $md : V(H) \to \mathbb{N}$. The notation md stands for maximum degree.

A *basic reducible configuration* is a configuration (H, md), where H is one of the plane graphs $C_{2,1}, \ldots, C_{10,3}$ depicted in Figures 7–12 and md of a vertex v can be read from the figure of H by adding the degree of v and the number of incident dashed edges. We denote the set of basic reducible configurations by $\mathcal{B}$.

Also note the graphs in Figures 7–12 are embedded in the plane and each face, except the outer face, bounded by a cycle C is labeled with a number ℓ equal to the size of C. If we are dealing with the outer face F, the cycle C does not form the entire boundary of F as F could also contain precolored vertices. In this case, ℓ corresponds to the length of the outer face of $G - \mathcal{P}$. Figures 7–12 are moved to the next section in order to make it easier for the reader to find them when they are actually needed.

Next we obtain the set of *reducible configurations* $\mathcal{R}$ by taking $\mathcal{B}$ together with configurations obtained from $(H, md) \in \mathcal{B}$; these are obtained by identification of two vertices of degree 1 into a new vertex w and defining $md(w) = 2$. We keep in $\mathcal{R}$ only configurations with plane graphs of girth at least 6.

We say that (H, md) *appears* in G if G contains a subgraph H' isomorphic to H and for every vertex v of H', $\deg_G(v) \le md(v)$, where md is defined on H' by its isomorphism to H.

We plan to show that no reducible configuration (H, md) appears in G. If (H, md) appears in G, we wish to obtain a contradiction by finding an injective L-coloring φ of $G - H'$, where H' is the isomorphic copy of H, and then by extending φ to an injective L-coloring to H'. To this end, we need to consider the subgraph W of $G^{(2)}$ induced by vertices of H'. It may happen that W is not isomorphic to $H'^{(2)}$ if H' is not an induced subgraph of G or some of the vertices in H'

have a common neighbor that is not in H'. We cover these cases by expanding the set of reducible configurations, as follows.

Let $\mathcal{A}$ be obtained from $\mathcal{R}$ by possibly repeating any of the following operations to configurations (H, md) already in $\mathcal{A}$:

- Add an edge between two vertices u and v where $md(u) > \deg(u)$ and $md(v) > \deg(v)$.
- Add a new vertex w adjacent to u and v where $md(u) > \deg(u)$ and $md(v) > \deg(v)$ and let $md(w) = 3$.

Notice that each operation decreases the number of vertices v in H where $\deg(v) < md(v)$, hence $\mathcal{A}$ is finite. We call $\mathcal{A}$ *all reducible configurations*. Note also that if $(H, md) \in \mathcal{R}$ appears in G then there exists $(H', md') \in \mathcal{A}$ such that G contains an isomorphic copy X of H' and $H'^{(2)}$ is isomorphic to $G^{(2)}$ restricted to the vertices of X.

Let $\mathcal{E} = \{X_1, X_2, X_3, X_4\}$ be the configurations from $\mathcal{A}$ that are depicted in Figures 3 and 4. The configurations in $\mathcal{E}$ are not injectively colorable from the depicted lists. We call these configurations *exceptions*.

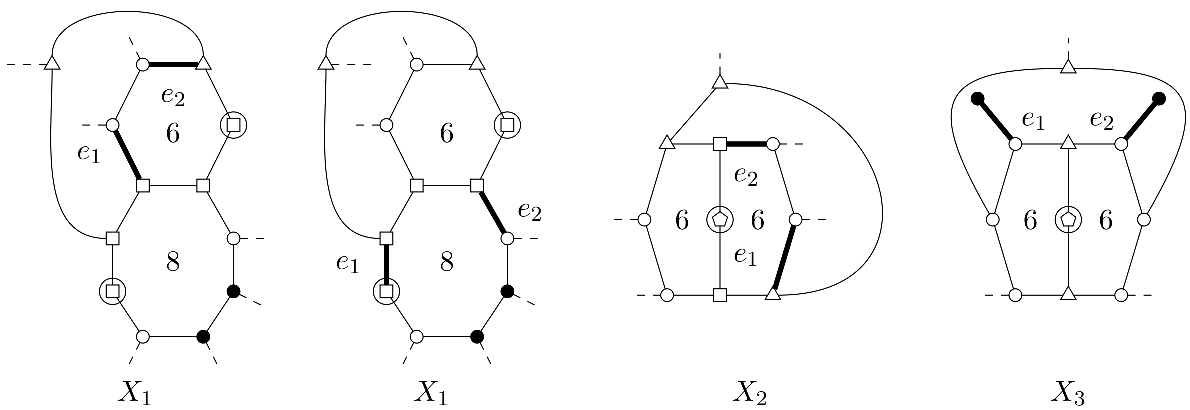


Figure 3: Exceptional graphs X_1, X_2 and X_3.

Claim 10. *If $(H, md) \in \mathcal{A} \setminus \mathcal{E}$ then (H, md) does not appear in G.*

Proof. Suppose for contradiction that some $(H, md) \in \mathcal{A}$ appears in G. Let H' be the isomorphic copy of H in G. Assume that H is as large as possible. That means H' is an induced subgraph of G and no pair of vertices of H' have a common neighbor in $G - H'$.

By the minimality of G, there exists an injective L-coloring φ of $G - H'$. Create a list assignment L' from L by removing the colors used on neighbors in $G^{(2)} - H'$, that is for every $v \in V(H')$,

$$L'(v) = L(v) \setminus \{\varphi(x) : vx \in E(G^{(2)}) \text{ and } x \in V(G) \setminus V(H')\}.$$

We depict $|L'(v)|$ for configurations in $\mathcal{B}$ in Figures 7–12 by the shape of vertices. Refer to Table 2 for shape meanings.

Using a computer program written in Sage, we verified that H' has an injective L'-coloring $\varphi_{H'}$. Notice $(G - H')^{(2)}$ is the same as $G^{(2)} - H'$ since every vertex in H' has at most one neighbor in $G - H'$. Hence $\varphi_{H'}$ and φ can be combined into an injective L-coloring of G, which is a contradiction.

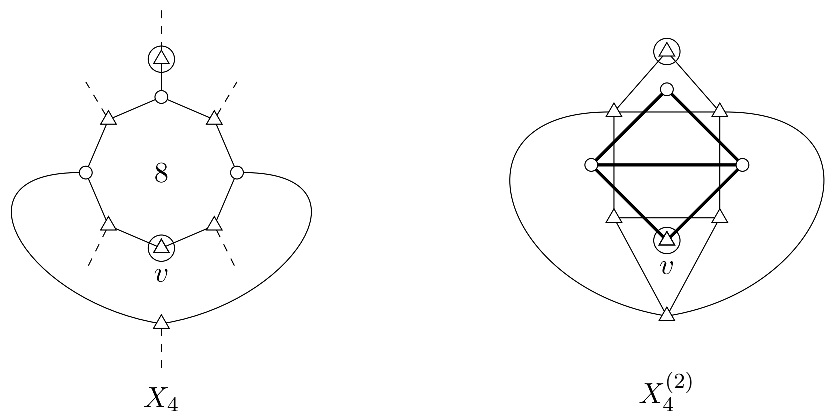


Figure 4: Configurations in X_4 and its neighbor graph with two components C_1 and C_2, where C_1 is drawn with thick edges.

The computer program verifies injective L'-colorability of H' by a greedy coloring or by finding an Alon-Tarsi orientation (cf. [1]) on the neighboring graph of H'. Both of these methods work only with list sizes and not with the actual contents of the lists. The method used is denoted next to the labels in Figures 7–12 by using AT for Alon-Tarsi and G for greedy. The program we used can be obtained at `http://orion.math.iastate.edu/lidicky/pub/injective6/`. □

Now we deal with the exceptions in $\mathcal{E}$. Let $C^{\star}_{2,2}$ be the configuration depicted in Figure 5.

Claim 11. *If $(H, md) \in \mathcal{E}$ and (H, md) appears in G, then it is $C^{\star}_{2,2}$.*

Proof. We use Claim 9 to show that if any of the configurations X_1, X_2, X_3 appear in G, then it must be $C^{\star}_{2,2}$. The thick edges in Figure 3 are edges e_1 and e_2 in Claim 9. Notice that $G - \{e_1, e_2\}$ is disconnected.

If neither e_1 nor e_2 is a bridge, then only one connected component contains precolored vertices and the vertices of e_1 and e_2 are not precolored. Also, notice that X_1 can be embedded in two different ways that are suggested by the dotted edges, but both cases contain the desired edges e_1 and e_2.

Suppose one of e_1 or e_2 is a bridge. By Claim 5, the other is also a bridge and both are incident to precolored vertices. This can happen only in X_3 and we obtain configuration $C^{\star}_{2,2}$. Finally, suppose (X_4, md) appears in G. Let X'_4 be the isomorphic copy of X_4 in G. Notice that X_4 is a bipartite graph, hence $X_4'^{(2)}$ has two connected components; see Figure 4, right. Let v be the vertex of degree 2 in X'_4 with $md(v) = 2$. Denote the bipartition of X'_4 by C_1 and C_2, where $v \in C_1$.

If we try to use the procedure of Claim 10 on (X_4, md), then a coloring of $G - X'_4$ does not have to extend to C_1 since $C_1^{(2)}$ contains a triangle with lists of size two at every vertex, but the procedure from Claim 10 does work on C_2. Notice that $C_2^{(2)}$ is 3-choosable; see Figure 4.

By the minimality of G we obtain an injective L-coloring φ of $G - v$. Since v has four neighbors in $G^{(2)}$ and $|L(v)| = 5$, φ can be extended to v. However, φ might not be an injective coloring of G as the neighbors of v in C_2 are not adjacent in $(G - v)^{(2)}$ but they are adjacent in $G^{(2)}$. Thus, we fix φ by uncoloring vertices in C_2 and recoloring them since $C_2^{(2)}$ is 3-choosable. □

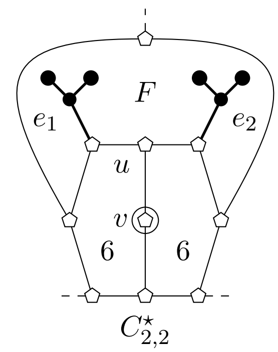


Figure 5: A non-reducible configuration $C^{\star}_{2,2}$ obtained from $C_{2,2}$. Black vertices are in $\mathcal{P}$ and the white vertices are in $G-\mathcal{P}$. The vertex v is a bad vertex and F is the outer face. The face F is not actually drawn as the outer face in order to show correspondence with X_3 in Figure 3.

3.3 Discharging Argument

Recall that G is a minimum counterexaple to Theorem 2. Hence G is a plane graph with maximum degree at most 3 and girth at least 6. In Section 3.2 we showed that G cannot contain any of the reducible configurations listed in Figures 7–12. Moreover, G contains at most two precolored paths, whose vertices are denoted by $\mathcal{P}$. Recall $|\mathcal{P}| \le 6$. Also, all precolored vertices are part of the outer face F.

For each vertex $v \in V(G)-\mathcal{P}$ and each face $f \in F(G)-\{F\}$, define initial charges $\mu_0(v) = 2\deg(v)-6$ and $\mu_0(f) = \ell(f)-6$. For precolored vertices v, define $\mu_0(v) = 0$ and for F define $\mu_0(F) = \ell(F)-5-|\mathcal{P}|$. Let $\mathcal{P}_1$ be the precolored vertices of degree one. Recall that $|\mathcal{P}_1| = \frac{2}{3}|\mathcal{P}| \le 4$. By using Euler's formula 1 and

$$\sum_{v\in V(G)} \deg(v) = 2|E| = \sum_{f\in F(G)} \ell(f),$$

we show that the sum of all charges is negative.

$$\begin{aligned} |V(G)| + |F(G)| &= |E(G)| + 2 \qquad (1)\\ 4|E(G)| - 6|V(G)| + 2|E(G)| - 6|F(G)| &= -12\\ \sum_{v\in V(G)} (2\deg(v)-6) + \sum_{f\in F(G)} (\ell(f)-6) &= -12\\ \sum_{v\in V(G)} \mu_0(v) + \sum_{f\in F(G)} \mu_0(f) - 4|\mathcal{P}_1| + |\mathcal{P}| - 1 &= -12\\ \sum_{v\in V(G)} \mu_0(v) + \sum_{f\in F(G)} \mu_0(f) &\le -1 \qquad (2)\end{aligned}$$

We sequentially apply the following four discharging rules; see Figure 6 for an illustration.

Let v be a 2-vertex and let f_1 and f_2 be the faces incident to v.

(R1) If $\ell(f_1) = 6$ and $\ell(f_2) \ge 8$, v pulls charge 2 from f_2.

(R2) If $\ell(f_1) \geq 7$ and $\ell(f_2) \geq 7$, v pulls charge 1 from f_1 and from f_2.

(R3) If $\ell(f_1) = 6$ and $\ell(f_2) = 7$, v pulls charge 1 from f_2.

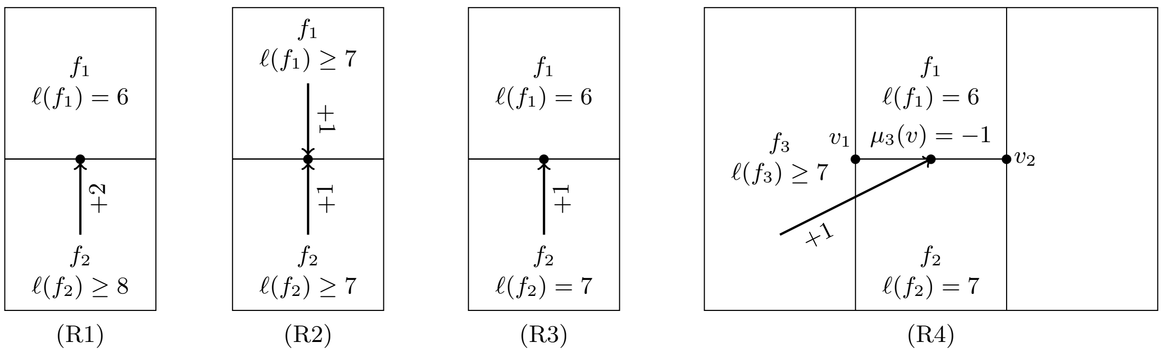


Figure 6: Rules (R1)–(R4).

Let μ_i be the charge after applying rules (R1) to (Ri) for $i \in \{3, 5\}$. Let f_3 be a face such that v (defined above) is a nearby vertex of f_3. Recall F is the outer face.

(R4) If $\mu_3(v) = -1$ and $\ell(f_3) \geq 7$, v pulls charge 1 from f_3.

(R5) If $\mu_3(v) = -2$, v pulls charge 2 from F.

Notice rules (R4) and (R5) apply only if v is incident to at least one 6-face. A 2-vertex u is *needy* if $\mu_3(u) = -1$. A 2-vertex u is *bad* if $\mu_3(u) = -2$.

Now we show that all vertices and faces in the minimal counterexample have nonnegative final charge μ_5, providing our contradiction with (2). Note that 3-vertices and 6-faces begin with charge 0 and never lose any charge due to the discharging rules, thus they end with nonnegative charge. By Claim 3, none of the precolored vertices have degree 2. Since precolored vertices begin with charge 0 and they are not affected by any of the discharging rules, their final charge is 0.

By Claim 7, the minimum vertex degree of $G - \mathcal{P}$ is 2.

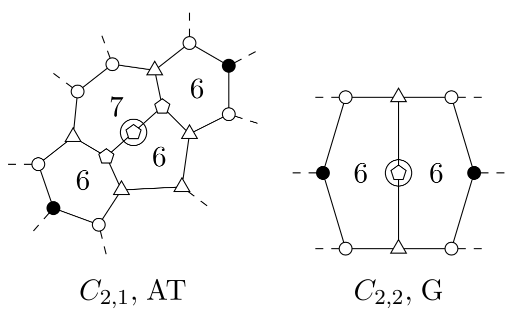


Figure 7: Configurations around a 2-vertex, where $C_{2,1}$ is reducible and $C_{2,2}$ is reducible except for the special case $C^{\star}_{2,2}$ depicted in Figure 5.

Let v be a 2-vertex with neighbors v_1 and v_2. Note the reducible configurations in Figure 7. The minimum distance between 2-vertices must be at least 4 by Claim 4. Hence v_1 and v_2 must

$\ell(f)$	7		8			9			10		
$\lvert I(f)\rvert$	1	0	2	1	0	2	1	0	2	1	0
$\max \lvert N(f)\rvert$	1	3	0	2	4	0	2	4	1	3	5

Table 3: Maximum number of nearby vertices of f when $\ell(f)$ and $|I(f)|$ are fixed.

be 3-vertices. Let f_1 and f_2 be the faces incident to v. Since v begins with charge $\mu_0(v) = -2$, we must show that it receives charge at least 2 during discharging. If $\ell(f_1) \geq 7$ and $\ell(f_2) \geq 7$, then v receives charge at least 1 from each face by (R2), resulting in $\mu_3(v) = \mu_5(v) = 0$. Assume then that $\ell(f_1) = 6$.

- If $\ell(f_2) \geq 8$, then v receives charge 2 from f_2 by (R1), resulting in $\mu_3(v) = \mu_5(v) = 0$.

- If $\ell(f_2) = 7$, then v receives charge 1 from f_2 by (R3). This gives $\mu_3(v) = -1$ and v is a needy vertex. Let f_3 and f_4 be the faces incident to v_1 and v_2, respectively, that are not incident to v. If $\max\{\ell(f_3), \ell(f_4)\} \geq 7$, then v receives an additional 1 charge by (R4). This results in $\mu_5(v) \geq 0$. The case $\ell(f_3) = \ell(f_4) = 6$ cannot happen, otherwise G would contain the reducible configuration $C_{2,1}$.

- If $\ell(f_2) = 6$, then $\mu_3(v) = -2$ and v receives charge 2 from F by (R5). Notice that G contains the configuration $C_{2,2}$ and the only possibility for $C_{2,2}$ is $C^{\star}_{2,2}$ from Figure 5 by Claim 11. Observe that there is at most one bad vertex v in G. Indeed, a neighbor u of a bad vertex is in F and both neighbors of u in F have neighbors in $\mathcal{P}$. Since G has girth 6, there can be at most one such vertex u; see Figure 5.

Hence, every 2-vertex has a nonnegative final charge.

Let f be a face. Due to the minimum distance between 2-vertices, f can be incident to at most $\left\lfloor \frac{\ell(f)}{4} \right\rfloor$ 2-vertices. The minimum distance restriction also limits the number of 2-vertices that are nearby f, as shown in Table 3. Note that this table only takes the minimum distance restriction into consideration and not any other reducible configurations. For each of the possible configurations in the table, we argue that f will either contain a reducible configuration or have nonnegative final charge. The case $\ell(f) \geq 11$ will be handled separately.

If $|\mathcal{P}| \geq 1$, then $\ell(F) \geq 12$ and F is included in the case $\ell(f) \geq 11$. If there are no precolored vertices, F is treated as any other face of G.

We distinguish the case based on $\ell(f)$. We also include charts to summarize the results based on the length of the face. Each cell will contain the argument for why the configuration is reducible, has a nonnegative final charge, denoted EC, or fails to meet the distance requirement, denoted DR.

- Suppose that $\ell(f) = 6$. We have $\mu_0(f) = 0 = \mu_5(f)$ since f does not send or receive any charge. Hence, if $\ell(f) = 6$, then f has nonnegative final charge.

- Suppose that $\ell(f) = 7$; $\mu_0(f) = 1$, and note the reducible configurations in Figure 8. Since f begins with charge $\mu_0(f) = 1$, we must show that it loses charge at most 1.

  Suppose that f is incident to one 2-vertex v. By configuration $C_{7,1}$, f cannot have a nearby 2-vertex. If f has no nearby 2-vertices, then it only loses charge 1 to v by (R2) or (R3), thus $\mu_5(f) \geq 0$.

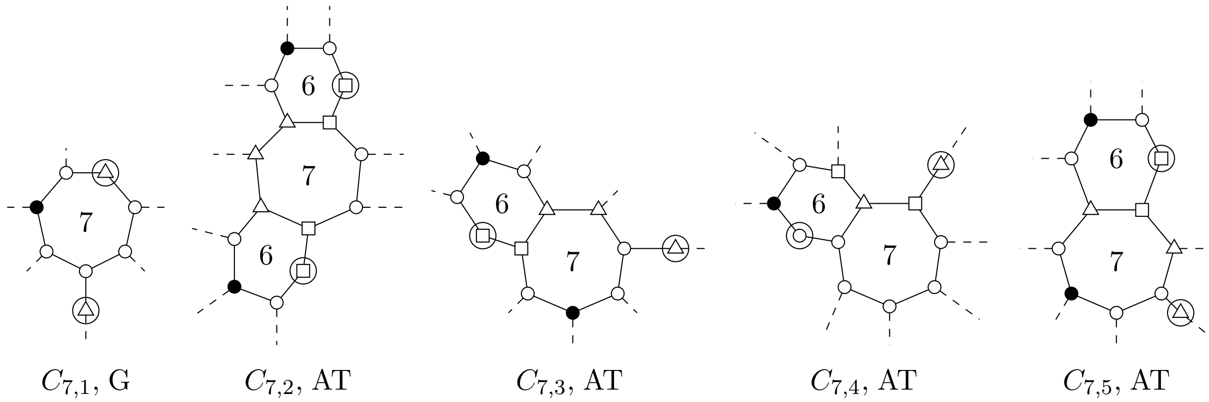


Figure 8: Reducible configurations around a 7-face.

Suppose that f is not incident to any 2-vertex. We show that at most one nearby 2-vertex of f is needy. Suppose for contradiction that v_1 and v_2 are two nearby vertices and both are needy. Since they are needy, each must be in a 6-face. If they are distance 5 apart, G contains $C_{7,2}$ or $C_{7,3}$, which is a contradiction. If they are distance 4 apart, G contains $C_{7,4}$ or $C_{7,5}$, which is a contradiction. Hence f is incident to at most one needy vertex. Therefore, (R4) applies to f at most once. Hence, if $\ell(f) = 7$, then f has nonnegative final charge.

$\ell(f) = 7$				
$\lvert I(f)\rvert \,/\, \lvert N(f)\rvert$	0	1	2	3
0	EC	EC	$C_{7,3}, C_{7,4}, C_{7,5}, C_{7,6}$	$C_{7,2}$
1	EC	$C_{7,1}$	DR	DR

- Suppose that $\ell(f) = 8$; $\mu_0(f) = 2$, and note the reducible configurations in Figure 9. By configuration $C_{8,1}$, f cannot be incident to two 2-vertices.

  Suppose that f is incident to one 2-vertex. By configuration $C_{8,2}$, f cannot have two nearby vertices. If f has one nearby 2-vertex, the distance between the incident 2-vertex and the nearby 2-vertex must be either 4 or 5, but by configuration $C_{8,3}$, this distance cannot be 5. If the incident 2-vertex and the nearby 2-vertex are distance 4 apart, then by $C_{8,4}$ and $C_{8,5}$ the nearby 2-vertex is not needy. Hence (R4) does not apply to f and f only loses at most charge 2 to the incident 2-vertex by (R1) or (R2). Therefore, the final charge of f is nonnegative.

  Suppose that f is not incident to any 2-vertex. By configuration $C_{8,6}$, f cannot have four nearby 2-vertices. If f has three nearby 2-vertices, then they are configured in one of the two ways shown in Figure 10. By configuration $C_{8,7}$, f cannot have the configuration in Figure 10(a). If f has the configuration in Figure 10(b), then by $C_{8,8}$ and $C_{8,9}$, neither v_1 nor v_3 is needy, thus f loses charge at most 1 to v_3 by (R4). If f has fewer than three nearby 2-vertices, then it loses charge at most 2: 1 to each nearby 2-vertex by (R4). Hence, if $\ell(f) = 8$, then f has nonnegative final charge.

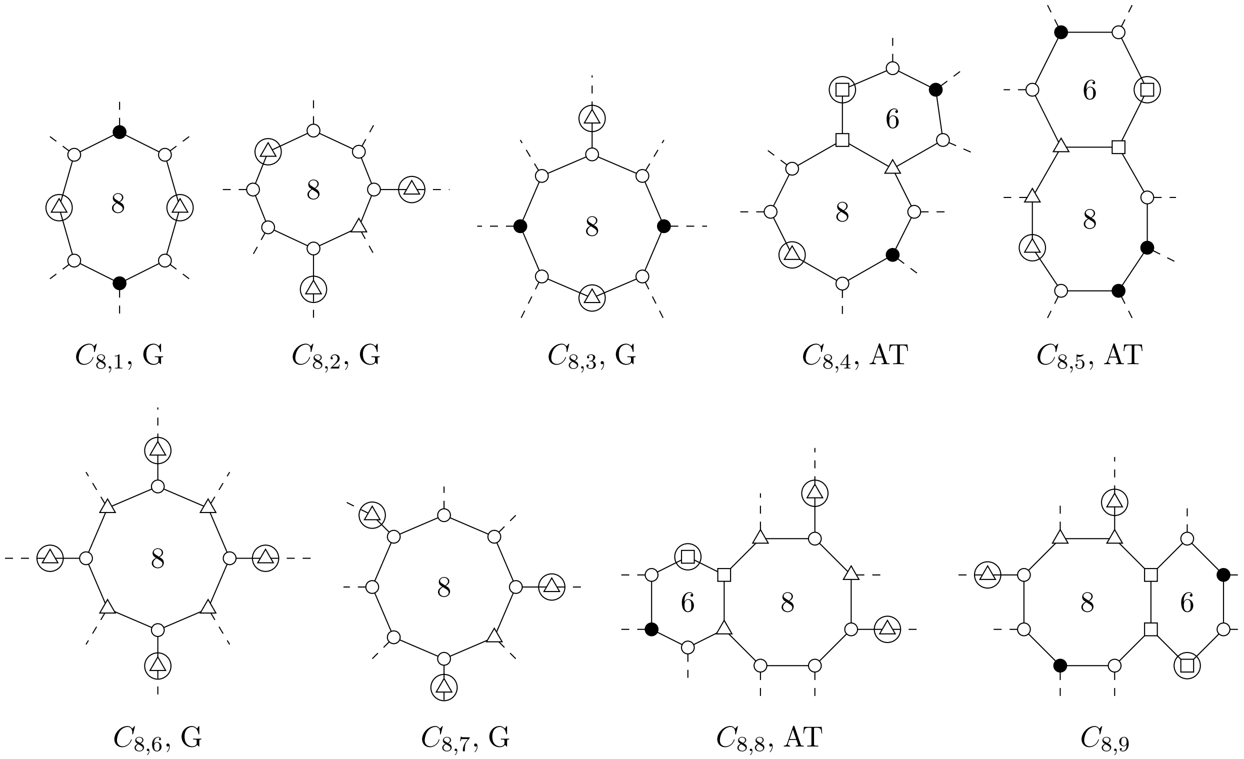
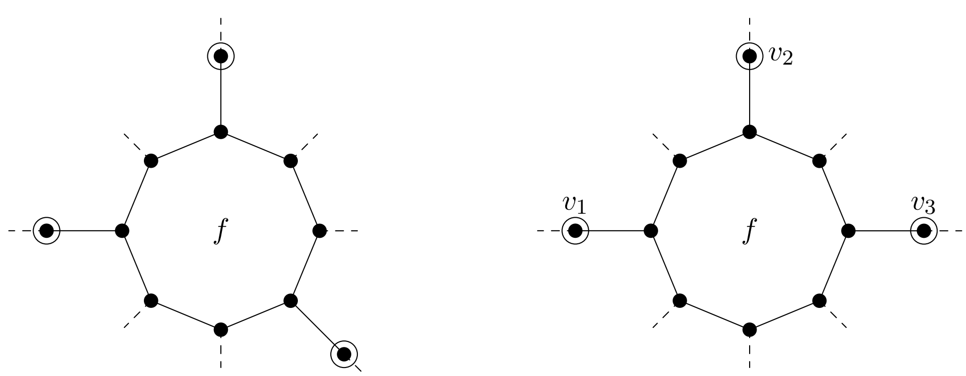
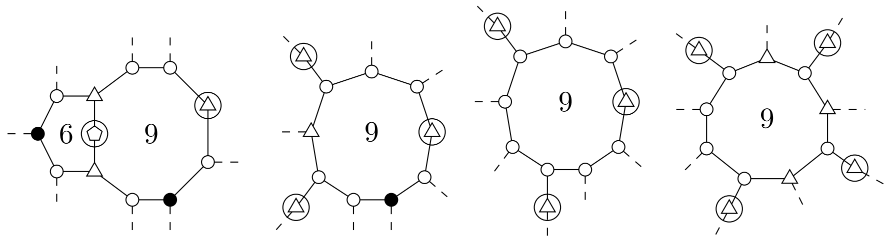


Figure 9: Reducible configurations around an 8-face.



(a) (b)

Figure 10: Two configurations of an 8-face f with three nearby 2-vertices



$C_{9,1}$, G $C_{9,2}$, G $C_{9,3}$, G $C_{9,4}$, G

Figure 11: Reducible configurations around a 9-face.

$\ell(f) = 8$					
$\|I(f)\|/\|N(f)\|$	0	1	2	3	4
0	EC	EC	EC	$C_{8,7}, C_{8,8}, C_{8,9}$	$C_{8,6}$
1	EC	$C_{8,3}, C_{8,4}, C_{8,5}$	$C_{8,2}$	DR	DR
2	$C_{8,1}$	DR	DR	DR	DR

- Suppose that $\ell(f) = 9$; $\mu_0(f) = 3$, and note the reducible configurations in Figure 11.

  If f is incident to two 2-vertices v_1 and v_2, then by configuration $C_{9,1}$ neither v_1 nor v_2 can be incident to a 6-face, thus f only loses charge 1 to each 2-vertex by (R2). By Claim 4, f does not have any nearby vertices. Hence the final charge of f is nonnegative.

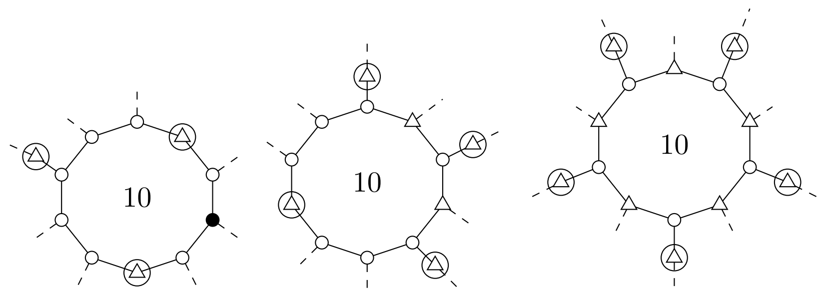
  Suppose that f is incident to one 2-vertex. By configurations $C_{9,2}$ and $C_{9,3}$, f cannot have two nearby 2-vertices. Since f has at most one 2-vertex v, it loses charge at most 3; at most 2 to the incident 2-vertex by (R1) and at most 1 to v by (R4). Hence, if f is incident to one 2-vertex, its final charge is nonnegative.

  Suppose that f is not incident to any 2-vertex. By configuration $C_{9,4}$, f cannot have four nearby 2-vertices. If f has at most three nearby 2-vertices, then it loses charge at most 3: at most 1 to each nearby 2-vertex by (R4). Hence, if $\ell(f) = 9$, then f has nonnegative final charge.

$\ell(f) = 9$					
$\|I(f)\|/\|N(f)\|$	0	1	2	3	4
0	EC	EC	EC	EC	$C_{9,4}$
1	EC	EC	$C_{9,2}, C_{9,3}$	DR	DR
2	$C_{9,1}$	DR	DR	DR	DR

- Suppose that $\ell(f) = 10$; $\mu_0(f) = 4$, and note the reducible configurations in Figure 12.

  If f is incident to two 2-vertices, then by configuration $C_{10,1}$, f has no nearby 2-vertices. Hence f twice loses charge at most 2 by (R1) or (R2). Hence the final charge of f is nonnegative.



$C_{10,1}$, G $\quad$ $C_{10,2}$, G $\quad$ $C_{10,3}$, G

Figure 12: Reducible configurations around a 10-face.

Suppose that f is incident to one 2-vertex. By configuration $C_{10,2}$, f can have at most two nearby 2-vertices. Hence f loses at most charge 2 by (R1) or (R2) once and loses at most twice charge 1 by (R4). Hence the final charge of f is nonnegative.

Suppose that f is not incident to any 2-vertex. By configuration $C_{10,3}$, f cannot have five nearby 2-vertices. Hence f loses at most four times charge 1 by (R4). Hence, if $\ell(f) = 9$, then f has nonnegative final charge.

$\ell(f) = 10$						
$\lvert I(f)\rvert/\lvert N(f)\rvert$	0	1	2	3	4	5
0	EC	EC	EC	EC	EC	$C_{10,3}$
1	EC	EC	EC	$C_{10,2}$	DR	DR
2	EC	$C_{10,1}$	DR	DR	DR	DR

- Suppose that $\ell(f) \geq 11$. We define sets of edges that are close to 2-vertices. For every 2-vertex u, define
$$W_u = \{e \in E(G) : \exists v \in N_G(u), v \in e\}.$$
If $u \in I(f)$ then $|W_u \cap f| \geq 4$. Similarly if $v \in N(f)$, then $|W_v \cap f| \geq 2$ as seen in Figure 13. By Claim 4, $W_u \cap W_v = \emptyset$ for any two distinct 2-vertices u and v. Hence $\ell(f) \geq 4|I(f)| + 2|N(f)|$. Every vertex in $I(f)$ receives charge at most 2 by (R1) and each vertex in $N(f)$ receives charge at most 1 by (R4).

  If f is not the outer face, we show that the final charge of f is nonnegative as follows:
$$\begin{aligned}\mu_5(f) &\geq \mu_0(f) - 2|I(f)| - |N(f)| = \ell(f) - 6 - 2|I(f)| - |N(f)| \\ &\geq \left(\left\lceil \frac{\ell(f)}{2} \right\rceil - 6\right) + \left(\left\lfloor \frac{\ell(f)}{2} \right\rfloor - 2|I(f)| - |N(f)|\right) \geq 0.\end{aligned}$$
  If f is the outer face F, we need a slightly better estimate. Notice that $F - \mathcal{P}$ is a cycle of length $\ell(F) - 2|\mathcal{P}|$. Let B be the set of bad vertices in G. Recall (R5) applies only to bad vertices. Notice $\ell(F) - 2|\mathcal{P}| \geq 4|I(f)| + 2|N(f)|$. Rules (R1), (R2), (R4), and (R5) may apply

and the computation of the final charge is

$$\mu_5(F) \geq \mu_0(F) - 2|I(F)| - |N(F)| - |B| = \ell(F) - 5 - |\mathcal{P}| - 2|I(F)| - |N(F)| - |B|$$
$$\geq \left(\left\lceil \frac{\ell(F)}{2} \right\rceil - 5 - |B|\right) + \left(\left\lfloor \frac{\ell(F)}{2} \right\rfloor - |\mathcal{P}| - 2|I(F)| - |N(F)|\right) \geq 0.$$

Observe that $C^{\star}_{2,2}$ can appear only once in G, so $|B| \leq 1$. Hence, if $\ell(f) \geq 11$, then f has a nonnegative final charge.

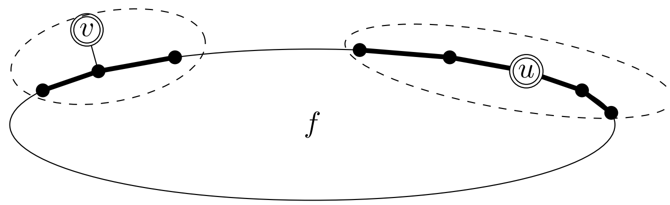


Figure 13: Edges on face f in W_u and W_v.

We conclude that $\sum_{v\in V(G)} \mu_0(v) + \sum_{f\in F(G)} \mu_0(f) \geq 0$, which is a contradiction with (2). This concludes the proof of Theorem 2.

4 Conclusion and Future Work

In this paper, we have shown through the method of discharging that subcubic planar graphs with girth at least 6 are injectively 5-choosable. This result improves several known bounds on the injective chromatic number and injective choosability in particular cases. However, it leaves the most interesting conjecture about injective 5-coloring of planar graphs open.

Conjecture 12 (Chen, Hahn, Raspaud and Wang [5])**.** *If a planar graph G has $\Delta(G) = 3$, then $\chi_i(G) \leq 5$.*

We believe it might be possible to answer the following question in the affirmative.

Question 13. *Is there a planar graph G with $\Delta(G) = 4$, $g(G) \geq 6$, and $\chi_i^{\ell}(G) = 6$?*

We are not aware of counterexamples to the following problems, which are closely related to our result.

Problem 14. *If a planar graph G has $\Delta(G) = 4$ and $g(G) \geq 6$, then $\chi_i^{\ell}(G) \leq 6$.*

Problem 15. *If a planar graph G has $\Delta(G) = 3$ and $g(G) \geq 5$, then $\chi_i^{\ell}(G) \leq 5$.*

Note that these conjectures on injective choosability are analogous to the conjecture on injective colorability of Lužar and Škrekovski, without requiring a girth restriction [15].

Acknowledgements

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References

[1] N. Alon and M. Tarsi. Colorings and orientations of graphs. *Combinatorica*, 12(2):125–134, 1992.

[2] O. V. Borodin and A. O. Ivanova. List 2-distance $(\Delta+2)$-coloring of planar graphs with girth six and $\Delta \geq 24$. *Sibirsk. Mat. Zh.*, 50(6):1216–1224, 2009.

[3] O. V. Borodin and A. O. Ivanova. Injective $(\Delta+1)$-coloring of planar graphs with girth 6. *Sibirsk. Mat. Zh.*, 52(1):30–38, 2011.

[4] O. V. Borodin, A. O. Ivanova, and T. K. Neustroeva. A prescribed 2-distance $(\Delta+1)$-coloring of planar graphs with a given girth. *Diskretn. Anal. Issled. Oper. Ser. 1*, 14(3):13–30, 2007.

[5] Y. Bu, D. Chen, A. Raspaud, and W. Wang. Injective coloring of planar graphs. *Discrete Appl. Math.*, 157(4):663–672, 2009.

[6] Y. Bu and K. Lu. List injective coloring of planar graphs with girth 5, 6, 8. *Discrete Appl. Math.*, 161(10-11):1367–1377, 2013.

[7] H.-Y. Chen and J.-L. Wu. List injective coloring of planar graphs with girth $g \geq 6$. *Discrete Math.*, 339(12):3043–3051, 2016.

[8] D. Cranston and D. B. West. A guide to the discharging method. arXiv:1306.4434 [math.CO].

[9] D. W. Cranston, S.-J. Kim, and G. Yu. Injective colorings of sparse graphs. *Discrete Math.*, 310(21):2965–2973, 2010.

[10] D. W. Cranston, S.-J. Kim, and G. Yu. Injective colorings of graphs with low average degree. *Algorithmica*, 60(3):553–568, 2011.

[11] W. Dong and W. Lin. Injective coloring of planar graphs with girth 6. *Discrete Math.*, 313(12):1302–1311, 2013.

[12] A. Doyon, G. Hahn, and A. Raspaud. Some bounds on the injective chromatic number of graphs. *Discrete Math.*, 310(3):585–590, 2010.

[13] G. Hahn, J. Kratochvíl, J. Širáň, and D. Sotteau. On the injective chromatic number of graphs. *Discrete Math.*, 256(1-2):179–192, 2002.

[14] R. Li and B. Xu. Injective choosability of planar graphs of girth five and six. *Discrete Math.*, 312(6):1260–1265, 2012.

[15] B. Lužar and R. Škrekovski. Counterexamples to a conjecture on injective colorings. *Ars Math. Contemp.*, 8(2):291–295, 2015.

[16] B. Lužar, R. Škrekovski, and M. Tancer. Injective colorings of planar graphs with few colors. *Discrete Math.*, 309(18):5636–5649, 2009.