

Positive co-degree densities and jumps

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Abstract

The *minimum positive co-degree* of a nonempty r -graph H , denoted by $\delta_{r-1}^+(H)$, is the largest integer k such that for every $(r-1)$ -set $S \subset V(H)$, if S is contained in a hyperedge of H , then S is contained in at least k hyperedges of H . Given a family \mathcal{F} of r -graphs, the *positive co-degree Turán function* $\text{co}^+\text{ex}(n, \mathcal{F})$ is the maximum of $\delta_{r-1}^+(H)$ over all n -vertex r -graphs H containing no member of \mathcal{F} . The *positive co-degree density* of \mathcal{F} is $\gamma^+(\mathcal{F}) = \lim_{n \rightarrow \infty} \frac{\text{co}^+\text{ex}(n, \mathcal{F})}{n}$. While the existence of $\gamma^+(\mathcal{F})$ is proved for all families \mathcal{F} , only few positive co-degree densities are known exactly.

For a fixed $r \geq 2$, we call $\alpha \in [0, 1]$ an *achievable value* if there exists a family of r -graphs \mathcal{F} with $\gamma^+(\mathcal{F}) = \alpha$, and call α a *jump* if for some $\delta > 0$, there is no family \mathcal{F} with $\gamma^+(\mathcal{F}) \in (\alpha, \alpha + \delta)$. Halfpap, Lemons, and Palmer [25] showed that every $\alpha \in [0, \frac{1}{r})$ is a jump. We extend this result by showing that every $\alpha \in [0, \frac{2}{2r-1})$ is a jump. We also show that for $r = 3$, the set of achievable values is infinite, more precisely, $\frac{k-2}{2k-3}$ for every $k \geq 4$ is achievable. Finally, we determine two additional achievable values for $r = 3$ using flag algebra calculations.

Keywords: positive co-degree, hypergraph, Turán, jump, flag algebra.

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1 Introduction

An r -graph is a hypergraph in which all hyperedges have size r . We often refer to the hyperedges of an r -graph as r -edges. Given a family of r -graphs \mathcal{F} , the *Turán number*

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$\text{ex}(n, \mathcal{F})$ is the maximum number of r -edges possible in an n -vertex r -graph that contains no member of \mathcal{F} as a subhypergraph. When $r = 2$, the function $\text{ex}(n, \mathcal{F})$ is well-studied and relatively well-understood. Given a set of r -graphs \mathcal{F} , we define the *Turán density* of \mathcal{F} to be

$$\pi(\mathcal{F}) := \lim_{n \rightarrow \infty} \frac{\text{ex}(n, \mathcal{F})}{\binom{n}{r}}.$$

The Erdős-Stone Theorem [16] as pointed out by Erdős-Simonovits [15] determines the Turán density of every 2-graph as a function of its chromatic number.

Theorem 1.1 (Erdős-Stone). *Let F be a 2-graph with $\chi(F) = k$. Then $\pi(F) = 1 - \frac{1}{k-1}$.*

Note that an extension of Theorem 1.1 also claims that for every family \mathcal{F} of 2-graphs, $\pi(\mathcal{F})$ is equal to the minimum of $1 - 1/(\chi(F) - 1)$ over $F \in \mathcal{F}$. While Theorem 1.1 does not give us perfect information about Turán numbers (in particular, for bipartite graphs F , it only demonstrates that $\text{ex}(n, F) = o(n^2)$), it yields a good “approximate” understanding of Turán numbers by fully describing Turán densities.

For $r \geq 3$, we do not have an analogue to Theorem 1.1, and much less is known about Turán densities. Even $\pi(\mathcal{F})$ could be smaller than $\min\{\pi(F) : F \in \mathcal{F}\}$, as observed by Balogh [4]. Not only do we lack a general theory, but the Turán densities of many small r -graphs remain unknown, despite great effort. Famously, the Turán density of the tetrahedron K_4^3 is still undetermined. The difficulty of determining Turán densities for hypergraphs has motivated the study of various other hypergraph extremal functions, typically maximizing some variant of the minimum degree. In particular, given an r -graph H , the *co-degree* of a set $S \in \binom{V(H)}{r-1}$ is the number of r -edges containing S , and the *minimum co-degree* of H , denoted by $\delta_{r-1}(H)$, is the smallest co-degree realized by an $(r - 1)$ -set contained in $V(H)$. The *co-degree Turán function* of a family of r -graphs \mathcal{F} , denoted by $\text{coex}(n, \mathcal{F})$, is the largest possible minimum co-degree of an n -vertex r -graph containing no member of \mathcal{F} as a subhypergraph.

Mubayi and Zhao [33] showed that the *co-degree density*

$$\gamma(\mathcal{F}) := \lim_{n \rightarrow \infty} \frac{\text{coex}(n, \mathcal{F})}{n}$$

exists for every family of r -graphs \mathcal{F} , and studied the general behavior of $\text{coex}(n, \mathcal{F})$. Note that for every family of 2-graphs \mathcal{F} we have $\gamma(\mathcal{F}) = \pi(\mathcal{F})$; however, for $r \geq 3$, co-degree Turán problems are not equivalent to Turán problems, and co-degree density does not behave in the manner suggested by Theorem 1.1. We first define the notion of a *jump* in density.

Definition 1.2. Fix $r \geq 2$. Suppose φ is a function that maps families of r -graphs to $[0, 1]$. We say that $\alpha \in [0, 1)$ is a φ -*jump* if there exists $\delta \in (0, 1 - \alpha)$ such that for no family \mathcal{F} of r -graphs, $\varphi(\mathcal{F}) \in (\alpha, \alpha + \delta)$.

While Theorem 1.1 shows that co-degree density (and Turán density) jumps everywhere when $r = 2$, Mubayi and Zhao showed that co-degree density does not jump when $r \geq 3$.

Theorem 1.3 (Mubayi-Zhao [33]). *For $r \geq 3$, no $\alpha \in [0, 1)$ is a γ -jump.*

This co-degree phenomenon was further investigated in [13, 37].

We remark that Theorem 1.3 suggests a substantial difference in behavior between Turán and co-degree Turán problems for hypergraphs. We know for every $r \geq 3$ that every $\alpha \in [0, r!/r^r)$ is a π -jump. On the other hand, π is also known to *not* jump in infinitely many places for $r \geq 3$; see [3, 22].

The *minimum positive co-degree* of an r -graph H is the largest integer k such that, whenever $S \in \binom{V(H)}{r-1}$ is contained in some r -edge of H , then S is contained in at least k r -edges of H . The edgeless r -graph is defined to have positive co-degree zero. We denote the minimum positive co-degree of H by $\delta_{r-1}^+(H)$. We define the *positive co-degree Turán number*, denoted by $\text{co}^+\text{ex}(n, \mathcal{F})$, to be the largest possible minimum positive co-degree of an n -vertex r -graph containing no member of \mathcal{F} as a subhypergraph.

Balogh, Lemons, and Palmer [6] introduced the minimum positive co-degree as an alternative notion of minimum degree in r -graphs. Since then this parameter has already been studied from several angles. The concept of $\text{co}^+\text{ex}(n, \mathcal{F})$ was recently introduced by Halfpap, Lemons, and Palmer [25]. The investigation of minimum positive co-degree as an extremal parameter is partially motivated by the admissibility of constructions that mimic the extremal graphs for classical questions. For example, given an r -graph H , the *t -blow-up* $H[t]$ of H is the r -graph obtained by replacing each vertex $v_i \in V(H)$ with a class V_i of t vertices, where a set of r -vertices spans an r -edge if and only if they belong to r distinct classes of $H[t]$ which correspond to an r -edge in H . In classical Turán theory, graph blow-ups yield extremal or nearly extremal constructions for all non-bipartite forbidden graphs. Blow-ups also occur as extremal examples for other types of thresholds—for instance, one of the constructions demonstrating the tightness of Dirac’s Theorem is a slightly unbalanced blow-up of an edge (i.e., a complete bipartite graph).

For $r \geq 3$ and every r -graph H a sufficiently large blow-up of H has minimum co-degree 0, which means that even after adding $o(n^3)$ hyperedges, blow-ups will not provide extremal constructions for minimum co-degree density problems. However, $H[t]$ inherits the *positive* co-degree properties of H . Thus, blow-ups (as well as other constructions with co-degree 0 sets, such as r -graphs containing large strongly independent sets or multiple components) are potential extremal examples for positive co-degree problems. Previous results suggest that extremal constructions for positive co-degree problems in fact *do* often look analogous to classical extremal constructions. See [25] for extremal constructions avoiding some small 3-graphs, and [26] on positive co-degree analogs of Dirac’s Theorem, for which the extremal constructions also naturally generalize the graph extremal examples (and have minimum co-degree 0). Due to the expanded range of potential constructions, $\text{co}^+\text{ex}(n, \mathcal{F})$ and $\text{coex}(n, \mathcal{F})$ are generally not equal, and they appear to behave differently.

Define the *positive co-degree density* of a family of r -graphs \mathcal{F} as the limit

$$\gamma^+(\mathcal{F}) := \lim_{n \rightarrow \infty} \frac{\text{co}^+\text{ex}(n, \mathcal{F})}{n}.$$

The existence of $\gamma^+(\mathcal{F})$ was established by Halfpap, Lemons, and Palmer [25] via a constructive argument, which can be generalized to finite families \mathcal{F} . Pikhurko [38] gave a

probabilistic argument establishing that $\gamma^+(\mathcal{F})$ exists for all families \mathcal{F} .

Proposition 1.4 (Halfpap-Lemons-Palmer [25]). *Fix $r \geq 2$ and let \mathcal{F} be a family of r -graphs. Then*

$$\gamma^+(\mathcal{F}) \in \{0\} \cup \left[\frac{1}{r}, 1 \right].$$

In other words, every $\alpha \in [0, 1/r)$ is a γ^+ -jump. Proposition 1.4 describes behavior similar to that of the classical Turán density. Every r -graph F that is contained in some blow-up of an r -edge can be shown to be “degenerate”, having $\text{co}^+\text{ex}(n, F) = o(n)$, so $\gamma^+(F) = 0$. On the other hand, if F is not contained in any blow-up of an r -edge, then $\gamma^+(F) \geq \frac{1}{r}$, since the balanced n -vertex blow-up of an r -edge has minimum positive co-degree approximately $\frac{n}{r}$. An r -graph F is k -partite if there is a partition of $V(F)$ into k classes such that each edge intersects each part at most once.

Although Proposition 1.4 suggests that γ and γ^+ exhibit fundamentally different behaviors, it is not clear what behavior to expect from γ^+ . Currently, we know the exact value $\gamma^+(F)$ only for very few 3-graphs F . Halfpap, Lemons, and Palmer [25] determined the values of $\gamma^+(F)$ for many small 3-graphs F , and bounded $\gamma^+(F)$ in some other instances. Various authors have reported improvements on several of these initial bounds, with the current best known values summarized in Table 1. For comparison, the best-known bounds on π and γ for these 3-graphs are also provided. Graphs not defined in Table 2 are defined by their edge sets as

$$\begin{aligned} K_4^{3-} &= \{123, 124, 134\}, & F_5 &= \{123, 124, 345\}, & \mathbb{F} &= \{123, 345, 156, 246, 147, 257, 367\}, \\ C_\ell &= \{123, 234, 345, \dots, (\ell-2)(\ell-1)\ell, (\ell-1)\ell 1, \ell 12\}, & C_\ell^- &= C_\ell - \{\ell 12\}. \end{aligned}$$

See [5, 25] for more details about 3-graphs in Table 1. Note that the 3-graph denoted here as K_4^{3-} is often called K_4^- in the literature; we adopt the notation K_4^{3-} to distinguish this 3-graph from the 2-graph obtained by deleting an edge from K_4 , which we denote by K_4^- .

Kamčev, Letzter, and Pokrovskiy [27] proved that the Turán density of longer tight cycles C_ℓ is $2\sqrt{3} - 3$, when ℓ is not multiple of three and sufficiently large (when ℓ is divisible by three, then C_ℓ is 3-partite, hence its Turán density is 0). Similarly, Balogh and Luo [7] proved for ℓ sufficiently large and not divisible by 3 that the Turán density of C_ℓ^- is $1/4$. Recently, this was proved for every $\ell \geq 5$ by Lidický, Mattes, and Pfender [29]. That the co-degree density is $1/3$ for tight cycles of length at least 10 and not divisible by 3 was proved by Piga, Sanhueza-Matamala, and Schacht [36] and Ma [30]. On the other hand, the fact that the co-degree density of C_ℓ^- is 0 is due to Piga, Sales, and Schülke [37].

Given $r \geq 2$, we call $\alpha \in [0, 1]$ an *achievable value* (for γ^+) if there exists a family \mathcal{F} of r -graphs such that $\gamma^+(\mathcal{F}) = \alpha$. Before our article, the only known achievable values of γ^+ for $r = 3$ were $0, 1/3, 1/2$, and $2/3$.

Our goal is to understand the positive co-degree density by demonstrating additional γ^+ -jumps for every r , as well as by expanding the known list of achievable values of γ^+ for $r = 3$. Our paper is organized as follows. In Section 2, we summarize our main results. In Section 3, we state some additional definitions and lemmas which will be used in our

F	$\leq \pi(F)$	$\pi(F) \leq$	$\leq \gamma(F)$	$\gamma(F) \leq$	$\leq \gamma^+(F)$	$\gamma^+(F) \leq$
K_4^{3-}	2/7 [21]	0.28689 [43]	1/4 [34]	1/4 [18]	1/3 [25]	1/3 [25]
F_5	2/9 [8]	2/9 [20]	0 [5]	0 [5]	1/3 [25]	1/3 [25]
$F_{3,2}$	4/9 [32]	4/9 [23]	1/3 [17]	1/3 [17]	1/2 [25]	1/2 [25]
\mathbb{F}	3/4 [41]	3/4 [12]	1/2 [31]	1/2 [31]	2/3 [25]	2/3 [25]
K_4^3	5/9 [42]	0.5615 [2]	1/2 [11]	0.529 [5]	1/2 [25]	0.543 [44]
$F_{3,3}$	3/4 [32]	3/4 [32]	1/2 [5]	0.604 [5]	3/5 [25]	0.616
C_5	$2\sqrt{3} - 3$ [32]	0.46829 [43]	1/3 [5]	0.3993 [5]	1/2 [25]	1/2 [45]
C_7	$2\sqrt{3} - 3$ [32]	0.464186	1/3 [5]	0.371	1/2 [24]	1/2 [24]
C_5^-	1/4 [32]	1/4 [29]	0 [5]	0 [35]	1/3 [25]	1/3 [45]
J_4	1/2 [9]	0.50409 [43]	1/4 [5]	0.473 [5]	4/7 [25]	4/7
$F_{4,2}$	4/9	0.4933328	1/3	0.4185	3/5	3/5

Table 1: Best-known density bounds for π , γ , and γ^+ .

proofs. In Section 4, we demonstrate further γ^+ -jumps for every r , and for $r = 3$ establish an infinite set of achievable values for γ^+ . In Section 2.4, we use flag algebras to exactly determine γ^+ for two additional 3-graphs, hence adding two values to the list of achievable values of γ^+ . Finally, in Section 6, we have some concluding remarks and list a wide variety of open problems in the area.

2 New results

Our first main theorem extends the range of jumps described in Proposition 1.4. We will need the following definition. An r -*triangle*, denoted by T^r , is an r -graph with $r + 1$ vertices and three r -edges. Notice that T^r can be obtained from the 2-graph triangle T^2 by adding $r - 2$ vertices and including them to each of the three edges. Such hypergraphs are sometimes called *daisies* in the literature. The T^2 part of a T^r is called the *base* of the T^r . Notice that T^3 is the same as K_4^{3-} .

Theorem 2.1. *Let \mathcal{F} be a family of r -graphs for $r \geq 2$. Then*

$$\gamma^+(\mathcal{F}) \in \left\{0, \frac{1}{r}\right\} \cup \left[\frac{2}{2r-1}, 1\right].$$

Thus, every $\alpha \in [0, \frac{2}{2r-1})$ is a γ^+ -jump.

Moreover, $\gamma^+(\mathcal{F}) = 0$ if and only if some member of \mathcal{F} is r -partite, and $\gamma^+(\mathcal{F}) = \frac{1}{r}$ if and only if no member of \mathcal{F} is r -partite, but some member of \mathcal{F} is contained in a blow-up of some T^r .

For $r = 3$, we also provide an infinite set of achievable values for γ^+ , based on forbidden families involving the following 3-graphs. For $k \geq 3$, let J_k be the $(k + 1)$ -vertex 3-graph,

K_4^3		
$F_{3,2}$		
$F_{4,2}$		
$F_{3,3}$		
J_4		
J_5		

Table 2: Small 3-uniform hypergraphs.

on vertex set $[k + 1]$, with 3-edges of the form $1ij$ for every $i, j \in \{2, \dots, k + 1\}$. Let K_4^3 denote the complete 4-vertex 3-graph, and let $F_{3,2}$ denote the 5-vertex 3-graph, on vertex set $\{1, 2, 3, 4, 5\}$, with edge set $\{123, 124, 125, 345\}$. See Table 2 for an illustration of these and other relevant 3-graphs.

Theorem 2.2. *For every $k \geq 4$,*

$$\gamma^+(\{K_4^3, F_{3,2}, J_k\}) = \frac{k - 2}{2k - 3}.$$

We also investigate whether the densities exhibited by Theorem 2.2 can be achieved by forbidding a single 3-graph. For $k = 4$, we have $\gamma^+(\{K_4^3, F_{3,2}, J_4\}) = \frac{2}{5}$. Let F_1 be a 7-vertex 3-graph with edges $\{125, 135, 235, 126, 146, 246, 347\}$, see Figure 2. We show that F_1 has positive co-degree density $\frac{2}{5}$.

Theorem 2.3. $\gamma^+(F_1) = \frac{2}{5}$.

Finally, we utilize flag algebras to determine the positive co-degree densities of two graphs, which exhibit new achievable values of γ^+ larger than $\frac{1}{2}$. We determine $\gamma^+(J_4)$, and show that the (asymptotic) extremal construction is the blow-up of the complement of the Fano plane. Notice that J_4 is a 3-daisy; see [14] for recent progress on Turán densities of r -daisies.

Theorem 2.4. $\gamma^+(J_4) = \frac{4}{7}$.

We introduce another 3-graph, which will have a different new density. Let $F_{4,2}$ be the 6-vertex 3-graph with edges $\{123, 124, 134, 156, 256, 356, 456\}$, depicted in Table 2. Note that in $F_{4,2}$ the common neighborhood of 5 and 6 is $\{1, 2, 3, 4\}$, and $\{1, 2, 3, 4\}$ spans a K_4^{3-} . We determine $\gamma^+(F_{4,2})$, and show that its (asymptotic) extremal construction is the balanced blow-up of K_5^3 .

Theorem 2.5. $\gamma^+(F_{4,2}) = \frac{3}{5}$.

We also include some non-tight results obtained using flag algebras.

Theorem 2.6. *The following bounds hold.*

$$\pi(F_{4,2}) \leq 0.4933327, \quad \gamma(F_{4,2}) \leq 0.4185, \quad \gamma^+(F_{3,3}) \leq 0.616.$$

The lower bounds in Table 1 for $\pi(F_{4,2})$ and $\gamma(F_{4,2})$ come from the lower bound constructions for $F_{3,2}$.

3 Preliminaries

We begin by stating some results related to supersaturation and a hypergraph removal lemma. The hypergraph removal lemma states that an r -graph containing only few copies of some subhypergraph F can be made F -free by the deletion of only few r -edges. For a discussion of removal lemmas, including the below formulation, see [10].

Lemma 3.1. *Fix $\alpha > 0$ and let F be an r -graph. There exists $\delta > 0$ such that if H is an n -vertex r -graph containing at most $\delta n^{|V(F)|}$ copies of F , then there exists $E' \subset E(H)$ such that $|E'| \leq \alpha n^r$ and $H - E'$ is F -free.*

Although an r -graph with $o(n^{|V(F)|})$ copies of F can be made F -free by deleting $o(n^r)$ r -edges, it is not obvious that the deletion would change the minimum positive co-degree by only $o(n)$. The following “clean-up” lemma due to Halfpap, Lemons, and Palmer [25] allows us to apply the hypergraph removal lemma to minimum positive co-degree problems. Roughly, this lemma guarantees that any positive co-degree drop arising from the deletion of a small set of r -edges can be mitigated by the deletion of another set of r -edges.

Lemma 3.2 (Halfpap-Lemons-Palmer [25]). *Let H be an n -vertex r -graph and fix $0 < \varepsilon < 1$ small enough that $(r+1)!\varepsilon^{1/2^{r-1}}n^r < |E(H)|$. Let H_1 be a subhypergraph of H obtained by the deletion of at most εn^r r -edges. Then H_1 has a subhypergraph H_2 with $\delta_{r-1}^+(H_2) \geq \delta_{r-1}^+(H) - 2^r r! \varepsilon^{1/2^{r-1}} n$.*

In practice it is not difficult to fulfill the condition in Lemma 3.2 that $(r+1)!\varepsilon^{1/2^{r-1}}n^r < |E(H)|$, since $\delta_{r-1}^+(H)$ can be used to give a lower bound on $|E(H)|$.

Lemma 3.3 (Halfpap-Lemons-Palmer [25]). *Fix $c > 0$ and suppose H is an r -graph with $\delta_{r-1}^+(H) \geq cn$. Then, for n large enough, $|E(H)| \geq \frac{1}{2} \frac{c^r}{r!} n^r$.*

Thus, for an n -vertex r -graph H with n sufficiently large and $\delta_{r-1}^+(H) \geq cn$, we can choose ε in Lemma 3.2 as a function of c and r alone.

Lemmas 3.1 and 3.2 can be used to prove supersaturation and related properties for minimum positive co-degree problems. In particular, we have the following basic formulation.

Theorem 3.4 (Halfpap-Lemons-Palmer [25]). *Fix $\varepsilon > 0$ and let F be an r -graph. Then there exists $\delta > 0$ such that, if H is an n -vertex r -graph with*

$$\delta_{r-1}^+(H) > \text{co}^+ \text{ex}(F) + \varepsilon n,$$

then H contains at least $\delta n^{|V(F)|}$ copies of F .

By a standard argument (see, e.g., [28]), if $\delta > 0$ and $t \in \mathbb{N}$ are fixed and n is sufficiently large, then every n -vertex r -graph H containing $\delta n^{|V(F)|}$ copies of F must contain $F[t]$. Thus, as an immediate consequence of Theorem 3.4, we have blow-up invariance for $\text{co}^+ \text{ex}(n, F)$.

Corollary 3.5 (Halfpap-Lemons-Palmer [25]). *Let F be an r -graph and t a positive integer. Then*

$$\text{co}^+ \text{ex}(n, F) \leq \text{co}^+ \text{ex}(n, F[t]) \leq \text{co}^+ \text{ex}(n, F) + o(n).$$

We remark that Corollary 3.5 is useful because it implies that if H and F are r -graphs such that F is contained in $H[t]$ for some t , then $\gamma^+(F) \leq \gamma^+(H)$. For example, when paired with the facts that a single 3-edge e has $\gamma^+(e) = 0$ and $\gamma^+(C_5) = \frac{1}{2}$, and with an appropriate lower bound construction, Corollary 3.5 implies that $\gamma^+(C_\ell) = 0$ if $\ell \equiv 0 \pmod{3}$ and $\gamma^+(C_\ell) = \frac{1}{2}$ if $\ell \not\equiv 0 \pmod{3}$ and C_ℓ is contained in a blow-up of C_5 . In fact, this resolves the positive co-degree density of every tight cycle except for $C_4 = K_4^3$ and C_7 . Halfpap [24] proved $\gamma^+(C_7) = 1/2$.

In Section 4, we will apply essentially the same idea, finding one construction whose blow-up contains another and then relating their positive co-degree densities. However, for our purposes, a somewhat different formulation from the above statements will be desirable. The proof ideas for Theorem 3.4 and Corollary 3.5 are used to derive the following lemma.

Lemma 3.6. *Let F be a fixed r -graph on f vertices and $\mathcal{F} = \{F_1, F_2, \dots, F_k\}$ a finite family of r -graphs such that $F_i[f]$ contains F for every $i \in [k]$. For every $d \in [0, 1)$, if $\gamma^+(F) > d$, then for some $\beta > 0$ and n sufficiently large there exists an n -vertex, $\{F\} \cup \mathcal{F}$ -free r -graph H with $\delta_{r-1}^+(H) > (d + \beta)n$.*

Proof. Given d as stated, choose $\beta > 0$ so that $\gamma^+(F) \geq d + 3\beta$, and $\alpha > 0$ such that $2^r r! \alpha^{1/2^{r-1}} < \beta$ and $(r+1)! \alpha^{1/2^{r-1}} n^r < \frac{1}{2} \frac{(2\beta)^r}{r!} n^r$. For each $i \in [k]$, take $\delta_i > 0$ as guaranteed by Lemma 3.1 such that any hypergraph H containing fewer than $\delta_i n^{|V(F_i)|}$ copies of F_i can be made F_i -free by the deletion of at most $\frac{\alpha}{k} |E(H)|$ r -edges. Choose $N \in \mathbb{N}$ sufficiently large such that for all $n \geq N$, we have:

- $\text{co}^+ \text{ex}(n, F) \geq (d + 2\beta)n$;
- for each $i \in [k]$, if H is an n -vertex graph containing $\delta_i n^{|V(F_i)|}$ copies of F_i , then H contains $F_i[f]$.

Fix $n \geq N$ and let H be an n -vertex r -graph with $\delta_{r-1}^+(H) = \text{co}^+ \text{ex}(n, F)$. Then H contains fewer than $\delta_i n^{|V(F_i)|}$ copies of F_i for every $i \in [k]$, and thus can be made \mathcal{F} -free by deletion of at most $\alpha |E(H)|$ edges by repeated application of Lemma 3.1. Since we have

$$(r+1)! \alpha^{1/2^{r-1}} n^r < \frac{1}{2} \frac{(2\beta)^r}{r!} n^r \leq |E(H)|$$

by Lemma 3.3 and the definition of α , we can now apply Lemma 3.2 to delete an additional set of r -edges, resulting in an $\{F\} \cup \mathcal{F}$ -free, n -vertex r -graph H' with

$$\delta_{r-1}^+(H') \geq \delta_{r-1}^+(H) - 2^r r! \alpha^{1/2^{r-1}} n > \delta_{r-1}^+(H) - \beta n \geq (d + \beta)n. \quad \square$$

Lemma 3.6 is an important tool in finding positive co-degree densities as it essentially allows us to expand our list of forbidden configurations.

We conclude this section with some relevant definitions and notation. We often consider 4-vertex cliques with one edge removed; these may be 2-uniform or 3-uniform. To avoid ambiguity, we denote by K_4^- the 2-graph on 4 vertices and 5 edges, and by K_4^{3-} the 3-graph on 4 vertices and three 3-edges.

Some of the hypergraphs we consider can be naturally described as arising from lower-uniformity hypergraphs. We define the following operation, which increases the uniformity of a hypergraph by one. Given an r -graph H , the *suspension* \widehat{H} is the $(r+1)$ -graph with vertex set consisting of $V(H)$ and one new vertex v , and $(r+1)$ -edges

$$E(\widehat{H}) = \{e \cup \{v\} : e \in E(H)\}.$$

We call $V(H)$ the r -graph *vertices* and v the *spike vertex*. Notice that the $(r+1)$ -triangle T^{r+1} is a suspension of the r -triangle T^r .

Let H be an r -graph. For a subset $X = \{x_1, \dots, x_{r-1}\}$ of size $r-1$ of vertices of H , denote by $N(X)$ the set of all vertices v such that $X \cup \{v\} \in E(H)$. We use $d(X) := |N(X)|$, which is the co-degree of X . To simplify notation we use $N(x_1, \dots, x_{r-1}) := N(X)$ and $d(x_1, \dots, x_{r-1}) := d(X)$. We call $N(X)$ the *neighborhood* of X .

In a 3-graph H , the *link graph* $L(v)$ of a vertex $v \in V(H)$ is the auxiliary 2-graph on $V(H) - \{v\}$ where xy is an edge if and only if vxy is a 3-edge of H .

For 3-graphs G and H the *density* of G in H , denoted by $d(G, H)$, is the number of subgraphs of H isomorphic to G divided by $\binom{|V(H)|}{|V(G)|}$. Notice that the density is always in $[0, 1]$.

4 Jumps and positive co-degree densities below $\frac{1}{2}$

Proof of Theorem 2.1. Let \mathcal{F} be a family of r -graphs. By Proposition 1.4, $\gamma^+(\mathcal{F}) \in \{0\} \cup [\frac{1}{r}, 1]$, so it is sufficient to show that there is no family \mathcal{F} with $\gamma^+(\mathcal{F}) \in (\frac{1}{r}, \frac{2}{2r-1})$.

First assume that forbidding \mathcal{F} implies that some blow-up $T^r[t]$ of T^r is also forbidden (recall, T^r denotes the triangle with three edges on $r + 1$ vertices). Corollary 3.5 gives $\gamma^+(T^r[t]) = \gamma^+(T^r)$. As

$$\gamma^+(\mathcal{F}) \leq \gamma^+(T^r[t]) = \gamma^+(T^r),$$

it is sufficient to show that $\gamma^+(T^r) \leq \frac{1}{r}$.

Suppose that H is an n -vertex r -graph with $\delta_{r-1}^+(H) > \frac{n}{r}$, and let $v_1v_2 \dots v_r$ be an r -edge of H . Consider the r vertex sets, each of size $r - 1$, contained in the r -edge $v_1v_2 \dots v_r$. Each of them is a set with positive co-degree, hence each has neighborhood of size greater than $\frac{n}{r}$. Since there are r such sets, there must be a vertex, say v_{r+1} , which is contained in at least two such neighborhoods. Relabeling if needed, we may assume that

$$v_{r+1} \in N(v_1, \dots, v_{r-2}, v_{r-1}) \cap N(v_1, \dots, v_{r-2}, v_r).$$

The three r -edges $v_1v_2 \dots v_{r-2}v_{r-1}v_r$, $v_1v_2 \dots v_{r-2}v_{r-1}v_{r+1}$, and $v_1v_2 \dots v_{r-2}v_rv_{r+1}$ form T^r . Thus, $\frac{1}{r} \geq \gamma^+(T^r) \geq \gamma^+(\mathcal{F})$. This implies that blow-ups of T^r are \mathcal{F} -free.

Now assume that forbidding \mathcal{F} does not exclude any blow-up of T^r . Consider the following n -vertex blow-up of T^r . The three vertices corresponding to the base triangle T^2 are blown up to classes of size $\frac{n}{2r-1}$. All other vertices are blown up to classes of size $\frac{2n}{2r-1}$. In total, we have 3 classes of size $\frac{n}{2r-1}$ and $r - 2$ classes of size $\frac{2n}{2r-1}$, for a total of n vertices, as desired. A set of $r - 1$ vertices whose intersection with any class has at least two vertices will have co-degree 0. A set of $r - 1$ vertices which intersects all three classes of size $\frac{n}{2r-1}$ will also have co-degree 0. All other sets of size $r - 1$ have neighborhood of size exactly $\frac{2n}{2r-1}$, corresponding either to two classes of size $\frac{n}{2r-1}$ or one class of size $\frac{2n}{2r-1}$. This construction implies $\gamma^+(\mathcal{F}) \geq \frac{2}{2r-1}$.

Now, we can characterize families of r -graphs \mathcal{F} with $\gamma^+(\mathcal{F}) \in \{0, \frac{1}{r}\}$. Corollary 3.5 establishes that if $F \in \mathcal{F}$ is r -partite, then $\gamma^+(F) = 0$, so $\gamma^+(\mathcal{F}) = 0$ as well. If no $F \in \mathcal{F}$ is r -partite, then any blow-up of an r -edge is \mathcal{F} -free, so $\gamma^+(\mathcal{F}) \geq \frac{1}{r}$. Thus, $\gamma^+(\mathcal{F}) = 0$ if and only if some $F \in \mathcal{F}$ is r -partite. Similarly, if an $F \in \mathcal{F}$ is contained in a blow-up of T^r , then

$$\gamma^+(\mathcal{F}) \leq \gamma^+(F) \leq \gamma^+(T^r) \leq \frac{1}{r}.$$

Thus, if some $F \in \mathcal{F}$ is contained in some T^r blow-up but no member of \mathcal{F} is r -partite, then we have $\gamma^+(F) = \frac{1}{r}$. On the other hand, if no member of \mathcal{F} is contained in any blow-up of T^r , then the above-described blow-up of T^r establishes that $\gamma^+(\mathcal{F}) \geq \frac{2}{2r-1}$. \square

Remark 4.1. The characterization in Theorem 2.1 implies the positive co-degree densities for a variety of natural 3-graphs. It is straightforward to verify that C_ℓ^- (the ℓ -vertex (tight) cycle with one edge deleted) is contained in a sufficiently large blow-up of K_4^{3-} . Moreover, C_ℓ^- is 3-partite if and only if $\ell \equiv 0 \pmod{3}$. Thus, $\gamma^+(C_\ell^-) = 0$ when $\ell \equiv 0 \pmod{3}$, and $\gamma^+(C_\ell^-) = \frac{1}{3}$ otherwise. This generalizes the result of Wu [45] that $\gamma^+(C_5^-) = \frac{1}{3}$.

The next natural question is whether Theorem 2.1 is best possible. That is, does there exist some family \mathcal{F} of r -graphs with $\gamma^+(\mathcal{F}) = \frac{2}{2r-1}$? We answer this question in the affirmative when $r = 3$. Furthermore, we show that an infinite number of densities in the interval $[\frac{2}{5}, \frac{1}{2}]$ are achievable when $r = 3$. We begin by exhibiting a family \mathcal{F} with $\gamma^+(\mathcal{F}) = \frac{2}{5}$. We first define $F_{3,2}^{+++}$ and $F_{3,2}^{++2}$, two superhypergraphs of $F_{3,2}$. Each is created by adding two edges to $F_{3,2}$. Let $F_{3,2}$ have vertex set $\{a, b, c, d, e\}$ and edge set $\{abc, abd, abe, cde\}$. Then we create $F_{3,2}^{+++}$ by adding acd and ace , and we create $F_{3,2}^{++2}$ by adding acd and bce , see Figure 1.

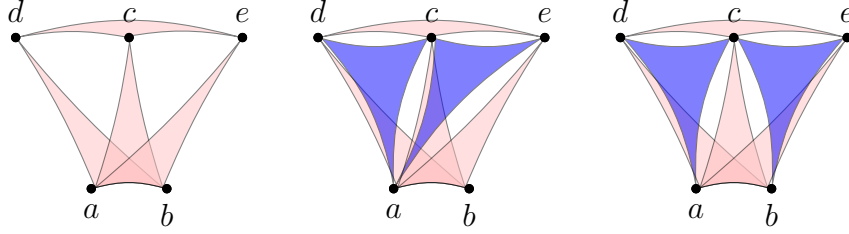


Figure 1: $F_{3,2}$ and two superhypergraphs, $F_{3,2}^{+++}$ and $F_{3,2}^{++2}$.

Proposition 4.2. For $\mathcal{F} = \{K_4^3, F_{3,2}^{+++}, F_{3,2}^{++2}, J_4\}$ and $\mathcal{F}' = \{K_4^3, F_{3,2}, J_4\}$ we have

$$\gamma^+(\mathcal{F}) = \gamma^+(\mathcal{F}') = \frac{2}{5}.$$

Proof. Since $F_{3,2}$ is a subgraph of both $F_{3,2}^{+++}, F_{3,2}^{++2}$, hence $\gamma^+(\mathcal{F}) \geq \gamma^+(\mathcal{F}')$. As noted in the proof of Theorem 2.1, an appropriately balanced n -vertex blow-up of $T^3 = K_4^{3-}$ has minimum positive co-degree $\frac{2n}{5} + O(1)$. Any blow-up of K_4^{3-} is J_4 -free and $F_{3,2}$ -free since K_4^{3-} is 4-partite and neither of $F_{3,2}$ and J_4 are. Since blow-ups K_4^{3-} are also K_4^3 -free, we have $\frac{2}{5} \leq \gamma^+(\mathcal{F}') \leq \gamma^+(\mathcal{F})$.

Next we show $\gamma^+(\mathcal{F}) \leq \frac{2}{5}$, which completes the proof. Fix an $\varepsilon > 0$ and suppose that H is an n -vertex 3-graph with $\delta_2^+(H) \geq (\frac{2}{5} + \varepsilon)n$ for sufficiently large n . Since $\gamma^+(K_4^{3-}) = \frac{1}{3}$, H contains K_4^{3-} , say with vertex set $\{a, b, c, d\}$ and edge set $\{abc, abd, acd\}$.

Consider the following five positive co-degree pairs: ab, ac, ad, bc , and bd . Since $\delta_2^+(H) \geq (\frac{2}{5} + \varepsilon)n$, there exists some vertex e that is in the neighborhood of at least three of these pairs. Note that c and d are symmetric; up to this symmetry, we have six cases based on which three of these five pairs form an edge with e .

- (1) If $abe, ace, ade \in E(H)$ then $abc, abd, acd, abe, ace, ade$ form a J_4 .
- (2) If $abe, ace, bce \in E(H)$ then abc, abe, ace, bce form a K_4^3 .
- (3) If $abe, ace, bde \in E(H)$ then acb, acd, ace, bde form an $F_{3,2}$. Moreover, abd and abe are 3-edges, so $\{a, b, c, d, e\}$ spans an $F_{3,2}^{+++}$.
- (4) If $ace, ade, bce \in E(H)$ then adb, adc, ade, bce form an $F_{3,2}$. Moreover, abc and ace are 3-edges, so $\{a, b, c, d, e\}$ spans an $F_{3,2}^{+++}$.
- (5) If $abe, bce, bde \in E(H)$ then bea, bec, bed, acd form an $F_{3,2}$. Moreover, bac and bad are 3-edges, so $\{a, b, c, d, e\}$ spans an $F_{3,2}^{+++}$.

- (6) If $ace, bce, bde \in E(H)$ then acb, acd, ace, bde form an $F_{3,2}$. Moreover, abd and cbe are 3-edges, so $\{a, b, c, d, e\}$ spans an $F_{3,2}^{+++}$.

This implies $\gamma^+(\mathcal{F}) \leq \frac{2}{5}$. □

We use Proposition 4.2 as the base of an inductive argument, showing that an infinite number of positive co-degree densities in $[\frac{2}{5}, \frac{1}{2}]$ are achievable by families of 3-graphs.

Proof of Theorem 2.2. For a lower bound, observe that any blow-up of J_{k-1} is $\{K_4^3, F_{3,2}, J_k\}$ -free, and that an appropriately balanced n -vertex blow-up of J_{k-1} (with class sizes $\frac{k-2}{2k-3}, \frac{1}{2k-3}, \dots, \frac{1}{2k-3}$) has minimum positive co-degree $(\frac{k-2}{2k-3})n + O(1)$. Thus, $\gamma^+(\{K_4^3, F_{3,2}, J_k\}) \geq \frac{k-2}{2k-3}$ for all $k \geq 4$.

To show that this lower bound is best-possible, we use induction on k . Proposition 4.2 yields the statement for $k = 4$. We assume the statement holds for $k \geq 4$, and prove it for $k + 1$. Suppose that H is an n -vertex 3-graph with minimum positive co-degree $\delta_2^+(H) > \frac{k-1}{2k-1}n$ where n is sufficiently large. If H contains one of K_4^3 or $F_{3,2}$, then we are done, so suppose not. Then by the inductive hypothesis, $\gamma^+(\{K_4^3, F_{3,2}, J_k\}) = \frac{k-2}{2k-3} < \frac{k-1}{2k-1}$, so H contains a J_k , say J (here we use that n is large enough). Let $V(J) = \{v_1, \dots, v_{k+1}\}$, where v_1 is the universal vertex of J (i.e., $E(J) = \{v_1 v_i v_j : 2 \leq i < j \leq k+1\}$). Define

$$S = \{(v_1, v_i) : 2 \leq i \leq k+1\} \cup \{(v_i, v_{i+1}) : 2 \leq i \leq k\}.$$

Observe that $|S| = 2k - 1$ and every vertex pair in S has positive co-degree in H . Since $\delta_2^+(H) > \frac{k-1}{2k-1}n$, there exists a vertex $u \in V(H)$ which is in the neighborhood of at least k pairs $(v_i, v_j) \in S$. Note that u may be an element of $V(J)$. However, if $(v_i, v_j) \in S$ is a pair such that $u \in N(v_i, v_j)$, then $u \notin \{v_i, v_j\}$.

If $u \in N(v_1, v_i)$ for every $i \in \{2, \dots, k+1\}$ then $u \notin V(J)$, and $V(J) \cup \{u\}$ spans J_{k+1} , with universal vertex v_1 .

Hence, we may assume that there is some $i \geq 2$ for which $u \in N(v_i, v_{i+1})$. Our goal in this case is to find $F_{3,2}$. Observe that there must be a j such that $u \in N(v_1, v_j)$. If there is such a $j \notin \{i, i+1\}$, then $u \notin \{v_1, v_i, v_{i+1}, v_j\}$, and $\{v_1, v_i, v_{i+1}, v_j, u\}$ will span an $F_{3,2}$, using 3-edges $v_1 v_j v_i, v_1 v_j v_{i+1}, v_1 v_j u, v_i v_{i+1} u$. Hence, we may assume that every such j is in $\{i, i+1\}$. In particular, u is in at most two neighborhoods of the form $N(v_1, v_j)$. Since $k \geq 4$ and u is in the neighborhood of at least k pairs from S , we have $u \in N(v_\ell, v_{\ell+1})$ for some $\ell \neq i$. Now, if $u \in N(v_1, v_i)$ and $u \in N(v_1, v_{i+1})$, then $\{v_1, v_i, v_{i+1}, u\}$ spans a K_4^3 . If not, then without loss of generality $u \in N(v_1, v_i)$ and u is in the neighborhood of the three pairs $(v_i, v_{i+1}), (v_\ell, v_{\ell+1}), (v_m, v_{m+1})$ for some ℓ, m . One of these pairs, say $(v_\ell, v_{\ell+1})$, must be disjoint from (v_1, v_i) , in which case we observe that $\{v_1, v_i, v_\ell, v_{\ell+1}, u\}$ will span an $F_{3,2}$. □

Although infinitely many positive co-degree densities in $[\frac{2}{5}, \frac{1}{2}]$ can be achieved by forbidden families of 3-graphs, it remains unclear whether all of these densities can be achieved by forbidding a single 3-graph. Using Proposition 4.2, we now show that $\frac{2}{5}$ is indeed achievable by a single 3-graph. We begin by defining two new 3-graphs F_1 and F_2 as follows.

$$\begin{aligned}
V(F_1) &= \{a, b, c, d, e, f, g\}, & E(F_1) &= \{abe, ace, bce, abf, adf, bdf, cdg\}, \\
V(F_2) &= \{a, b, c, d, e, g\}, & E(F_2) &= \{abe, ace, bce, ade, bde, cdg\}.
\end{aligned}$$

We depict F_1 and F_2 in Figure 2. We remark that each of F_1, F_2 can be viewed as a partial identification of two K_4^{3-} copies, along with an extra 3-edge (using g) which ensures that c and d have positive co-degree.

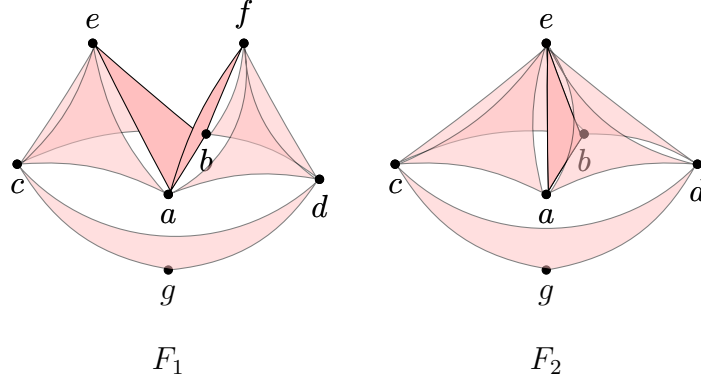


Figure 2: F_1 and F_2 .

Note that in both F_1 and F_2 , the vertex g has very little structural interaction with the other vertices. Suppose H is isomorphic to the subhypergraph of F_1 (resp. F_2) induced on $V(F_1) \setminus \{g\}$ (resp. $V(F_2) \setminus \{g\}$). Then H is guaranteed to extend to F_1 (resp. F_2) if $d(c, d) > 4$. We will be working in 3-graphs with minimum positive co-degree much larger than 4, so to find copies of F_1 or F_2 , it will suffice to find $F_1 \setminus \{g\}$ or $F_2 \setminus \{g\}$ and to demonstrate that the pair $\{c, d\}$ has positive co-degree.

Note that F_2 is 5-partite, so $\gamma^+(F_2) \geq \frac{1}{2}$. However, F_1 is 4-partite, and is contained in a blow-up of J_4 , but it is not contained in a blow-up of $J_3 = K_4^{3-}$. Thus, $\gamma^+(F_1) \geq \frac{2}{5}$. We now prove that $\gamma^+(F_1) = \frac{2}{5}$.

Proof of Theorem 2.3. For a contradiction, assume that there exists $\varepsilon > 0$ such that $\gamma^+(F_1) \geq \frac{2}{5} + 3\varepsilon$. We begin with a claim that will allow us to expand our forbidden family.

Claim 4.3. *The 3-blow-up of each member of the following family*

$$\mathcal{F} = \{K_4^3, J_4, F_{3,2}^{+++}, F_2\}$$

contains F_1 .

Proof. Observe that F_2 can be obtained from F_1 by identifying e and f . Using the notation

$$\begin{aligned}
V(K_4^3) &= \{1, 2, 3, 4\}, & E(K_4^3) &= \{123, 124, 134, 234\}, \\
V(J_4) &= \{1, 2, 3, 4, 5\}, & E(J_4) &= \{123, 124, 125, 134, 135, 145\}, \\
V(F_{3,2}^{+++}) &= \{1, 2, 3, 4, 5\}, & E(F_{3,2}^{+++}) &= \{123, 124, 125, 345, 134, 135\}
\end{aligned}$$

the following maps prove the claim for K_4^3 , J_4 , and $F_{3,2}^{+++1}$.

$$f_1 : F_1 \rightarrow K_4^3 : a \rightarrow 1, b \rightarrow 2, c \rightarrow 3, d \rightarrow 4, e \rightarrow 4, f \rightarrow 3, g \rightarrow 1.$$

$$f_2 : F_1 \rightarrow J_4 : a \rightarrow 2, b \rightarrow 3, c \rightarrow 4, d \rightarrow 5, e \rightarrow 1, f \rightarrow 1, g \rightarrow 1.$$

$$f_3 : F_1 \rightarrow F_{3,2}^{+++1} : a \rightarrow 2, b \rightarrow 3, c \rightarrow 4, d \rightarrow 5, e \rightarrow 1, f \rightarrow 1, g \rightarrow 3.$$

□

Thus, by Lemma 3.6, for n large enough there exists a 3-graph H with $\delta_2^+(H) \geq (\frac{2}{5} + \varepsilon)n$ which is $\{F_1\} \cup \mathcal{F}$ -free. By Proposition 4.2,

$$\gamma^+(\{K_4^3, J_4, F_{3,2}^{+++1}, F_{3,2}^{+++2}\}) = \frac{2}{5},$$

so H contains $F_{3,2}^{+++2}$, which we shall call F . Put

$$V(F) = \{a, b, c, d, e\}, \quad \text{with} \quad E(F) = \{abc, abd, abe, cde, acd, bce\}.$$

Observe that every pair of vertices in $V(F)$ has positive co-degree. Since $\delta_2^+(H) \geq (\frac{2}{5} + \varepsilon)n$, there exists some vertex f that is in the neighborhood of at least five pairs from $V(F)$. Let G_f be the link graph of vertex f induced on vertex set $V(F)$ (i.e. $G_f = L(f)[V(F)]$).

Claim 4.4. *The following statements hold:*

- (i) *For every $xyz \in E(H)$, at most two of xy, xz, yz are in $E(G_f)$.*
- (ii) *There is no isolated vertex in G_f .*
- (iii) *At most one of ad, cd is in $E(G_f)$ and at most one of be, ce is in $E(G_f)$.*
- (iv) *At most one of cd, de is in $E(G_f)$ and at most one of ce, de is in $E(G_f)$.*
- (v) *At most one of ab, ac is in $E(G_f)$ and at most one of ab, bc is in $E(G_f)$.*
- (vi) *At most one of ab, ad is in $E(G_f)$ and at most one of ab, be is in $E(G_f)$.*
- (vii) *At most one of ac, ad is in $E(G_f)$ and at most one of bc, be is in $E(G_f)$.*
- (viii) *G_f is triangle-free.*

Proof. To prove each statement, we shall assume that the statement does not hold and use this assumption to find some forbidden structure in H . Notice that the two parts in each of the statements (iii)–(vii) are symmetric so it is sufficient to prove only the first part of each of those claims. Here the symmetry is coming from the fact that $a \rightarrow b, b \rightarrow a, c \rightarrow c, d \rightarrow e, e \rightarrow d$ is an automorphism of F .

For (i), observe that if $xyz \in E(H)$ and $xy, xz, yz \in E(G_f)$, then $\{x, y, z, f\}$ spans a K_4^3 in H .

For (ii), assume that G_f contains a vertex of degree 0. Then the other four vertices in $V(F)$ span (at least) five edges of G_f , so G_f contains a K_4^- . As illustrated in Figure 3, a K_4^-

in G_f implies that H contains F_2 if the appropriate pair of vertices (x and y , in the figure) has positive co-degree. Since all pairs of vertices in F have positive co-degree, a K_4^- in G_f indeed implies that H contains F_2 .

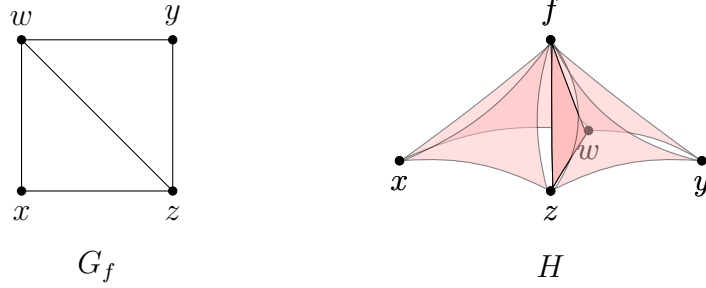


Figure 3: K_4^- in G_f and the resulting structure in H .

For (iii), observe that if $ad, cd \in E(G_f)$, then $\{a, c, d, f\}$ spans K_4^{3-} with spike vertex d , and $\{a, b, c, e\}$ spans K_4^{3-} with spike vertex b , so $\{a, b, c, d, e, f\}$ spans F_1 (since $d(e, f) > 0$ by (ii)).

For (iv), if $cd, de \in E(G_f)$, then $\{c, d, e, f\}$ spans K_4^{3-} with spike vertex d , and $\{a, b, c, e\}$ spans K_4^{3-} with spike vertex b , so $\{a, b, c, d, e, f\}$ spans F_1 (since $d(a, f) > 0$ by (ii)).

For (v), if $ab, ac \in E(G_f)$, then $\{a, b, c, d\}$ and $\{a, b, c, f\}$ span copies of K_4^{3-} with spike vertex a , so $\{a, b, c, d, f\}$ spans F_2 (since $d(d, f) > 0$ by (ii)).

For (vi), if $ab, ad \in E(G_f)$, then $\{a, b, c, d\}$ and $\{a, b, d, f\}$ span copies of K_4^{3-} with spike vertex a , so $\{a, b, c, d, f\}$ spans F_2 (since $d(c, f) > 0$ by (ii)).

For (vii), if $ac, ad \in E(G_f)$, then $\{a, b, c, d\}$ and $\{a, c, d, f\}$ span copies of K_4^{3-} with spike vertex a , so $\{a, b, c, d, f\}$ spans F_2 (since $d(b, f) > 0$ by (ii)).

Finally, we show (viii). From (i), we know that $\{c, d, e\}$ does not form a triangle in G_f , and from (v) and (vi) it follows that ab is not contained in a triangle in G_f . Thus, any triangle in G_f must include one edge spanned by $\{c, d, e\}$, and two edges with one vertex in $\{a, b\}$ and the other in $\{c, d, e\}$. By (i), $\{a, c, d\}$ does not span a triangle (nor does $\{b, c, e\}$). Up to symmetry, there are two other potential triangles: $\{a, d, e\}$, and $\{a, c, e\}$. Observe that if $\{a, d, e\}$ spans a triangle in G_f , then $\{a, d, e, f\}$ spans K_4^{3-} with spike vertex f , and $\{a, b, c, e\}$ spans K_4^{3-} with spike vertex b , so $\{a, b, c, d, e, f\}$ spans F_1 (since $d(c, d) > 0$).

Next, suppose $\{a, c, e\}$ spans a triangle in G_f . By (iii), (iv), (v), and (vii), none of be, de, ab, ad are in $E(G_f)$. Thus, two of bc, bd, cd are in $E(G_f)$. We shall show that this is impossible. Observe that if $bc \in E(G_f)$, then $\{a, b, c, f\}$ and $\{b, c, e, f\}$ each spans a K_4^{3-} with spike vertex c , so $\{a, b, c, e, f\}$ spans F_2 (since $d(a, e) > 0$). Moreover, if $cd \in E(G_f)$, then $\{a, c, d, f\}$ and $\{c, d, e, f\}$ span K_4^{3-} with spike vertex c , so $\{a, c, d, e, f\}$ spans F_2 (since $d(a, e) > 0$). Thus, G_f is triangle-free. \square

With Claim 4.4 established, we are ready to determine the structure of G_f . By (i), at most two out of five edges of G_f are spanned by $\{c, d, e\}$, so one of a, b has degree at least 2. Without loss of generality, $d(a) \geq d(b)$ and $d(a) \geq 2$.

Observation 4.5. $ab \notin E(G_f)$.

Proof. Suppose to the contrary that $ab \in E(G_f)$. By (v) and (vi), ac and ad are not in $E(G_f)$, so we must have $ae \in E(G_f)$. By (v) and (vi), we also have $bc, be \notin E(G_f)$. Since G_f

has five edges, at least three of bd, cd, ce, de are in $E(G_f)$. By (iv), at most one of ce, de is in $E(G_f)$, so $bd, cd \in E(G_f)$. Also by (iv), if $cd \in E(G_f)$, then $de \notin E(G_f)$, so $ce \in E(G_f)$. Since $ae \in E(G_f)$, there exists a vertex x not in $V(F)$ that is a common neighbor of a and e . The set of edges $\{adb, dce, fab, fbd, fdc, fce, aex\}$ forms F_1 . See Figure 4 for an illustration of G_f, F , and F_1 . \square

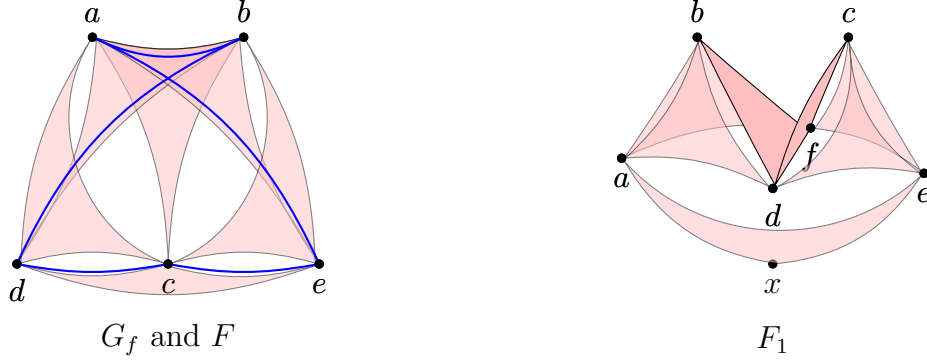


Figure 4: The configuration when $ab \in E(G_f)$, and the resulting copy of F_1 .

Since $d(a) \geq 2$, two of ac, ad, ae are in $E(G_f)$. By (vii), at most one of ac, ad is in $E(G_f)$, so $E(G_f)$ contains ae and exactly one of ac, ad . Suppose first that $ad \in E(G_f)$. By (iii), $cd \notin E(G_f)$, and by (viii), $de \notin E(G_f)$. Since $d(b) \leq d(a) = 2$, we must have $ce \in E(G_f)$. Now, by (iii), $be \notin E(G_f)$, so we must have $bc, bd \in E(G_f)$. Since $ae \in E(G_f)$, there exists a vertex x not in $V(F)$ that is a common neighbor of a and e . The set of edges $\{adb, bce, fad, fbd, fbc, fce, aex\}$ forms F_1 . See Figure 5 for an illustration of G_f, F , and F_1 .

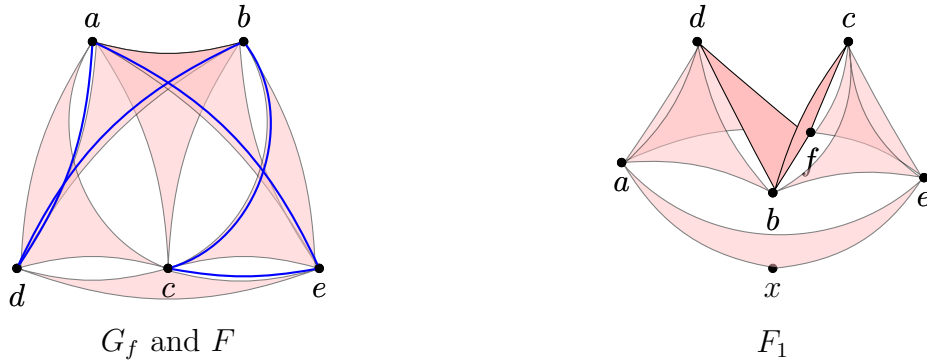


Figure 5: The configuration when $ad, ae \in E(G_f)$, and the resulting copy of F_1 .

Thus, we have $ac, ae \in E(G_f)$ and $ad \notin E(G_f)$. By (viii), $ce \notin E(G_f)$, and by (iv), at most one of cd, de is in $E(G_f)$. Thus, $d(b) = 2$. By (vii), at most one of bc, be is in $E(G_f)$, hence $bd \in E(G_f)$. Suppose for a contradiction that $be \in E(G_f)$. Then by (viii), $de \notin E(G_f)$. Since G_f has at least 5 edges, $dc \in E(G_f)$. Since $bd \in E(G_f)$, there exists a vertex x not in $V(F)$ that is a common neighbor of b and d . The set of edges

$\{abe, adc, fbe, fae, fac, fcd, bdx\}$ forms F_1 . See Figure 6 for an illustration of G_f , F , and F_1 .

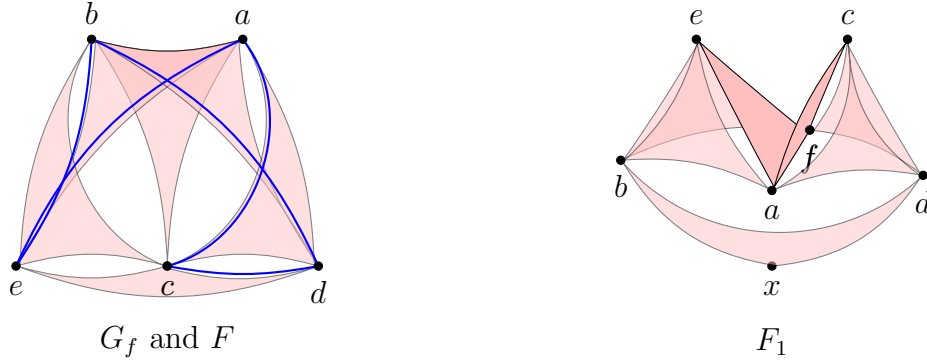


Figure 6: The configuration when $be \in E(G_f)$, and the resulting copy of F_1 .

Thus, $bc \in E(G_f)$. By (viii), $cd \notin E(G_f)$, so we must have $de \in E(G_f)$. Hence $E(G_f) = \{ae, ac, bc, bd, de\}$; see Figure 7 for an illustration.

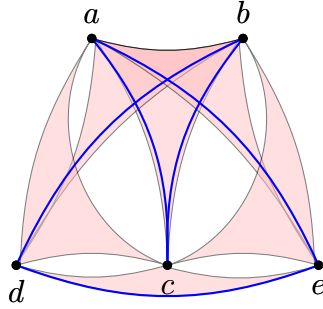


Figure 7: F and G_f when $ac, ae \in E(G_f)$.

Unlike in the previous cases, we cannot immediately find F_1 (or any other forbidden hypergraph) in Figure 7. However, we now have that the subhypergraph of H induced on $\{a, b, c, d, e, f\}$ has edge set $\{abc, abd, abe, adc, bce, cde, fac, fcb, fbd, fde, fea\}$. We call this subhypergraph F' , and we shall use F' to find some forbidden hypergraph.

Observe first that each of the 15 pairs of vertices in F' have positive co-degree. Thus, there exists some $g \in V(H)$ that is in the neighborhood of at least $15 \left(\frac{2}{5} + \varepsilon\right) > 6$ pairs. Let G_g be the link graph of vertex g induced on $V(F')$ (i.e. $G_g = L(g)[V(F')]$). Again, we begin with some observations on G_g .

Claim 4.6. $\delta(G_g) \geq 1$. Moreover, in G_g , $N(f) = \{c, d, e\}$.

Proof. As in the proof of Claim 4.4, since all pairs from $\{a, b, c, d, e, f\}$ have positive co-degree, G_g cannot contain K_4^- .

If G_g has an isolated vertex v , then $G_g - v$ is a component on 5 vertices and 7 edges that necessarily contains a K_4^- . Thus, every vertex of G_g has positive degree.

Next, observe that the previous analysis of G_f in fact shows that if some vertex $v \in V(H) \setminus V(F)$ has positive co-degree with at least 5 pairs from $V(F)$, then the link graph of v induced on $V(F)$ must be equal to G_f . In particular, either G_g contains G_f or g has positive co-degree with at most 4 pairs from $V(F)$. Observe that if G_g contains G_f , then $\{a, b, c, f\}$ and $\{a, b, c, g\}$ span copies of K_4^{3-} with spike vertex c . This implies that $\{a, b, c, f, g\}$ spans F_2 , since we have argued that $d(f, g) > 0$. Thus, g has positive co-degree with at most 4 pairs from $V(F)$, which implies that f has degree at least 3 in G_g , since $|E(G_g)| \geq 7$.

Finally, to determine $N(f)$ in G_g , we consider the interaction between G_g and G_f . Suppose that x, y are neighbors of f in G_g such that $xy \in E(G_f)$. Then $\{x, y, f, g\}$ forms K_4^{3-} with spike vertex f . We consider the possible values of x, y ; we know $xy \in \{ac, ae, bc, bd, de\}$. Since $\{a, b, c, d\}$ spans K_4^{3-} with spike vertex a , and we have established that g has positive co-degree with every vertex in $V(F)$, we can find F_1 in H on vertex set $\{a, b, c, d, f, g\}$ if $xy \in \{bc, bd\}$. Similarly, if $xy \in \{ac, ae\}$, then we can find F_1 on $\{a, b, c, e, f, g\}$. Thus, in G_g , either $N(f)$ is an independent set or $N(f)$ contains precisely the edge de . Recall that $|N(f)| \geq 3$, so the first outcome is impossible, as the independence number of G_f is 2. The second outcome occurs only if $N(f) = \{c, d, e\}$. \square

With Claim 4.6 established, we conclude that $|E(G_g)| = 7$, with exactly 4 edges of G_g spanned by $\{a, b, c, d, e\}$. Given that $cf, df, ef \in E(G_g)$, we shall prove that this is not possible.

First, observe that none of cd, ce, de are in $E(G_g)$. Indeed, if $de \in E(G_g)$, then $\{d, e, f, g\}$ spans K_4^3 (recall that $de \in E(G_f)$). If either cd or ce is in $E(G_g)$, then either $\{c, d, f, g\}$ or $\{c, e, f, g\}$ spans K_4^{3-} with spike vertex g ; if $\{c, d, f, g\}$ spans K_4^{3-} , then $\{a, b, c, d, f, g\}$ spans an F_1 , while if $\{c, e, f, g\}$ spans K_4^{3-} , then $\{a, b, c, e, f, g\}$ spans an F_1 .

Notice that Claim 4.4 applies also to G_g since Claim 4.6 proves (ii) and it is the only statement that used the number of edges of G_f . Next, we prove that $ab \notin E(G_g)$. Indeed, by (v) and (vi), if $ab \in E(G_g)$, then ac, bc, ad, be are not in $E(G_g)$. Recall cd, ce, de are not in $E(G_g)$. So if $ab \in E(G_g)$, then at most three edges of G_g are spanned by $\{a, b, c, d, e\}$. However, by Claim 4.6, $d(f) = 3$ in G_g , so we have $|E(G_g)| \leq 6$, a contradiction. Thus, $ab \notin E(G_g)$.

Finally, by (vii), at most two of ac, ad, bc, be are in $E(G_g)$. So in order to have $|E(G_g)| = 7$, both ae and bd must be in $E(G_g)$. However, this results in F_1 on $\{a, b, d, e, g, f\}$ with edges $\{age, afe, abc, gfe, gfd, gbd, bdf\}$ as depicted in Figure 8. \square

5 Positive co-degree densities from flag algebras

This section contains calculations using flag algebras, as introduced by Razborov [39]. For an introduction to flag algebras and formal definitions, see [18, 19, 39]. Computer code is available at <https://lidicky.name/pub/pc/>.

We use flag algebras to prove Theorem 2.4 that $\gamma^+(J_4) = 4/7$, with the asymptotically unique construction being the balanced blow-up of the complement of the Fano plane. In Figure 9, we illustrate (a blow-up of) the Fano plane, with a particular labeling of classes

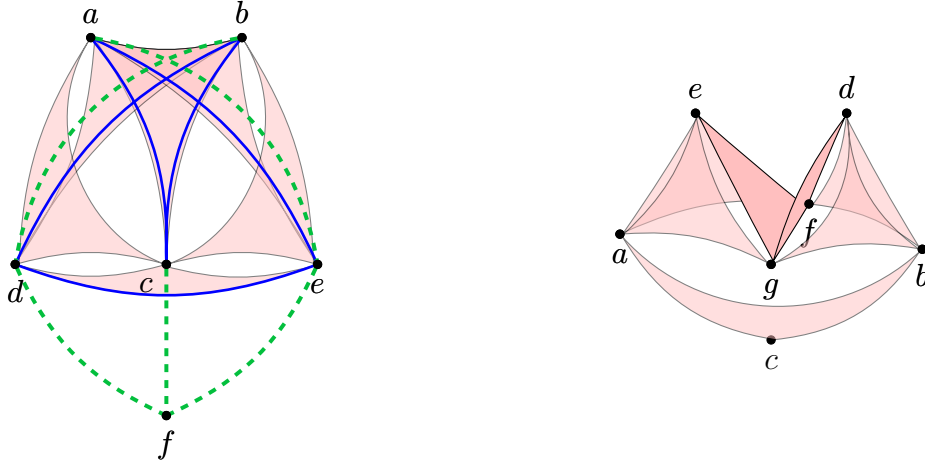


Figure 8: F' (edges involving f indicated by G_f in thick blue) and a configuration in G_g (in dashed green) yielding F_1 .

to which we will later refer. We denote the complement of the Fano plane by $\overline{\mathbb{F}}$ and the n -vertex, balanced blow-up of the complement of the Fano plane by $\overline{\mathbb{F}}_n$.

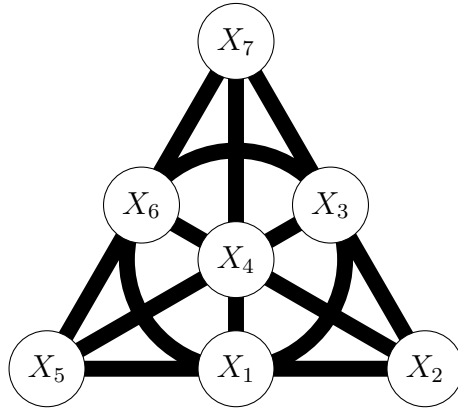


Figure 9: A blow-up of the Fano plane. In $\overline{\mathbb{F}}_n$, 3-edges span triples of classes which do not span 3-edges in the blow-up of \mathbb{F} .

Let \mathcal{F} be the family of thirteen 3-graphs 6-vertex induced subgraphs of $\overline{\mathbb{F}}_n$, depicted in Figure 10. We include labels to indicate which classes of $\overline{\mathbb{F}}_n$ each vertex belongs to, corresponding to the labeling in Figure 9. Note that due to the symmetries of $\overline{\mathbb{F}}_n$, the indicated labelings are not the unique ways a subgraph can be obtained.

Let \mathcal{F}_6 be the family of J_4 -free 6-vertex 3-graphs.

The following claim shows that if a J_4 -free n -vertex 3-graph G has minimum positive co-degree at least $\frac{4}{7}n$ then every 3-graph on six vertices with positive density in G is in \mathcal{F} .

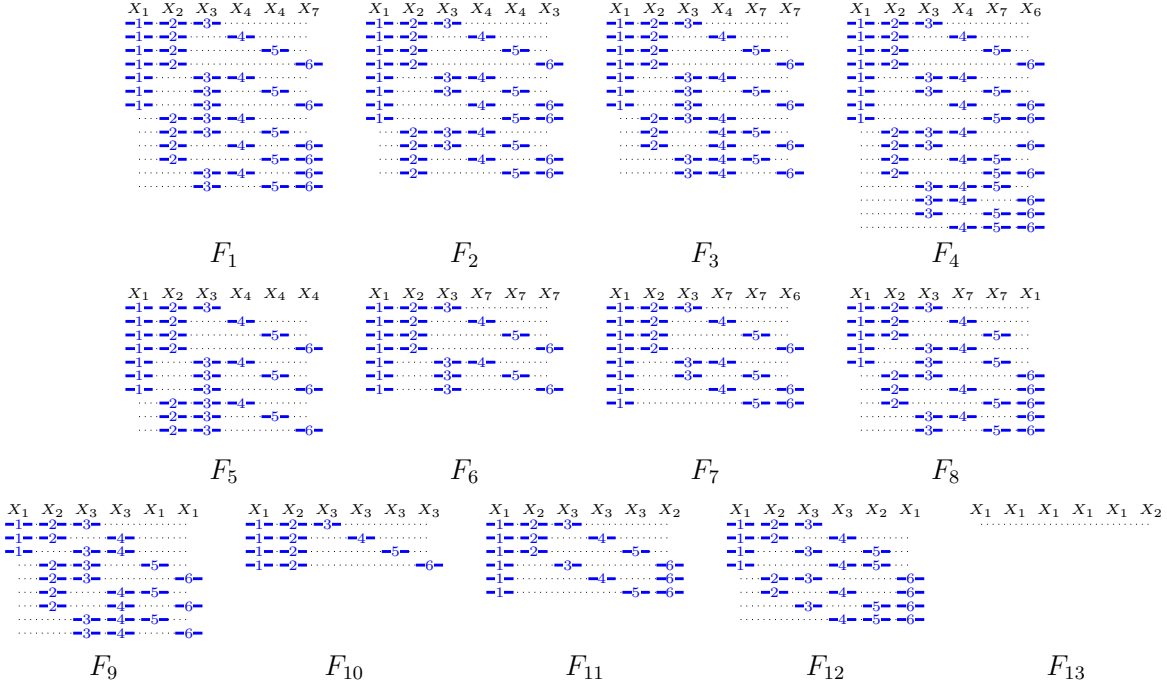


Figure 10: The family \mathcal{F} of thirteen 6-vertex 3-graphs.

Claim 5.1. *For every fixed $\delta > 0$, there exists n_0 such that for every $n \geq n_0$, if G_n is a J_4 -free n -vertex 3-graph with $\delta_2^+(G_n) \geq \frac{4n}{7}$, then*

$$\sum_{H \in \mathcal{F}_6 \setminus \mathcal{F}} d(H, G_n) \leq \delta.$$

Proof. The claim is proved by a standard application of flag algebras. The notable part is encoding the condition $\delta_2^+(G_n) \geq \frac{4n}{7}$ into flag algebras by (5.1).

$$0 \leq \begin{array}{c} \blacksquare \blacksquare \bullet \\ \hline \text{---} \text{---} \text{---} \\ \hline \text{---} \text{---} \text{---} \\ \hline \text{---} \text{---} \text{---} \\ \hline \text{---} \text{---} \text{---} \\ \hline \text{---} \text{---} \text{---} \end{array} \times \left(7 \begin{array}{c} \blacksquare \blacksquare \bullet \\ \hline \text{---} \text{---} \text{---} \\ \hline \text{---} \text{---} \text{---} \\ \hline \text{---} \text{---} \text{---} \\ \hline \text{---} \text{---} \text{---} \\ \hline \text{---} \text{---} \text{---} \end{array} - 4 \begin{array}{c} \blacksquare \blacksquare \\ \hline \text{---} \text{---} \end{array} \right). \quad (5.1)$$

If two labeled vertices 1 and 2 have zero co-degree, the right-hand side of (5.1) is zero because of the first term. If they have positive co-degree, both terms in the product are non-negative.

The calculation is computer-assisted and too large to fit in this paper. The details are available at <https://lidicky.name/pub/pc/>. \square

In the proof of Theorem 2.4 we will use an induced removal lemma.

Theorem 5.2 (Induced Removal Lemma, [1,40]). *Let $r, C \in \mathbb{Z}$ and $\varepsilon > 0$ be fixed. For every family of r -graphs \mathcal{F} on at most C vertices, there exists $\delta > 0$ such that every sufficiently large n , every r -graph on n vertices, which contains at most $\delta n^{|V(F)|}$ induced copies of F for every $F \in \mathcal{F}$, can be made induced \mathcal{F} -free by adding and/or deleting at most εn^r hyperedges.*

Claim 5.1 together with Theorem 5.2 forces 3-graphs with positive co-degree at least $\frac{4}{7}n$ to be highly structured, which is the core of the proof of Theorem 2.4.

Proof of Theorem 2.4. Recall that $\overline{\mathbb{F}}_n$ is the blow-up of the complement of the Fano plane (see Figure 9) on vertices x_1, \dots, x_7 , where each vertex x_i is blown-up to $\frac{n}{7}$ vertices into the set X_i . Let $x \in X_i$ and $y \in X_j$ for some $i, j \in [7]$. If $i = j$ then $d(x, y) = 0$. If $i \neq j$ then $d(x, y) = \frac{4}{7}n$. As $J_4 \not\subseteq \overline{\mathbb{F}}_n$, $\text{co}^+\text{ex}(n, J_4) \geq \frac{4}{7}n + o(n)$, which implies $\gamma^+(J_4) \geq 4/7$.

Next, we show that $\gamma^+(J_4) \leq 4/7$. Fix $\beta > 0$ and $\varepsilon > 0$ small enough such that $24\varepsilon^{1/4} < \frac{1}{12} \left(\frac{4}{7}\right)^3 \approx 0.015$ and $48\varepsilon^{1/4} < \beta$. We shall fix n sufficiently large such that Claim 5.1 and Theorem 5.2 imply that every J_4 -free, n -vertex 3-graph G has a subgraph G' at edit distance at most εn^3 from G such that every 6-vertex induced subgraph of G' is in \mathcal{F} . We will also take n sufficiently large that Lemma 3.3 may be applied with $c = \frac{4}{7}$ and any n -vertex 3-graph H with $\delta_2^+(H) \geq \left(\frac{4}{7} - 48\varepsilon^{1/4}\right)n$ contains a K_4^3 . This last condition is possible because $\gamma^+(K_4^3) \leq 0.543$ (see Table 1), and $0.543 < \frac{4}{7} - 48\varepsilon^{1/4}$ by the choice of ε .

Fix an n -vertex J_4 -free 3-graph G_n with $\delta_2^+(G_n) \geq \frac{4}{7}n$. Our goal is to show that $\delta_2^+(G_n) < \left(\frac{4}{7} + \beta\right)n$, which will establish $\gamma^+(J_4) \leq \frac{4}{7}$. The proof will also imply that in fact, the balanced blow-up of the complement of the Fano plane is the asymptotically unique extremal construction.

By Lemma 3.3 and the choice of ε , G_n contains more than $24\varepsilon^{1/4}n^3$ hyperedges. We can thus apply Lemma 3.2 to conclude that G'_n contains an n -vertex subgraph G''_n with $\delta_2^+(G''_n) \geq \left(\frac{4}{7} - 48\varepsilon^{1/4}\right)n$. Since G''_n may not be a subgraph of G_n , we apply Lemma 3.2 to the subgraph of G'_n obtained by deleting any 3-edges that were added to G_n by the application of Theorem 5.2. Thus, G''_n contains K_4^3 , say on vertices v_1, v_2, v_3, v_4 . Hence $\{v_1, v_2, v_3, v_4\}$ also spans K_4^3 in G'_n . A search through \mathcal{F} shows that there are only two possible subgraphs on 5 vertices containing K_4^3 , which we label A and B below.



We now partition $V(G'_n)$ into 7 sets X_1, \dots, X_7 as follows. Put $v_i \in X_i$ for $i \in [4]$. For $v \in V(G'_n) \setminus \{v_1, v_2, v_3, v_4\}$, define $G_v := G_n[v_1, v_2, v_3, v_4, v]$. If G_v is isomorphic to A then put $v \in X_i$ where $d_{G_v}(v, v_i) = 0$ for some v_i . If G_v is isomorphic to B then put $v \in X_5$ if $v_1v_2v, v_3v_4v \notin E(G_v)$ or $v \in X_6$ if $v_1v_3v, v_2v_4v \notin E(G_v)$ or $v \in X_7$ if $v_1v_4v, v_2v_3v \notin E(G_v)$. This addresses every vertex v , so we have a partition of $V(G'_n)$ into X_1, \dots, X_7 . This labeling matches that in Figure 9.

Now, we show that G'_n is in fact a blow-up of the complement of the Fano plane with classes X_1, \dots, X_7 .

Claim 5.3. G'_n satisfies the following conditions.

- (i) For every $i \in [7]$, no edge of G'_n intersects X_i in more than one vertex.
- (ii) If $a \in X_i, b \in X_j, c \in X_k$ for classes X_i, X_j, X_k which correspond to an edge of the Fano plane as labeled in Figure 9, then $abc \notin E(G'_n)$.

(iii) If $a \in X_i, b \in X_j, c \in X_k$ for distinct classes X_i, X_j, X_k which do not correspond to an edge of the Fano plane as labeled in Figure 9, then $abc \in E(G'_n)$.

In particular, G'_n is a blow-up of the complement of the Fano plane.

Proof. We prove conditions (i), (ii), and (iii) one by one; in each case, we will argue that if the condition is not satisfied, then G'_n must contain some subgraph that is not in \mathcal{F} . Throughout, refer to Figure 10 for the labeled members of \mathcal{F} . We implemented this check by a computer to reduce the number of cases needed to be done by hand.

For (i), suppose for a contradiction that there exists an edge abc such that a, b are in the same class of G'_n . First note that by the definition of X_1, \dots, X_7 , if abc either intersects $\bigcup_{i=1}^4 X_i$ in at most two vertices or abc is contained in X_i for some $i \in [4]$, then there exists $\{i, j, k\} \subset [4]$ such that two of $N(x_i x_j)$, $N(x_i x_k)$, and $N(x_j x_k)$ contain a, b, c . Up to symmetry, we may assume $a, b, c \in N(x_1 x_2)$ and $N(x_1 x_3)$. The set of edges of an induced subgraph H of G'_n on $\{x_1, x_2, x_3, a, b, c\}$ includes

$$E = \{x_1 x_2 x_3, x_1 x_2 a, x_1 x_2 b, x_1 x_2 c, x_1 x_3 a, x_1 x_3 b, x_1 x_3 c, abc\}.$$

Since no graph of \mathcal{F} contains a subset of edges isomorphic to E , we have $H \notin \mathcal{F}$, a contradiction. We use an analogous claim repeatedly. We use computer for verification as well as arguments by hand.

Thus, if edge abc exists, we must have $a, b \in X_i$ and $c \in X_j$ for some $i \neq j$ with $i, j \in [4]$. Note that at most one of a, b, c is in $\{x_1, x_2, x_3, x_4\}$, since by the definition of the classes, no edge containing x_i and x_j for $\{i, j\} \in [4]$ intersects X_i or X_j . We have (up to symmetry of the classes) two cases.

Case 1: $a, b \in X_3, c \in X_4$, and $x_3 \notin \{a, b\}$. Define $H = G'_n[x_1, x_2, x_3, a, b, c]$. The set of edges of H includes

$$E = \{x_1 x_2 x_3, x_1 x_2 a, x_1 x_2 b, x_1 x_2 c, x_1 x_3 c, abc\}$$

and avoids edges $N = \{x_1 x_3 a, x_1 x_3 b, x_2 x_3 a, x_2 x_3 b\}$. Since no graph in \mathcal{F} contains a subset of edges isomorphic to E and avoids N , we have $H \notin \mathcal{F}$, a contradiction.

Case 2: $a = x_3, b \in X_3$, and $c \in X_4$. Consider the subgraph H of G'_n induced on $\{x_1, x_2, x_3, x_4, b, c\}$. The set of edges of H includes

$$E = \{x_1 x_2 x_3, x_1 x_2 x_4, x_1 x_3 x_4, x_2 x_3 x_4, x_1 x_2 b, x_1 x_4 b, x_2 x_4 b, x_1 x_2 c, x_1 x_3 c, x_2 x_3 c, abc\}.$$

Since no graph in \mathcal{F} contains a subset of edges isomorphic to E , we have $H \notin \mathcal{F}$, a contradiction.

For (ii), suppose for a contradiction that there exists an edge abc spanning three classes that correspond to an edge in Figure 9. Up to symmetry, there are two cases.

Case 1: The edge abc intersects $\bigcup_{i=1}^4 X_i$ in two vertices. Without loss of generality, $a \in X_1, b \in X_4$, and $c \in X_7$. Observe that we cannot have both $a = x_1$ and $b = x_4$ as otherwise $c \notin X_7$ by definition of the classes, $N(x_1, x_4)$ is disjoint from X_7 . Without loss of

generality, $b \neq x_4$. We consider the subgraph H of G'_n induced on $\{x_2, x_3, x_4, a, b, c\}$. The set of edges of H includes

$$E = \{x_2x_3x_4, x_2x_3a, x_2x_4a, x_3x_4a, x_2x_3b, x_2x_4c, x_3x_4c, abc\}$$

and avoids edges containing both x_4 and b . Since no graph in \mathcal{F} contains a subset of edges isomorphic to E while avoiding edges containing both x_4 and b , we have $H \notin \mathcal{F}$, a contradiction.

Case 2: The edge abc does not intersect $\bigcup_{i=1}^4 X_i$. Without loss of generality, $a \in X_5, b \in X_6$, and $c \in X_7$. We define the subgraph H of G'_n induced on $\{x_1, x_2, x_3, a, b, c\}$. The set of edges of H includes

$$E = \{x_1x_2x_3, x_1x_2b, x_1x_2c, x_1x_3a, x_1x_3c, x_2x_3a, x_2x_3b, abc\}$$

and avoids edges $N = \{x_1x_2a, x_2x_3c, x_1x_3b\}$. Since no graph in \mathcal{F} contains a subset of edges isomorphic to E while avoiding N we have $H \notin \mathcal{F}$, a contradiction.

Finally, for (iii), suppose for a contradiction that there exist vertices $a \in X_i, b \in X_j, c \in X_k$ such that X_i, X_j, X_k do not correspond to an edge in Figure 9 and $abc \notin E(G'_n)$. Up to symmetry of classes, there are three cases.

Case 1: We have $i, j, k \in [4]$. Without loss of generality, $a \in X_1, b \in X_2$, and $c \in X_3$. By the definition of X_1, X_2, X_3 , note that at most one of a, b, c is in $\{x_1, x_2, x_3\}$. Without loss of generality, $b \neq x_2$ and $c \neq x_3$. We define the subgraph H of G'_n induced on $\{x_2, x_3, x_4, a, b, c\}$. The set of edges of H includes

$$E = \{x_2x_3x_4, x_2x_3a, x_2x_4a, x_3x_4a, x_3x_4b, x_2x_4c\}$$

and avoids edges $N = \{abc, x_2x_3b, x_2x_4b, x_2x_3c, x_3x_4c\}$. Since, no graph of \mathcal{F} contains a subset of edges isomorphic to E while avoiding edges in N , $H \notin \mathcal{F}$, a contradiction.

Case 2: Precisely two of i, j, k are in $[4]$. Without loss of generality, $a \in X_1, b \in X_2$, and $c \in X_6$. Observe that by the definition of X_6 , at most one of a, b is in $\{x_1, x_2\}$; without loss of generality, $b \neq x_2$. We define the subgraph H of G'_n induced on $\{x_2, x_3, x_4, a, b, c\}$. The set of edges of H includes

$$E = \{x_2x_3x_4, x_2x_3a, x_2x_4a, x_3x_4a, x_3x_4b, x_2x_3c, x_3x_4c\}$$

and avoids edges $N = \{abc, x_2x_3b, x_2x_4b, x_2x_4c\}$. Since no graph in \mathcal{F} contains a subset of edges isomorphic to E while avoiding edges in N , we have $H \notin \mathcal{F}$, a contradiction.

Case 3: Exactly one of i, j, k is in $[4]$. Without loss of generality, $a \in X_1, b \in X_5$, and $c \in X_6$.

We define the subgraph H of G'_n induced on $\{x_2, x_3, x_4, a, b, c\}$. The set of edges of H includes

$$E = \{x_2x_3x_4, x_2x_3a, x_2x_4a, x_3x_4a, x_2x_3b, x_2x_4b, x_3x_4b, x_2x_3c, x_3x_4c\}$$

and avoids edges $N = \{abc, x_2x_4c, x_3x_4b\}$. Since, no graph of \mathcal{F} contains a subset of edges isomorphic to E while avoiding edges in N , $H \notin \mathcal{F}$, a contradiction.

We conclude that all conditions (i), (ii), and (iii) hold, i.e., G'_n is a blow-up of $\overline{\mathbb{F}}$. \square

Finally, we show that G'_n is almost balanced. Recall that G'_n contains a subgraph G''_n with $\delta_2^+(G''_n) \geq (\frac{4}{7} - 48\varepsilon^{1/4})n$. Fix $\alpha \geq 0$ so that a largest class in G'_n has size at least $(\frac{1}{7} + \alpha)n$. Without loss of generality, X_1 is a largest class. We bound the co-degree in G''_n of pairs containing vertices in X_1 . Observe that there are three sets of classes disjoint from X_1 that appear as neighborhoods of vertex pairs in G''_n . Namely,

$$N(x_1, x_2) \subseteq X_3 \cup X_4 \cup X_6 \cup X_7;$$

$$N(x_1, x_3) \subseteq X_2 \cup X_4 \cup X_5 \cup X_7;$$

$$N(x_1, x_4) \subseteq X_2 \cup X_3 \cup X_5 \cup X_6.$$

Thus, in G''_n , we have

$$d(x_1, x_2) + d(x_1, x_3) + d(x_1, x_4) \leq \sum_{i=2}^7 2|X_i| \leq \left(\frac{12}{7} - 2\alpha\right)n.$$

By averaging, one of $d(x_1, x_2), d(x_1, x_3), d(x_1, x_4)$ is at most $(\frac{4}{7} - \frac{2\alpha}{3})n$. Thus, $\delta_2^+(G''_n) \leq \frac{4n}{7}$, a contradiction if $\delta_2^+(G_n) \geq (\frac{4}{7} + \beta)n$. We conclude that $\gamma^+(J_4) \leq \frac{4}{7}$. To see that G''_n should be approximately balanced, note that by the minimum positive co-degree condition on G''_n , we thus must have $\frac{2\alpha}{3} \leq 48\varepsilon^{1/4}$, i.e., G''_n (and G'_n) contains no class of size larger than $(\frac{1}{7} + 72\varepsilon^{1/4})n$. This upper bound implies that G'_n contains no class of size smaller than $(\frac{1}{7} - 432\varepsilon^{1/4})n$. \square

We determine the positive co-degree density and the asymptotically unique extremal construction for $F_{4,2}$. Since the proof is analogous to (but simpler than) the proof of Theorem 2.4, we only include a sketch.

Sketch of the proof of Theorem 2.4. Observe first that $F_{4,2}$ is 6-partite, so it is not contained in the balanced blow-up of K_5^3 , which implies $\gamma^+(F_{4,2}) \geq \frac{3}{5}$. We now show that $\gamma^+(F_{4,2}) \leq \frac{3}{5}$. Let \mathcal{F} be the family of seven 6-vertex 3-graphs depicted in Figure 11.

Observe that \mathcal{F} consists of the empty 3-graph and the 6-vertex blow-ups of K_3^3, K_3^4 , and K_5^3 . Thus, \mathcal{F} is exactly the set of 6-vertex induced subgraphs of the balanced blow-up of K_5^3 .

Using flag algebras, we show that for sufficiently large n , if G_n is a $F_{4,2}$ -free n -vertex 3-graph with $\delta_2^+(G_n) \geq \frac{3n}{5}$, then the 6-vertex 3-graphs with positive density in G_n are in \mathcal{F} . The details of the calculations by computer are available at <https://lidicky.name/pub/pco/>. Using the induced removal lemma (Theorem 5.2), G_n has a small edit distance to G'_n , where every 6-vertex subgraph vertices belongs to \mathcal{F} .

Recall that $\gamma^+(K_4^3) \leq 0.543 < \frac{3}{5}$, so by Lemma 3.2, G'_n contains a subgraph with minimum positive co-degree larger than $(0.543 + \varepsilon)n$ for some $\varepsilon > 0$. Thus, we can assume G'_n contains K_4^3 , say on vertices v_1, v_2, v_3, v_4 . A search through \mathcal{F} shows that there are only two

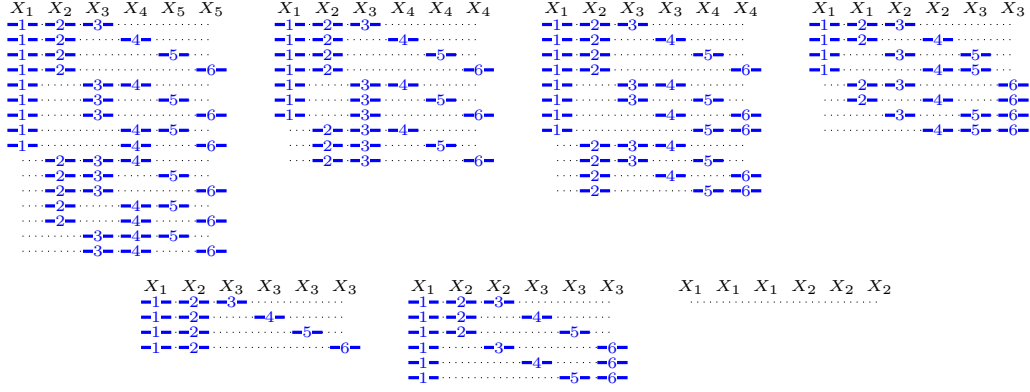


Figure 11: The family \mathcal{F} of seven 6-vertex 3-graphs.

possible subgraphs on 5 vertices containing K_4^3 .



Note that A is the (unique) 5-vertex blow-up of K_4^3 and $B = K_5^3$. We now partition $V(G'_n)$ into five sets X_1, \dots, X_5 as follows. Put $v_i \in X_i$ for $i \in [4]$. For $v \in V(G'_n) \setminus \{v_1, v_2, v_3, v_4\}$, define $G_v := G_n[v_1, v_2, v_3, v_4, v]$. If G_v is isomorphic to A , then put $v \in X_i$, where $d_{G_v}(v, v_i) = 0$. If G_v is isomorphic to B , then put $v \in X_5$.

An inspection of cases establishes the following claim.

Claim 5.4. G'_n satisfies the following conditions.

- (i) For every $i \in [5]$, no edge of G'_n intersects X_i in more than one vertex.
- (ii) For every $a \in X_i, b \in X_j$, and $c \in X_k$ with i, j, k pairwise distinct, $abc \in E(G'_n)$.

In particular, G'_n is a blow-up of K_5^3 . From here, the positive co-degree condition can be used to establish that the vertex-partition of G'_n is essentially balanced. \square

Finally, Theorem 2.6 was proved using flag algebras. The certificates for the proofs are available at <https://lidicky.name/pub/pco/>. These bounds are unlikely to be tight.

6 Concluding remarks

While we significantly expand the known sets of jumps and achievable values for γ^+ , the general behavior of γ^+ remains mysterious, even for $r = 3$. It is unclear whether γ^+ has a jump everywhere, though we conjecture that more jumps exist than are characterized in Theorem 2.1.

Question 1. For $r \geq 3$, which values of $\alpha \in [\frac{2}{2r-1}, 1]$ are γ^+ -jumps? Does there exist an $\alpha \in [\frac{2}{2r-1}, 1]$ which is *not* a γ^+ -jump?

Many more concrete questions could be asked when $r = 3$. For example, it is unclear, how far Theorem 2.2 is from completely characterizing achievable densities in the range $[0, \frac{1}{2}]$ when $r = 3$.

Question 2. Are there achievable values of γ^+ in $[\frac{2}{5}, \frac{1}{2}]$ that are not of the form $\frac{k-2}{2k-3}$, when $r = 3$?

A negative answer to Question 2 would suggest some similarity between γ^+ and π , since the extremal constructions in Theorem 2.2 are a fairly natural analogue of Turán graphs. However, note that there is no hope for an Erdős-Stone-Simonovits-type result giving values of γ^+ . Indeed, since a balanced blow-up of K_4^3 has positive co-degree density $\frac{1}{2}$, every 3-graph F with $0 < \gamma^+(F) < \frac{1}{2}$ is 4-partite.

To begin addressing either Question 1 or 2, it seems natural to start with the next interval between known achievable values. Even this next case seems difficult.

Question 3. For $r = 3$, is every $\alpha \in [\frac{2}{5}, \frac{3}{7})$ a γ^+ -jump? Is there a family \mathcal{F} with $\gamma^+(\mathcal{F}) \in (\frac{2}{5}, \frac{3}{7})$?

The use of flag algebras introduces the potential for new approaches to positive co-degree questions, particularly when combined with proving the existence of γ^+ -jumps. As illustrated by Theorems 2.4 and 2.5, flag algebra calculations have the potential to directly determine values of γ^+ , and even inexact bounds (e.g., $\gamma^+(K_4^3) \leq 0.543$) given by flag algebras are sometimes substantially better and more useful than what seems tractable by hand. When combined with known jumps of the function γ^+ , flag algebra bounds also have the potential to produce exact results. For instance, any 3-graph F that can be shown via a flag calculation to have $\gamma^+(F) < \frac{2}{5}$ must have $\gamma^+(F) \in \{0, \frac{1}{3}\}$. Thus, obtaining estimates via flags and “rounding down” via known jumps is a very efficient way to determine the densities of many small 3-graphs. Since we can directly characterize those 3-graphs with positive co-degree density in $\{0, \frac{1}{3}\}$, it is now also possible to directly determine by inspection whether a fixed 3-graph F has $\gamma^+(F) \in \{0, \frac{1}{3}\}$; however, it seems unlikely that the set

$$\mathcal{F}(r, d) := \{F \text{ an } r\text{-graph} : \gamma^+(F) = d\}$$

can be as simply characterized for other values of d , even when $r = 3$. We are interested to see whether further understanding of jumps will include characterizations of this type. If they do not, the potential combination of estimated densities with the theory of jumps is an appealing approach to determining densities exactly.

It is also open whether every achievable value of γ^+ can be achieved as the density of a single r -graph. Theorem 2.3 shows that $\frac{2}{5}$ is achievable by a single 3-graph, as is every achievable density known outside the interval $(\frac{2}{5}, \frac{1}{2})$.

Question 4. For a fixed $k \geq 5$, is there a 3-graph F_k such that $\gamma^+(F_k) = \frac{k-2}{2k-3}$? More generally, if \mathcal{F} is a family of r -graphs with $\gamma^+(\mathcal{F}) = \alpha$, does there always exist a single r -graph F with $\gamma^+(F) = \alpha$?

Every 3-graph that is not the subgraph of a (blow-up of a) suspension will have positive co-degree density at least $\frac{1}{2}$. The complete list of known achievable densities at least $\frac{1}{2}$ is as follows: $\frac{1}{2}$ (achieved by $F_{3,2}$), $\frac{4}{7}$ (achieved by J_4), $\frac{3}{5}$ (achieved by $F_{4,2}$), and $\frac{2}{3}$ (achieved by the Fano plane).

Question 5. For $r = 3$, find other achievable values for γ^+ larger than $\frac{1}{2}$. Is there an $\alpha \in [\frac{1}{2}, 1)$ which is a γ^+ -jump? Is there an $\alpha \in [\frac{1}{2}, 1)$ for which we can characterize the 3-graphs with $\gamma^+(F) = \alpha$?

Very little is known about the γ^+ function for r -graphs when $r \geq 4$. A natural starting point would be to study extensions of 3-graphs whose positive co-degree densities are known. For example, the r -daisy \mathcal{D}_r is the 6-edge r -graph on $r+2$ vertices whose all six edges contain the same $r-2$ vertices and each pair of the remaining 4 vertices is in one edge. There was a recent breakthrough on the Turán density of r -daisies [14]. Note that J_4 is the 3-daisy.

Question 6. What is $\gamma^+(\mathcal{D}_r)$ for $r \geq 4$?

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