Polychromatic Colorings on the Hypercube

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Abstract

Given a subgraph G of the hypercube Q_n, a coloring of the edges of Q_n such that every embedding of G contains an edge of every color is called a G-polychromatic coloring. The maximum number of colors with which it is possible to G-polychromatically color the edges of *any* hypercube is called the polychromatic number of G. To determine polychromatic numbers, it is only necessary to consider a structured class of colorings, which we call simple. The main tool for finding upper bounds on polychromatic numbers is to translate the question of polychromatically coloring the hypercube so every embedding of a graph G contains every color into a question of coloring the 2-dimensional grid so that every so-called *shape sequence* corresponding to G contains every color. After surveying the tools for finding polychromatic numbers, we apply these techniques to find polychromatic numbers of a class of graphs called punctured hypercubes. We also consider the problem of finding polychromatic numbers in the setting where larger subcubes of the hypercube are colored. We exhibit two new constructions which show that this problem is not a straightforward generalization of the edge coloring problem.

Keywords. Polychromatic, Coloring, Hypercube, coloring, Turán.

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1 Introduction

For $n \in \mathbb{Z}$, $n \geq 1$, the *n-dimensional hypercube*, denoted by Q_n, is the graph with $V(Q_n) = \{0,1\}^n$, and edges between vertices which differ in exactly one coordinate. For any graphs G, H, a subgraph of H isomorphic to G is called an *embedding* of G in H. Given a set R of r colors, an edge coloring of a graph G with r colors is a surjective function $\chi : E(G) \to R$ assigning a color to each edge of G. All colorings of graphs will refer to edge colorings, unless otherwise noted. Given a fixed graph G, an edge coloring of a hypercube with r colors such that every embedding of G contains an edge of every color is called a *G-polychromatic r-coloring.* Given a graph G, denote by $p(G)$ the maximum number of colors with which it is possible to G-polychromatically color the edges of *any* hypercube. Call $p(G)$ the *polychromatic number* of G.

Motivated by Turán type problems on the hypercube, Alon, Krech, and Szabó [1] introduced the notion of polychromatic coloring on the hypercube and proved bounds for the polychromatic number of Q_d.

Theorem 1 (Alon, Krech, and Szabó [1]) *For all $d \geq 1$,*

$$\binom{d+1}{2} \geq p(Q_d) \geq \begin{cases} \frac{(d+1)^2}{4} & \text{if } d \text{ is odd} \\ \frac{d(d+2)}{4} & \text{if } d \text{ is even.} \end{cases}$$

The exact value of the polychromatic number of Q_d was determined in [5].

Theorem 2 (Offner [5]) *For all $d \geq 1$,*

$$p(Q_d) = \begin{cases} \frac{(d+1)^2}{4} & \text{if } d \text{ is odd} \\ \frac{d(d+2)}{4} & \text{if } d \text{ is even.} \end{cases}$$

Prior to the work of Alon, Krech, and Szabó [1], coloring arguments had also been used to give bounds on Turán type problems on the hypercube, for example by Conder [4] and Axenovich and Martin [2]. In [6], a condition was given which, if satisfied by a graph G, implies $p(G) \geq 3$.

In this paper we begin by surveying what is known about polychromatic colorings on the hypercube. In Section 2 we establish that when studying polychromatic colorings, we need only consider a specific type of coloring called a *simple* coloring (such colorings were called *Ramsey* in [5]). In Section 3, we use the idea of simple coloring to transform the problem of edge coloring the hypercube so that a given subgraph is polychromatic to one of coloring a rectangular grid so that a collection of subsets is polychromatic. In this context, Lemma 5 provides the key insight to prove upper bounds on polychromatic numbers. All known lower bounds for polychromatic numbers come from explicit constructions, and at the end of the section we give an example by proving the lower bounds in Theorems 1 and 2.

Following this survey, we show in Section 4 how to use these methods to determine the value of $p(G)$ for some graphs G where $p(G)$ was not previously known, for example hypercubes with one edge or vertex deleted. We call these graphs punctured hypercubes. Theorem 10 gives the polychromatic number for any odd-dimensional punctured hypercube, and Theorems 11 and 12 give the polychromatic number for punctured Q_4's. For even-dimensional punctured hypercubes with dimension greater than 4, Theorem 13 provides a lower bound on the polychromatic number, but we can not determine it exactly. The current best bounds are summarized in Corollary 14. The section concludes with suggestions for future research.

Section 5 concerns a generalization of the problem proposed by Alon, Krech, and Szabó [1] where instead of edges, subcubes of a fixed dimension are colored. Previously, Özkahya and Stanton [7] had generalized the bounds given in Theorem 1 to this setting. If this more general problem were a straightforward generalization of the edge-coloring problem, the polychromatic number would be equal to the lower bound. However Theorems 20 and 21 provide two constructions that show this is not the case, and thus new ideas will be required to determine polychromatic numbers in this setting.

1.1 Notation for Hypercubes

We refer to the n coordinates of a vertex as *bits*, and given an edge $\{x, y\}$, we refer to the unique bit where $x_i \neq y_i$ as the *flip bit*. We represent an edge of Q_n by an n-bit vector with a star in the flip bit. For example, in Q_4, we represent the edge between vertices [0100] and [0101] by [010∗]. Similarly, we represent an embedding of Q_d in Q_n by an n-bit vector with stars in d coordinates. For instance $[1*00*]$ is the embedding of Q_2 in Q_5 with vertices $\{[10000], [11000], [10001], [11001]\}$ and edges $\{[1*000], [1000*], [1*001], [1100*]\}$. We call edges with the same flip bit *parallel*, and the class of edges with flip bit i the i^{th} *parallel class*. For an edge $e \in E(Q_n)$ with flip bit j define the prefix sum $l(e) = \sum_{i=1}^{j-1} x_i$ and postfix sum $r(e) = \sum_{i=j+1}^{n} x_i$.

2 Simple Colorings

Recall that for an edge $e \in E(Q_n)$, $l(e)$ is the number of 1's to the left of the star in e, and $r(e)$ is the number of 1's to the right. Call a coloring χ of the hypercube *simple* if $\chi(e)$ is determined by $l(e)$ and $r(e)$ (such colorings were called *Ramsey* in [5]). The following lemma tells us that when studying polychromatic colorings on the hypercube, we need only consider simple ones. The proof is essentially from [5], building on ideas from [1].

Lemma 3 *Let $k \geq 1$ and G be a subgraph of Q_k. If $p(G) = r$, then there is a simple G-polychromatic r-coloring on Q_k.*

Proof: Fix k. We show that if n is sufficiently large and Q_n has a G-polychromatic r-coloring, then it contains a subgraph Q_k with a simple coloring.

Suppose that n is large and χ is a G-polychromatic r-coloring of Q_n. We will use Ramsey's theorem for k-uniform hypergraphs with $r^{k2^{k-1}}$ colors. We define a $r^{k2^{k-1}}$-coloring of the k-subsets of $[n]$. Fix an arbitrary ordering of the edges of Q_k. For an arbitrary subset S of the indices, define $cube(S)$ to be the subcube whose $*$ coordinates are at the positions of S and all other coordinates are 0. Let S be a k-subset of $[n]$, and define the color of S to be the vector whose coordinates are the χ-values of the edges of the k-dimensional subcube $cube(S)$ (according to our fixed ordering of the edges of Q_k). By Ramsey's theorem, if n is large enough, there is a set $T \subseteq [n]$ of $k^2 + k - 1$ coordinates such that the color-vector is the same for any k-subset of T. Fix a set S of k particular coordinates from T: those which are the (ik)th elements of T for $i \in [k]$.

We show the coloring of $cube(S)$ is simple. Let e_1 and e_2 be two edges of $cube(S)$ such that $l(e_1) = l(e_2)$ and $r(e_1) = r(e_2)$. Since there are at least $k - 1$ elements in T in between each coordinate of S, as well as $k - 1$ elements to the left of the first coordinate of S and to the right of the last coordinate of S, there is a set of k coordinates $S' \subseteq T$ and an edge e_3 of $cube(S')$ such that

(i) e_3 is the same edge when restricted to S as e_1 and

(ii) e_3 occupies the same position in the ordering of edges in $cube(S')$ as e_2 occupies in $cube(S)$.

Thus $\chi(e_1) = \chi(e_3) = \chi(e_2)$, so the coloring of $cube(S)$ is a simple G-polychromatic r-coloring.

For example, suppose $k = 6$ and ignoring all coordinates not in T, suppose

$$e_1 = xxxxx0xxxxx1xxxxx1xxxxx*xxxxx0xxxxx1xxxxx$$
$$e_2 = xxxxx1xxxxx1xxxxx*xxxxx0xxxxx1xxxxx0xxxxx$$

where the coordinates in T but not S are represented by x. Then a possibility for e_3 is

$$e_3 = xxxxxxxxxxx1xxxxx1xxxxx*xxxxx0xxxxx10xxxx.$$

■

3 Techniques for Finding Bounds on $p(G)$

In a simple coloring of the hypercube, we refer to all edges e with the same value of $(l(e), r(e))$ as a *color class*. For example, all edges e with $l(e) = 2$ and $r(e) = 5$ are in color class $(2, 5)$. We begin with an elementary example of how Lemma 3 allows us to prove upper bounds on polychromatic numbers.

(0,0)						
(0,1)	(1,0)					
(0,2)	(1,1)	(2,0)				
(0,3)	(1,2)	(2,1)	(3,0)			
(0,4)	(1,3)	(2,2)	(3,1)	(4,0)		
(0,5)	(1,4)	(2,3)	(3,2)	(4,1)	(5,0)	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Figure 1: Initial part of the grid of color classes.

Proposition 4 *Denote by $Q_3 \setminus v$ the graph Q_3 with one vertex deleted. Then $p(Q_3 \setminus v) \leq 3$.*

Proof: By Lemma 3, we need to consider only simple colorings. Consider the embedding of $Q_3 \setminus v$ with the vertex [1110000...] deleted from the cube [***0000...]. This graph has edges in only three color classes, $(0,0)$, $(1,0)$, and $(0,1)$, and thus can only contain three colors in a simple coloring. ■

This example illustrates a general scheme for proving upper bounds on $p(G)$: Given a graph G, show that in an arbitrary simple coloring there is some embedding of G in Q_n that contains edges in only a small number of color classes. For instance, applying the argument of Proposition 4 to Q_d gives the upper bound in Theorem 1. To do better, we need Lemma 5.

Arrange the set of color classes in a rectangular grid, with the ith *row* containing the color classes (a,b), with $a+b=i$, and the ith *column* containing classes of the form (i,j), as shown in Figure 1. We translate the question of polychromatically coloring the hypercube so every embedding of a graph G contains every color into a question of coloring the grid of color classes so that every so-called *shape sequence* corresponding to G (which is defined below) contains every color.

Define a *region* of the grid to be all color classes contained in some consecutive rows and consecutive columns. A *shape* is a finite set of elements of the grid. Two shapes are *congruent* if one is a translation of the other, i.e. if $S = \{(a_1,b_1),(a_2,b_2),\ldots,(a_k,b_k)\}$ then $S' \cong S$ if and only if $S' = \{(a_1+i,b_1+j),(a_2+i,b_2+j),\ldots,(a_k+i,b_k+j)\}$ for some $i,j \in \mathbb{Z}$. The width $w(S)$ of a shape $S = \{(a_1,b_1),(a_2,b_2),\ldots,(a_k,b_k)\}$ is given by $\max_{i,j}|a_i - a_j|$. We say S is located at the column of its leftmost element, i.e. S is located at column $\min_i(a_i)$. A *shape list* is a finite list of shapes $S_1,\ldots,S_k$, with the

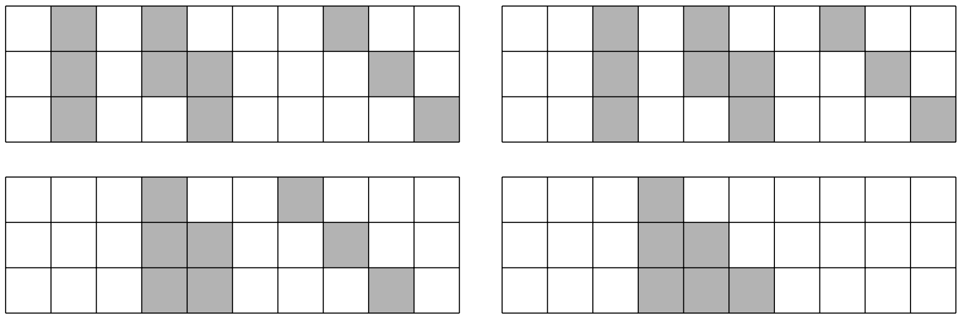

Figure 2: Four instances of a given shape sequence (for Q_3). In the two instances at the bottom, the shapes overlap, which is allowed as long as they remain in order.

restriction that if $i < j$ then S_i is not to the right of S_j. Two shape lists are *congruent* if each contains the same number of shapes, and corresponding shapes in the lists are congruent and are horizontal translations of each other. A *shape sequence* $\mathcal{S}$ is the set of all shape lists congruent to a specific list. An *instance* of a shape sequence $\mathcal{S}$ is one particular list–when the context is clear, we will not always distinguish between a shape sequence and an instance of a shape sequence, since specifying any instance determines all other instances of the sequence (see Figure 2). Let the width $w(\mathcal{S})$ of a sequence equal the width of its widest shape. For the height $h(\mathcal{S})$ of a sequence, if i_s is the smallest row index where some shape in $\mathcal{S}$ contains some element, and if i_l is the largest row index where some shape in $\mathcal{S}$ contains some element, let $h(\mathcal{S}) = i_l - i_s + 1$. Finally, given a shape sequence $\mathcal{S}$, let $p(\mathcal{S})$ be the maximum number of colors such that for any rectangular grid, there is a coloring of the elements of the grid so that every instance of $\mathcal{S}$ contains an element of every color.

Lemma 5 *Consider a shape sequence $\mathcal{S}$ of shapes $S_1, \ldots, S_k$, with elements in rows $i_s, \ldots, i_l$. Let X_j^i be the number of elements in S_j in row i, and let $X^i = \max_j X_j^i$. Then*

$$p(\mathcal{S}) \leq \sum_{i=i_s}^{i_l} X^i.$$

Proof: Consider a region R with $h(\mathcal{S})$ rows and n columns colored with colors $\{1, 2, \ldots, p(\mathcal{S})\}$. Assume that every instance of $\mathcal{S}$ in this region contains every color, i.e. it is not possible to find an instance of $\mathcal{S}$ in these rows where every shape in the sequence lacks a particular color. Thus for each color l, $1 \leq l \leq p(\mathcal{S})$, we can partition the interval $[1, n]$ into k_l intervals $[1, c_1^l), [c_1^l, c_2^l), \ldots, [c_{k_l-1}^l, n]$, with the property that $k_l \leq k$ and all copies of S_j located at columns in the jth interval contain color l. We adopt the convention that $c_0^l = 1$, $c_{k_l}^l = n$ and if $c_{j-1}^l = c_j^l$ then the interval $[c_{j-1}^l, c_j^l)$ is empty. The following procedure describes how to do this for a given color l.

1. Set $\alpha = 1$.

2. If all copies of S_α at locations $\geq c^l_{\alpha-1}$ contain color l, then
   - Set $c^l_\alpha = n$
   - Set $k_l = \alpha$.
   - STOP.
3. Else
   - Let c^l_α be the smallest number such that $c^l_\alpha \geq c^l_{\alpha-1}$ and the copy of S_α at column c^l_α does not contain color l.
   - Increment α by 1.
   - Return to Step 2.

The condition that every instance of $\mathcal{S}$ contains every color guarantees that this procedure returns a partition: If the procedure reaches a state where $\alpha = k$, then there are values $c^l_1 \leq c^l_2 \leq \cdots \leq c^l_{k-1}$ such that the shape S_j at location c^l_j does not contain color l. Thus since every instance of $\mathcal{S}$ contains l, every copy of S_k at location $\geq c^l_{k-1}$ must contain color l, and the procedure will terminate in Step 2. Let C be the set $\{1, c^1_1, c^1_2, \ldots, c^1_{k_1}, c^2_1, c^2_2, \ldots, c^2_{k_2}, \ldots, c^{p(\mathcal{S})}_1, c^{p(\mathcal{S})}_2, \ldots, c^{p(\mathcal{S})}_{k_{p(\mathcal{S})}}\}$. Relabel the elements of C so that $C = \{c_1, c_2, \ldots, c_q\}$ and $c_1 \leq c_2 \leq \cdots \leq c_q$. Since $q \leq k \cdot p(\mathcal{S}) + 1$ and we can choose n as large as we want, we can find a difference $c_p - c_{p-1}$ as large as we want. Choose n large enough so that this number $m = c_p - c_{p-1}$ is much bigger than $w(\mathcal{S})$. This difference corresponds to a region R' with m columns (the columns in the interval $[c_{p-1}, c_p)$), where for each color l there is a shape $S^l \in \mathcal{S}$ such that every copy of S^l contains the color l.

For each color l, and all $1 \leq i \leq h(\mathcal{S})$ let l_i be the number of times the color l appears in the ith row in R'. Any appearance of l in the ith row can be contained in at most X^i copies of S^l. There are at least $m - w(\mathcal{S})$ copies of S^l in the region, and thus

$$l_1 X^1 + l_2 X^2 + \cdots + l_{h(\mathcal{S})} X^{h(\mathcal{S})} \geq m - w(\mathcal{S}).$$

Since there are m columns in the region R', $1_i + 2_i + \cdots + p(\mathcal{S})_i = m$, and if we add up the equations for each color, we get

$$X^1 m + X^2 m + \cdots + X^{h(\mathcal{S})} m \geq (m - w(\mathcal{S})) p(\mathcal{S}).$$

To finish the proof, divide both sides by m, and note that by making m large, $(m - w(\mathcal{S}))/m$ can be as close to 1 as desired. ■

Now to prove upper bounds on $p(G)$, we translate problems about polychromatically coloring graphs into problems about polychromatically coloring shape sequences. We consider an arbitrary simple coloring of an enormous hypercube. Then we note that

(0,10)	(1,9)	(2,8)	(3,7)	(4,6)	(5,5)	(6,4)	(7,3)	(8,2)	(9,1)	(10,0)			
(0,11)	(1,10)	(2,9)	(3,8)	(4,7)	(5,6)	(6,5)	(7,4)	(8,3)	(9,2)	(10,1)	(11,0)		
(0,12)	(1,11)	(2,10)	(3,9)	(4,8)	(5,7)	(6,6)	(7,5)	(8,4)	(9,3)	(10,2)	(11,1)	(12,0)	
(0,13)	(1,12)	(2,11)	(3,10)	(4,9)	(5,8)	(6,7)	(7,6)	(8,5)	(9,4)	(10,3)	(11,2)	(12,1)	(13,0)

Figure 3: The shape sequence corresponding to the embedding [1101*100010*111*00101*] of Q_4 in Q_{22}.

the color classes covered by an embedding of G are a shape sequence $\mathcal{S}$ in the grid of color classes. Further, any instance of $\mathcal{S}$ corresponds to the color classes covered by the edges of some embedding of G. Since Lemma 5 gives an upper bound on $p(\mathcal{S})$, we get an upper bound on $p(G)$.

As an example of how to apply Lemma 5, we now prove the upper bound on $p(Q_d)$ in Theorem 2. This result was originally proved in [5], but this proof is more streamlined, and will provide useful preparation for proving new results later. Define an $i \times j$ *parallelogram* to be a set of color classes of the following form: $\{(a+\alpha, b+\beta) : 0 \leq \alpha < j, 0 \leq \beta < i\}$. We say that a color class is at coordinate (α, β) in such a parallelogram if it is of the form $(a+\alpha, b+\beta)$.

For an example of a shape sequence corresponding to an embedding of a graph, consider the embedding [1101*100010*111*00101*] of Q_4 in Q_{22}, see Figure 3. Edges using the leftmost star are in color classes (3,7), (3,8), (3,9), and (3,10), a 4×1 parallelogram. Edges using the second star from the left are in color classes (5,5), (5,6), (5,7), (6,5), (6,6), and (6,7), a 3×2 parallelogram. Edges using the third star from the left are in color classes (8,2), (8,3), (9,2), (9,3), (10,2), and (10,3), a 2×3 parallelogram. Edges using the fourth star from the left are in color classes (10,0), (11,0), (12,0), and (13,0), a 1×4 parallelogram. Thus the shape sequence corresponding to Q_4 consists of four parallelograms, all occupying the same four rows, where each parallelogram corresponds to the edges using one of the four stars. Further, we can create any other instance of this shape sequence in the same four rows by rearranging some of the stars. For example, [1101*100010*111**00101] would have the first three shapes identical, with the fourth shape shifted two columns to the left. These observations are generalized for Q_d in the following fact.

Fact 6 *Let $n \geq d \geq 1$. Every shape sequence for an embedding of Q_d in Q_n consists of d shapes $S_1, \ldots, S_d$ where S_i is a $(d-i+1) \times i$ parallelogram, and each shape occupies the same d rows. The color classes in S_i correspond to the edges using the*

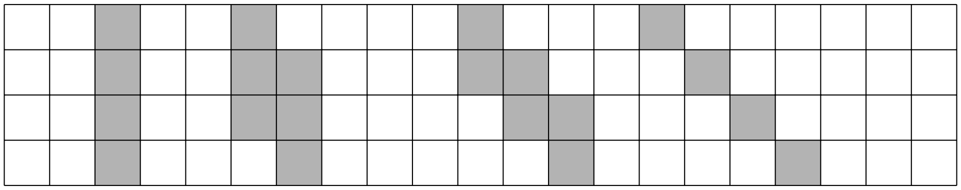

Figure 4: A shape sequence for Q_4.

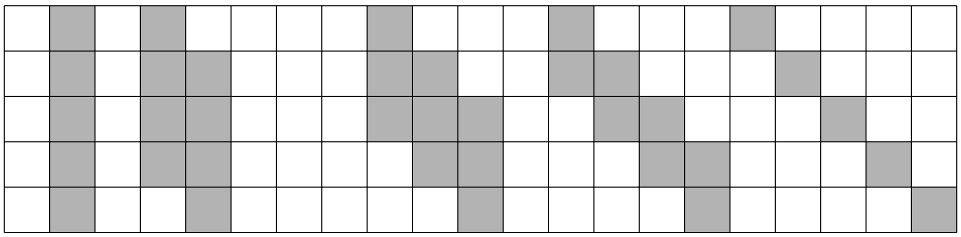

Figure 5: A shape sequence for Q_5.

ith star from the left. Conversely, every instance of such a shape sequence corresponds to some embedding of Q_d in Q_n.

See Figures 2, 4, and 5 for examples of shape sequences corresponding to Q_3, Q_4, and Q_5, respectively.

Proof of Theorem 2, upper bound: By Lemma 3, we may consider a simple Q_d-polychromatic $p(Q_d)$-coloring on an arbitrarily large hypercube. Fact 6 describes the shape sequence for Q_d. For any of the parallelograms in the shape sequence, the maximum number of color classes in the ith row is $\min\{i, d-i+1\}$. Thus, using the notation of Lemma 5, in the shape sequence for Q_d, $X^i = \max_j X^i_j = \min\{i, d-i+1\}$. Applying Lemma 5, we get $p(Q_d) \leq 1 + 2 + \cdots + \lceil d/2 \rceil + \cdots + 2 + 1 = (d+1)^2/4$ if d is odd, and $d(d+2)/4$ if d is even. ■

We now turn our attention to lower bounds. To prove lower bounds on $p(G)$, we explicitly describe a simple coloring of the hypercube by assigning a particular color to each color class. Then we analyze what color classes must be contained in the shape sequence of any embedding of the graph G, and show it contains all colors. For instance, here is a proof of the lower bound for Theorems 1 and 2:

Proof of Theorems 1 and 2, lower bound: Consider the simple coloring χ where $\chi(e) = \lceil \frac{d+1}{2} \rceil \cdot l(e) + r(e) \pmod q$, where $q = \frac{(d+1)^2}{4}$ if d is odd, and $q = \frac{d(d+2)}{4}$ if d is even. Then the $\lceil \frac{d}{2} \rceil$th shape in the shape sequence for Q_d is a $\lceil \frac{d+1}{2} \rceil \times \lceil \frac{d}{2} \rceil$ parallelogram, and thus contains q color classes. The elements in each column in this parallelogram contain $\lceil \frac{d+1}{2} \rceil$ consecutive colors $\pmod q$, and no two columns of the parallelogram share any colors, so this shape contains all q colors regardless of its position in the grid of color classes. Thus any shape sequence corresponding to Q_d contains all colors. ■

4 Polychromatic Numbers for Punctured Cubes

We now use the techniques of Section 3 to determine $p(G)$ for other subgraphs G of the hypercube besides subcubes. In each case, we explicitly describe a simple coloring to prove a lower bound on $p(G)$, and examine possible shape sequences corresponding to G and apply Lemma 5 to prove an upper bound.

We consider graphs which are obtained by deleting a vertex or edge from a hypercube. We call such graphs *punctured hypercubes* or *punctured cubes* for short. Let $Q_d \setminus v$ and $Q_d \setminus e$ denote the graphs obtained by deleting one vertex or one edge from Q_d, respectively. If d is odd we call these *odd punctured cubes*, and if d is even we call them *even punctured cubes.* When embedding a punctured cube in a larger hypercube, we can think of generating each embedding by first embedding Q_d, then deleting a given vertex or edge from the embedded graph (not from the base graph of course). The shape sequence corresponding to such an embedding depends on which vertex or edge is deleted. Since an embedding of Q_d in Q_n can be represented as an n-bit vector with d stars, we say that deleting the vertex corresponding to some d-bit string corresponds to deleting from the embedding of Q_d that vertex in Q_n which replaces the d stars with that given d-bit string. We also use the same notion for edges. For example, we might consider the subgraph $Q_3 \setminus v \subseteq Q_6$ corresponding to the embedding [01*0**] of Q_3 with the vertex corresponding to [011] deleted, so the embedded subgraph will not contain the vertex [010011] or any incident edges.

Proposition 7 *If G is a subgraph of H, and both are subgraphs of the hypercube, then $p(G) \leq p(H)$.*

Corollary 8 *For any $d \geq 2$,*

$$p(Q_{d-1}) \leq p(Q_d \setminus v) \leq p(Q_d \setminus e) \leq p(Q_d).$$

Letting $d = 2k$ when d is even, or $d = 2k - 1$ when d is odd, Theorem 2 states that $p(Q_{2k-1}) = k^2$ and $p(Q_{2k}) = k^2 + k$. Thus Corollary 8 implies for odd punctured cubes

$$k^2 - k = p(Q_{2k-2}) \leq p(Q_{2k-1} \setminus v) \leq p(Q_{2k-1} \setminus e) \leq p(Q_{2k-1}) = k^2,$$

and for even punctured cubes

$$k^2 = p(Q_{2k-1}) \leq p(Q_{2k} \setminus v) \leq p(Q_{2k} \setminus e) \leq p(Q_{2k}) = k^2 + k.$$

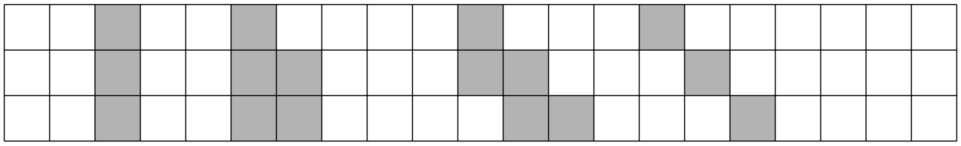

Figure 6: A shape sequence for $Q_4 \setminus v$ where v corresponds to [1111].

In the case of odd punctured cubes, we can determine the polychromatic number exactly, while in the case of even punctured cubes, we give upper and lower bounds, but we do not know the exact polychromatic number when $2k \geq 6$.

Lemma 9 *For all $d \geq 2$, $p(Q_d \setminus v) \leq p(Q_d) - 1$.*

Proof: Recall that the shape sequence corresponding to Q_d is a sequence of d parallelograms as described in Fact 6. If the vertex corresponding to $[11\ldots1]$ is deleted from Q_d, then the shape sequence corresponding to this embedding is identical to the shape sequence for Q_d except the single element in the dth row is deleted from each parallelogram. For example, for $Q_4 \setminus v$, the resulting shape sequence is shown in Figure 6. Since the rest of the rows are not altered, the result follows from Lemma 5. ■

Theorem 10 *For all $k \geq 2$, $p(Q_{2k-1} \setminus v) = p(Q_{2k-1} \setminus e) = k^2 - 1$.*

Proof: By Corollary 8 it suffices to show that $k^2 - 1 \leq p(Q_{2k-1} \setminus v)$ and $p(Q_{2k-1} \setminus e) \leq k^2 - 1$.

To prove the first inequality, we show that the simple coloring $\chi : E(Q_n) \to [k^2 - 1]$ where $\chi(e) = k \cdot l(e) + r(e) \pmod{k^2 - 1}$ is $(Q_{2k-1} \setminus v)$-polychromatic.

The kth shape in the shape sequence for Q_{2k-1} is a $k \times k$ parallelogram. Since each column in this parallelogram contains k consecutive colors $\pmod{k^2 - 1}$, and only the first and last columns in the parallelogram share any colors (the color at coordinate (0,0) and the color at coordinate $(k-1, k-1)$ are the same), this shape contains all $k^2 - 1$ colors. Since all the edges in the color classes in this parallelogram form a matching, deleting a vertex can remove at most one edge in any of these color classes. Every color class except the ones at coordinates (0,0), $(k-1, k-1)$, $(0, k-1)$, and $(k-1, 0)$ contains at least two edges, and since the colors at coordinates (0,0) and $(k-1, k-1)$ are the same, there are only four vertices which can be deleted from Q_{2k-1} where in the corresponding shape sequence for $Q_{2k-1} \setminus v$ the kth shape will not contain all colors. In these vertices, the first $k-1$ entries are all 0's or all 1's, and the last $k-1$ entries are all 1's or all 0's, opposite from the first $k-1$ entries. The kth entry can be 1 or a 0. For instance, if $k = 3$, the vertices in question correspond to [00011], [11100], [00111], and [11000].

Since $k(l+1) + (r-1) - (kl + r) = k - 1$, and since $(k-1)$ divides $k^2 - 1$, the coloring $\chi(e) = k \cdot l(e) + r(e) \pmod{k^2 - 1}$ has the property that there are exactly $k+1$ colors

0	2	4	6	0	2	4	6	0	2	4	6	0	2	4	6	0	2	4	6	0
1	3	5	7	1	3	5	7	1	3	5	7	1	3	5	7	1	3	5	7	1
2	4	6	0	2	4	6	0	2	4	6	0	2	4	6	0	2	4	6	0	2
3	5	7	1	3	5	7	1	3	5	7	1	3	5	7	1	3	5	7	1	3
4	6	0	2	4	6	0	2	4	6	0	2	4	6	0	2	4	6	0	2	4

Figure 7: A shape sequence for $Q_5 \setminus v$, where v corresponds to [11100], with the coloring $\chi(e) = 3 \cdot l(e) + r(e) \pmod 8$.

in each row, each congruent $\pmod{k-1}$. Further, consecutive rows have colors in consecutive congruence classes. See Figures 7 and 8 for examples where $k = 3$.

Consider an embedding of $Q_{2k-1} \setminus v$ and assume v is one of the four vertices where the kth shape does not contain all colors. This choice of v deletes one color class from the kth row of the kth shape. In the shape sequence for $Q_{2k-1} \setminus v$ the colors in the kth row are also found only in the first and $(2k-1)$th row. Without loss of generality assume these colors are congruent to $0 \pmod{k-1}$.

In the two cases where the kth entry of v is the same as the entries before it, the $(k-1)$st shape will remain intact (a $(k+1) \times (k-1)$ parallelogram). In the two cases where the kth entry of v is the same as the entries after it, the $(k+1)$st shape will remain intact (a $(k-1) \times (k+1)$ parallelogram). Thus it suffices to show that both of these parallelograms contain all colors congruent to $0 \pmod{k-1}$. In the $(k-1)$st shape, a $(k+1) \times (k-1)$ parallelogram, the colors at coordinates (0,0), $(0, k-1)$, $(1, k-2)$, …, $(k-2, 1)$, $(k-2, k)$ are $(k+1)$ consecutive multiples of $k-1 \pmod{k^2-1}$. In the $(k+1)$st shape, the colors at coordinates (0,0), $(k, k-2)$, $(1, k-2)$, $(2, k-3)$, …, $(k-1, 0)$ are $(k+1)$ consecutive multiples of $k-1 \pmod{k^2-1}$. Hence the coloring χ is $(Q_{2k-1} \setminus v)$-polychromatic, as claimed.

We now prove the inequality $p(Q_{2k-1} \setminus e) \leq k^2 - 1$. If the edge corresponding to the edge with a star in the kth position, 0's to the left and 1's to the right (e.g. for $k = 3$, $e = [00*11]$) is deleted, then a single color class in the kth row of the kth shape is deleted from the shape sequence for Q_{2k-1} (Figure 9 shows the shape sequence for $k = 3$). This reduces the maximum width of the shapes in the kth row by one, leaving the other rows intact, and the inequality follows from Lemma 5. ■

We now turn our attention to even punctured cubes.

Theorem 11 $p(Q_4 \setminus v) = 5$.

Proof: Since $p(Q_4) = 6$, the upper bound follows from Lemma 9.

For the lower bound, we show that the simple coloring $\chi : E(Q_n) \to [5]$ where $\chi(e) = 3 \cdot l(e) + r(e) \pmod 5$ is $(Q_4 \setminus v)$-polychromatic, see Figure 10. To see this, note that the shape sequence for Q_4 contains all five colors in both the second shape

0	2	4	6	0	2	4	6	0	2	4	6	0	2	4	6	0	2	4	6	0
1	3	5	7	1	3	5	7	1	3	5	7	1	3	5	7	1	3	5	7	1
2	4	6	0	2	4	6	0	2	4	6	0	2	4	6	0	2	4	6	0	2
3	5	7	1	3	5	7	1	3	5	7	1	3	5	7	1	3	5	7	1	3
4	6	0	2	4	6	0	2	4	6	0	2	4	6	0	2	4	6	0	2	4

Figure 8: A shape sequence for $Q_5 \setminus v$, where v corresponds to [11000], with the coloring $\chi(e) = 3 \cdot l(e) + r(e) \pmod 8$.

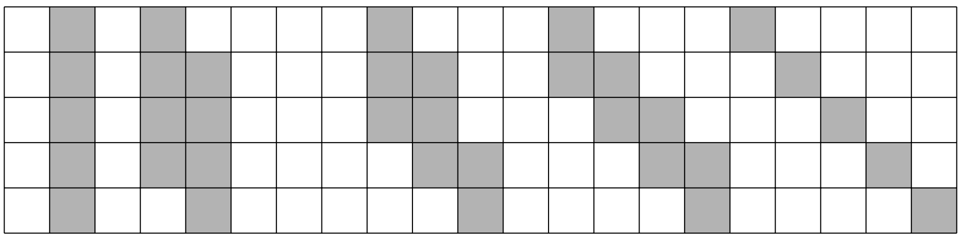

Figure 9: A shape sequence for $Q_5 \setminus e$, where e corresponds to [11*00].

and the third shape regardless of where those shapes appear. If a vertex other than one corresponding to [0000], [1111], [0011], or [1100] is deleted, then the shape sequence corresponding to $Q_4 \setminus v$ will have the second or third shape still intact, and thus be polychromatic. If either of the first two of these are deleted, then even though the second shape will lose one element, it will still contain all five colors, and if either of the last two are deleted, the same holds for the third shape. ■

Theorem 12 $p(Q_4 \setminus e) = 6$.

Proof: Since $p(Q_4) = 6$, the upper bound follows from Corollary 8.

For the lower bound, we show that the simple coloring $\chi : E(Q_n) \to [6]$ where $\chi(e) = 4 \cdot l(e) + r(e) \pmod 6$ is $(Q_4 \setminus e)$-polychromatic, see Figure 11. Since deleting an edge affects at most one color class, it suffices to show that each color is present

0	2	4	1	3	0	2	4	1	3	0	2	4	1	3	0	2	4
1	3	0	2	4	1	3	0	2	4	1	3	0	2	4	1	3	0
2	4	1	3	0	2	4	1	3	0	2	4	1	3	0	2	4	1
3	0	2	4	1	3	0	2	4	1	3	0	2	4	1	3	0	2

Figure 10: A shape sequence for Q_4 with the coloring $\chi(e) = 3 \cdot l(e) + r(e) \pmod 5$.

0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3
1	4	1	4	1	4	1	4	1	4	1	4	1	4	1	4	1	4
2	5	2	5	2	5	2	5	2	5	2	5	2	5	2	5	2	5
3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0

Figure 11: A shape sequence for Q_4 with the coloring $\chi(e) = 4 \cdot l(e) + r(e) \pmod 6$.

in two shapes of the shape sequence for Q_4. For this coloring, each row contains only two colors, which are congruent $\pmod 3$, and adjacent rows contain consecutive congruence classes. In the four rows containing a shape sequence for Q_4, the two colors in the first and fourth rows are the first and third shapes, and the colors in the two middle rows are in the second and third shapes. ■

Theorem 13 *For all $k \geq 3$, $p(Q_{2k} \setminus v) \geq k^2 + k - 2 = (k-1)(k+2)$.*

Proof: We show that the simple coloring $\chi : E(Q_n) \to [(k-1)(k+2)]$ where $\chi(e) = k \cdot l(e) + r(e) \pmod{(k-1)(k+2)}$ is $(Q_4 \setminus v)$-polychromatic. In this coloring, each row only contains $k+2$ colors, which are all congruent $\pmod{k-1}$. Adjacent rows contain colors in consecutive congruence classes.

First note that a $k \times (k+1)$ parallelogram, the $(k+1)$st shape in the shape sequence for Q_{2k} contains all colors. All color classes in this parallelogram except for the ones at coordinates $(0,0)$, $(k,0)$, $(0,k-1)$, and $(k,k-1)$ have at least two parallel edges, and thus deleting a vertex from Q_{2k} will leave at least one edge in each of these color classes. Further, the edges in the classes at coordinates $(0,0)$ and $(k,k-1)$ have the same color. Thus the only four vertices which will cause this shape to lose a color are those with first k coordinates all 0 or all 1, and last $k-1$ coordinates all 1 or all 0, opposite from the first. For one of these vertices, if the first k coordinates are all 1, the color class at coordinate $(k,0)$ is deleted. If the first k coordinates are all 0, the color class at coordinate $(0,k-1)$ is deleted. Without loss of generality, assume that the color classes at coordinates $(0,k-1)$ and $(k,0)$ are in rows containing colors congruent to 0 and 1 $\pmod{k-1}$ respectively. See Figure 12 for an example with $k = 4$ and $v = [11110000]$.

In either of these cases, the kth shape in the shape sequence for $Q_{2k} \setminus v$ must contain the deleted color. In the kth shape, the color classes at coordinates $(0,0)$, $(0,k-1)$, $(1,k-2)$, $\ldots$, $(k-2,1)$, $(k-2,k)$, and $(k-1,k-1)$ contain $k+2$ consecutive multiples of $k-1 \pmod{(k-1)(k+2)}$ and thus contain all colors congruent to 0 $\pmod{k-1}$. Similarly, the color classes at coordinates $(0,1)$, $(1,0)$, $(1,k-1)$, $(2,k-2)$, $\ldots$, $(k-1,1)$, and $(k-1,k)$, contain $k+2$ consecutive numbers $\pmod{(k-1)(k+2)}$ that are congruent to 1 $\pmod{k-1}$ and thus contain all colors congruent to 1 $\pmod{k-1}$. None of these color classes can be deleted by deleting any of the four

0	3	6	9	12	15	0	3	6	9	12	15	0	3	6	9	12	15	0	3	6
1	4	7	10	13	16	1	4	7	10	13	16	1	4	7	10	13	16	1	4	7
2	5	8	11	14	17	2	5	8	11	14	17	2	5	8	11	14	17	2	5	8
3	6	9	12	15	0	3	6	9	12	15	0	3	6	9	12	15	0	3	6	9
4	7	10	13	16	1	4	7	10	13	16	1	4	7	10	13	16	1	4	7	10
5	8	11	14	17	2	5	8	11	14	17	2	5	8	11	14	17	2	5	8	11
6	9	12	15	0	3	6	9	12	15	0	3	6	9	12	15	0	3	6	9	12
7	10	13	16	1	4	7	10	13	16	1	4	7	10	13	16	1	4	7	10	13

Figure 12: The fourth and fifth shapes in the shape sequence for $Q_8 \setminus v$, where v corresponds to [11110000], with the coloring $\chi(e) = 4 \cdot l(e) + r(e) \pmod{18}$.

vertices under consideration. ■

Corollary 14 *For all $k \geq 3$,*

- $k^2 + k - 2 \leq p(Q_{2k} \setminus v) \leq k^2 + k - 1$.
- $k^2 + k - 2 \leq p(Q_{2k} \setminus e) \leq k^2 + k$.

4.1 Open Problems

We conclude with some directions for future research on polychromatic edge colorings.

Problem 15 *For which other graphs G can we determine $p(G)$?*

A first step might be to determine the polychromatic numbers for even punctured cubes, starting with $p(Q_6 \setminus v)$ and $p(Q_6 \setminus e)$. If these polychromatic numbers are smaller than the upper bounds given in Corollary 14, it would provide our first example of a polychromatic number not equal to the bound given by Lemma 5. In turn, such a result might give insight into how Lemma 5 might be strengthened–for example, its proof only uses horizontal translations, but shape sequences can be translated in any direction–or whether other ideas are necessary.

Problem 16 *For each $r \geq 2$, is there some G s.t. $p(G) = r$?*

The smallest open case for Problem 16 is $r = 7$. It would be interesting even to show that the gaps between polychromatic numbers are bounded. It might also be interesting to investigate whether (and under what circumstances) polychromatic numbers can "jump." In other words, if an edge is deleted from a graph G, by how much can the polychromatic number of the resulting graph differ from $p(G)$?

Bialostocki [3] proved that any subgraph of the hypercube not containing Q_2 as a subgraph and intersecting every Q_2 has at most $(n + \sqrt{n})2^{n-2}$ edges. This implies for large n, every Q_2-polychromatic 2-coloring has approximately half the edges of each color. Is it possible to generalize Bialostocki's theorem?

Problem 17 *For large n, given a G-polychromatic coloring of Q_n with $p(G)$ colors, is it true that the proportion of edges in each color class must approach $1/p(G)$?*

5 Coloring Subcubes of Higher Dimension

Alon, Krech, and Szabó [1] suggested a generalization of the problem of finding polychromatic numbers on the hypercube where instead of edges (which themselves can be thought of as one-dimensional hypercubes), subcubes of a fixed dimension of the hypercube are colored. Let $p^i(G)$ be the polychromatic number of the graph G if Q_i's are colored, i.e. $p^i(G)$ is the largest number of colors with which it is possible to color the Q_i's in any hypercube so that every embedding of G contains a Q_i of every color. To determine these polychromatic numbers, many of the ideas from edge colorings can be directly generalized. However in Theorems 20 and 21 we show that not everything about edge coloring generalizes in a straightforward way.

When Q_i's are colored, a simple coloring is one where the color of a Q_i is determined by the vector of length $(i+1)$ where the first coordinate is the number of 1's to the left of the first star, the $(i+1)$st coordinate is the number of 1's to the right of the ith star, and for $1 < j < i+1$, the jth coordinate is the number of 1's between the $(j-1)$st and jth stars. With this definition, a proof almost identical to that of Lemma 3 (see Özkahya and Stanton [7]) gives the following generalization.

Lemma 18 *Let $k \geq i \geq 1$ and G be a subgraph of Q_k. If $p^i(G) = r$, then there is a simple G-polychromatic r-coloring of the Q_i's in Q_k.*

Thus we restrict our attention to simple colorings, and consider color classes in an $(i+1)$-dimensional grid. As with edge colorings, we refer to all Q_i's with the same vector in a simple coloring as a color class. For example, the embedding $[01110*0*11*01001*11011]$ of Q_4 in Q_{22} would be in color class $(3,0,2,2,4)$. Define two color classes in a $(i+1)$-dimensional grid to be on the same *level* if their entries have the same sum. Define a $j_1 \times j_2 \times \cdots \times j_l$ *parallelepiped* to be a set of color classes of the following form: $\{(a_1+\alpha_1, a_2+\alpha_2, \ldots, a_l+\alpha_l) : 0 \leq \alpha_k < j_k\}$. For the remainder of the section, we restrict our attention to the case where $G = Q_d$. Shape sequences for Q_d are characterized by the following generalization of Fact 6.

Fact 19 *Let $n \geq d \geq i \geq 1$. Every shape sequence for an embedding of Q_d in Q_n consists of $\binom{d}{i}$ shapes where each shape is a $j_1 \times j_2 \times \cdots \times j_{i+1}$ parallelepiped where $j_1 + j_2 + \cdots + j_{i+1} = d+1$, and each shape occupies the same d levels. The color classes in each shape correspond to the Q_i's using the same set of i stars. Conversely, every instance of such a shape sequence where the shapes are arranged in a proper relative position corresponds to some embedding of Q_d in Q_n.*

Sets of two stars	Q_2's using those stars	Color classes	Dimensions of shape
3rd and 4th	$[11110011010110*101*001]$	$(9,2,1)$	$3\times 1\times 1$
	$[11110011110110*101*001]$	$(10,2,1)$	
	$[11111011010110*101*001]$	$(10,2,1)$	
	$[11111011110110*101*001]$	$(11,2,1)$	
2nd and 4th	$[11110011*101100101*001]$	$(6,5,1)$	$2\times 2\times 1$
	$[11110011*101101101*001]$	$(6,6,1)$	
	$[11111011*101100101*001]$	$(7,5,1)$	
	$[11111011*101101101*001]$	$(7,6,1)$	
1st and 4th	$[1111*0110101100101*001]$	$(4,7,1)$	$1\times 3\times 1$
	$[1111*0110101101101*001]$	$(4,8,1)$	
	$[1111*0111101100101*001]$	$(4,8,1)$	
	$[1111*0111101101101*001]$	$(4,9,1)$	
1st and 3rd	$[1111*011010110*1010001]$	$(4,5,3)$	$1\times 2\times 2$
	$[1111*011010110*1011001]$	$(4,5,4)$	
	$[1111*011110110*1010001]$	$(4,6,3)$	
	$[1111*011110110*1011001]$	$(4,6,4)$	
2nd and 3rd	$[11110011*10110*1010001]$	$(6,3,3)$	$2\times 1\times 2$
	$[11110011*10110*1011001]$	$(6,3,4)$	
	$[11111011*10110*1010001]$	$(7,3,3)$	
	$[11111011*10110*1011001]$	$(7,3,4)$	
1st and 2nd	$[1111*011*1011001010001]$	$(4,2,6)$	$1\times 1\times 3$
	$[1111*011*1011001011001]$	$(4,2,7)$	
	$[1111*011*1011011010001]$	$(4,2,7)$	
	$[1111*011*1011011011001]$	$(4,2,8)$	

Figure 13: The six shapes for the embedding $[1111*011*10110*101*001]$ of Q_4 in Q_{22}.

For example, if Q_2's are colored, the embedding $[1111*011*10110*101*001]$ of Q_4 in Q_{22} contains 24 Q_2's, partitioned into $\binom{4}{2}=6$ sets according to which two stars they use. These six sets give rise to six shapes, all parallelepipeds containing color classes in levels 12, 13, and 14 (see Figure 13).

Özkahya and Stanton [7] proved lower bounds on $p^i(Q_d)$ precisely analogous to those in Theorem 1. Suppose the shape with the most elements in the shape sequence for Q_d contains r color classes. Then there is an r-coloring of the Q_i's where the largest shape in the shape sequence contains all r colors for every copy of Q_d. The number of colors of this shape is the largest product of $i+1$ natural numbers that sum to $d+1$, and is obtained by minimizing the difference between these numbers. The exact value depends on the remainder of $d+1 \pmod{i+1}$; if the remainder is zero, the number of colors is $\left(\frac{d+1}{i+1}\right)^{i+1}$. From this, one obtains lower bounds of $p^2(Q_3)\geq 1\cdot 1\cdot 2=2$,

Shape	Color Classes
S_{12}	$(0, a_2, a_3)$, $(0, a_2, a_3 + 1)$
S_{13}	$(0, a_2 + a_3, 0)$, $(0, a_2 + a_3 + 1, 0)$
S_{23}	$(a_2, a_3, 0)$, $(a_2 + 1, a_3, 0)$

a_2	a_3	S_{12}	S_{13}	S_{23}
0	0	0,1	0,2	0,1
0	1	0,1	1	1,2
1	0	1,2	1	0,1
1	1	1,2	0,2	1,2

Figure 14: The colors of the Q_2's contained in each shape for the 4 possibilities for a_2 and $a_3 \pmod 2$ in an embedding of Q_3, using the coloring χ from Theorem 20.

$p^2(Q_4) \geq 1 \cdot 2 \cdot 2 = 4$, $p^2(Q_5) \geq 2 \cdot 2 \cdot 2 = 8$, $p^2(Q_6) \geq 2 \cdot 2 \cdot 3 = 12$, and so forth.

Özkahya and Stanton [7] also proved the upper bound analogous to Theorem 1, which is $p^i(Q_d) \leq \binom{d+1}{i+1}$. This is the number of color classes covered by an embedding of Q_d with d stars and all other entries 0. For example, if Q_2's are colored, the embedding of Q_3 represented by [***00000...] has Q_2's only in color classes (0,0,0), (1,0,0), (0,1,0), and (0,0,1), and so $p^2(Q_3) \leq 4 = \binom{3+1}{2+1}$.

In the case of edge colorings, the polychromatic number for Q_d turned out to be the lower bound in Theorem 1, but this is not the case for colorings of larger subcubes; the polychromatic number may be larger than the size of the largest shape. We show this first in the case of $p^2(Q_3)$. All shapes in the shape sequence for Q_3 when Q_2's are colored have two elements, and the bounds corresponding to Theorem 1 are $2 \leq p^2(Q_3) \leq 4$.

Theorem 20 $p^2(Q_3) = 3$.

Proof: Lower bound: Consider the simple 3-coloring χ that assigns colors to any Q_2 in color class (x_1, x_2, x_3) the color $\chi(x_1, x_2, x_3)$, where

$$\chi(x_1, x_2, x_3) = \begin{cases} x_1 + x_2 + x_3 \pmod 3 & \text{if } x_2 \equiv 0 \pmod 2 \\ x_1 + x_2 + x_3 + 1 \pmod 3 & \text{if } x_2 \equiv 1 \pmod 2. \end{cases}$$

Consider an embedding of Q_3 where there are a_1 1's to the left of the first star, a_2 1's between the first and second stars, a_3 1's between the second and third stars, and a_4 1's to the right of the third star. We show that it contains all three colors. Without loss of generality assume $a_1 + a_2 + a_3 + a_4 \equiv 0 \pmod 3$, and $a_1 = a_4 = 0$. For $1 \leq i < j \leq 3$, denote by S_{ij} the shape using stars i and j. The tables in Figure 14 list the color classes in each of the three shapes, and the colors in each shape for the four possibilities for a_2 and $a_3 \pmod 2$. For each possibility, each of the three colors is in at least one of the three shapes.

Upper bound: A computer search found that no Q_3-polychromatic 4-coloring, simple or otherwise, is possible on Q_5.

For a human-checkable proof, we know that if the Q_2's in Q_n can be 4-colored so that every Q_3 is polychromatic, then it can be done with a simple coloring. We try to construct a simple coloring χ on Q_5, eventually showing that it is impossible.

The embedding [***00] of Q_3 contains only color classes 000, 100, 010, and 001, so these color classes must all be distinct, and without loss of generality we assign them the colors 1, 2, 3, and 4, respectively.

To assign colors to the color classes 110, 101, 011, 200, 020, and 002, we examine the embeddings [1***0], [*1**0], [**1*0], and [***10]. A straightforward but tedious examination shows there are only five simple 4-colorings of the Q_4 [****0], falling into two patterns: One is χ_1, below. The other four are χ_2, where $\chi_2(020)$ can be chosen to be any color.

Color class	000	100	010	001	101	011	002	110	020	200
χ_1	1	2	3	4	1	3	2	3	1	4
χ_2	1	2	3	4	3	1	2	1	*	4

We now attempt to extend these colorings to the Q_4's [1****] and [****1]. In the case of [1****], we already know the colors of 100, 200, 110, and 101, so the colors for the color classes for [1****] must fit one of the following four patterns. For $i, j \in \{1, 2\}$, χ_{ij} is the coloring generated by extending the χ_i coloring with the χ_j pattern.

Color class	100	200	110	101	201	111	102	210	120	300
χ_{11}	2	4	3	1	2	3	4	3	2	1
χ_{12}	2	4	3	1	3	2	4	2	*	1
χ_{21}	2	4	1	3	2	1	4	1	2	3
χ_{22}	2	4	1	3	1	2	4	2	*	3

In the Q_4 [****1], color classes 001, 101, 011, 002, 102, 111, and 201 are already assigned. It is impossible to extend χ_{12} and χ_{22} consistent with the patterns χ_1 or χ_2, while χ_{11} and χ_{21} extend uniquely as follows.

Color class	001	101	011	002	102	012	003	111	021	201
χ_{11}	4	1	3	2	4	3	1	3	4	2
χ_{21}	4	3	1	2	4	1	3	1	4	2

Now every color class except for 030 is assigned a color. However in either of the colorings, the embedding [*1*1*] (containing color classes 110, 210, 020, 030, 011, and 012) has at most two colors before 030 is assigned. Thus it cannot contain four colors regardless of the choice for the color of 030. ■

Theorem 21 $p^2(Q_4) \geq 5$.

Proof: Consider the simple 5-coloring χ that assigns to any Q_2 in color class (x_1, x_2, x_3) the color $\chi(x_1, x_2, x_3)$, where

$$\chi(x_1,x_2,x_3) = \begin{cases} x_1+x_2+x_3 \pmod 5 & \text{if } x_2 \equiv 0 \pmod 3 \\ x_1+x_2+x_3+1 \pmod 5 & \text{if } x_2 \equiv 1 \pmod 3 \\ x_1+x_2+x_3+2 \pmod 5 & \text{if } x_2 \equiv 2 \pmod 3. \end{cases}$$

Consider an embedding of Q_4 where there are a_1 1's to the left of the first star, a_2 1's between the first and second stars, a_3 1's between the second and third stars, a_4 1's between the third and fourth stars, and a_5 1's to the right of the fourth star. We show that it contains all five colors. Without loss of generality assume $a_1+a_2+a_3+a_4+a_5 \equiv 0 \pmod 5$ and $a_1 = a_5 = 0$. For $1 \le i < j \le 4$, denote by S_{ij} the shape using stars i and j. The following table lists the color classes in each of the six shapes.

Shape	Color Classes
S_{12}	$(0, a_2, a_3 + a_4)$, $(0, a_2, a_3 + a_4 + 1)$, $(0, a_2, a_3 + a_4 + 2)$
S_{13}	$(0, a_2 + a_3, a_4)$, $(0, a_2 + a_3 + 1, a_4)$, $(0, a_2 + a_3, a_4 + 1)$, $(0, a_2 + a_3 + 1, a_4 + 1)$
S_{14}	$(0, a_2 + a_3 + a_4, 0)$, $(0, a_2 + a_3 + a_4 + 1, 0)$, $(0, a_2 + a_3 + a_4 + 2, 0)$
S_{24}	$(a_2, a_3 + a_4, 0)$, $(a_2 + 1, a_3 + a_4, 0)$, $(a_2, a_3 + a_4 + 1, 0)$, $(a_2 + 1, a_3 + a_4 + 1, 0)$
S_{34}	$(a_2 + a_3, a_4, 0)$, $(a_2 + a_3 + 1, a_4, 0)$, $(a_2 + a_3 + 2, a_4, 0)$
S_{23}	(a_2, a_3, a_4), $(a_2 + 1, a_3, a_4)$, $(a_2, a_3, a_4 + 1)$, $(a_2 + 1, a_3, a_4 + 1)$

Figure 15 lists the colors contained in each shape for the 27 possibilities for a_2, a_3, and $a_4 \pmod 3$. For each possibility, each of the five colors is in at least one of the six shapes. ■

Theorem 22 *For all $d \ge i \ge 1$, $j \ge 1$, $p^{i+j}(Q_{d+j}) \ge p^i(Q_d)$.*

Proof: Suppose χ is a Q_d-polychromatic k-coloring of the Q_i's in Q_n, where $n \ge d + j$. Consider the k-coloring of the Q_{i+j}'s in Q_n given by

$$\chi'(x_1, x_2, \ldots, x_{i+1}, \ldots, x_{i+j+1}) = \chi(x_1, x_2, \ldots, x_{i+1}).$$

We show χ' is Q_{d+j}-polychromatic.

Let G_{d+j} be an embedding of Q_{d+j} in Q_n represented by an n-bit vector with $d+j$ stars. Let G_d be the embedding of Q_d in Q_n represented by the same vector with the $(d+1)$st star and every coordinate to the right replaced by zeros. For example, if $d = 4$, $j = 2$, and $G_{d+j} = [0*111*1011**110*0*01]$, then $G_d = [0*111*1011**11000000]$. The $i+1$ coordinates of the color classes for G_d are identical to the first $i+1$ coordinates of the color classes for G_{d+j} in the shapes that use the last j stars. Since χ is Q_d-polychromatic, G_d contains Q_i's of each of the k colors, and G_{d+j} must also contain a Q_{i+j} of each color with the coloring χ'. ■

Corollary 23 *For all $d \ge 2$, $p^d(Q_{d+1}) \ge p^2(Q_3) = 3$.*

a_2	a_3	a_4	S_{12}	S_{13}	S_{14}	S_{24}	S_{34}	S_{23}
0	0	0	0,1,2	0,1,2,3	0,2,4	0,1,2,3	0,1,2	0,1,2
1	1	1	1,2,3	2,1,3	0,2,4	2,1,3	1,2,3	1,2,3
2	2	2	2,3,4	1,2,3,4	0,2,4	1,2,3,4	2,3,4	2,3,4
1	0	0	1,2,3	1,2,3,4	1,3,2	0,1,2,3	0,1,2	0,1,2
0	1	0	0,1,2	1,2,3,4	1,3,2	1,2,3,4	0,1,2	1,2,3
0	0	1	0,1,2	0,1,2,3	1,3,2	1,2,3,4	1,2,3	0,1,2
1	1	0	1,2,3	2,1,3	2,1,3	1,2,3,4	0,1,2	1,2,3
1	0	1	1,2,3	1,2,3,4	2,1,3	1,2,3,4	1,2,3	0,1,2
0	1	1	0,1,2	1,2,3,4	2,1,3	2,1,3	1,2,3	1,2,3
2	0	0	2,3,4	2,1,3	2,1,3	0,1,2,3	0,1,2	0,1,2
0	2	0	0,1,2	2,1,3	2,1,3	2,1,3	0,1,2	2,3,4
0	0	2	0,1,2	0,1,2,3	2,1,3	2,1,3	2,3,4	0,1,2
2	1	0	2,3,4	2,1,3	0,2,4	1,2,3,4	0,1,2	1,2,3
1	2	0	1,2,3	0,1,2,3	0,2,4	1,2,3,4	0,1,2	2,3,4
2	0	1	2,3,4	2,1,3	0,2,4	1,2,3,4	1,2,3	0,1,2
1	0	2	1,2,3	1,2,3,4	0,2,4	2,1,3	2,3,4	0,1,2
0	2	1	0,1,2	2,1,3	0,2,4	0,1,2,3	1,2,3	2,3,4
0	1	2	0,1,2	1,2,3,4	0,2,4	0,1,2,3	2,3,4	1,2,3
2	1	1	2,3,4	0,1,2,3	1,3,2	2,1,3	1,2,3	1,2,3
1	2	1	1,2,3	0,1,2,3	1,3,2	0,1,2,3	1,2,3	2,3,4
1	1	2	1,2,3	2,1,3	2,1,3	0,1,2,3	2,3,4	1,2,3
2	2	0	2,3,4	1,2,3,4	1,3,2	2,1,3	0,1,2	2,3,4
2	0	2	2,3,4	2,1,3	1,3,2	2,1,3	2,3,4	0,1,2
0	2	2	0,1,2	2,1,3	1,3,2	2,1,3	2,3,4	2,3,4
1	2	2	1,2,3	0,1,2,3	1,3,2	1,2,3,4	2,3,4	2,3,4
2	1	2	2,3,4	0,1,2,3	1,3,2	0,1,2,3	2,3,4	1,2,3
2	2	1	2,3,4	1,2,3,4	1,3,2	0,1,2,3	1,2,3	2,3,4

Figure 15: The colors of the Q_2's contained in each shape for the 27 possibilities for a_2, a_3, and $a_4 \pmod 3$ in an embedding of Q_4, using the coloring χ from Theorem 21.

It is not clear whether the colorings in Theorems 20 and 21 have natural generalizations, or whether they are sporadic small cases. Thus almost any progress in determining polychromatic numbers when larger subcubes are colored would be interesting.

Problem 24 *Improve any of the following bounds:*

- *For* $d \geq 3$, $d + 2 \geq p^d(Q_{d+1}) \geq 3$.
- $10 \geq p^2(Q_4) \geq 5$.
- *For* $d \geq 5$, $\binom{d+1}{3} \geq p^2(Q_d) \geq p(Q_{d-1})$.

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