Decomposing graphs into edges and triangles*

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Abstract

We prove the following 30-year old conjecture of Győri and Tuza: the edges of every n-vertex graph G can be decomposed into complete graphs $C_1, \dots, C_\ell$ of orders two and three such that $|C_1| + \cdots + |C_\ell| \le (1/2 + o(1))n^2$. This result implies the asymptotic version of the old result of Erdős, Goodman and Pósa that asserts the existence of such a decomposition with $\ell \le n^2/4$.

1 Introduction

Results on the existence of edge-disjoint copies of specific subgraphs in graphs is one of the most classical themes in extremal graph theory. Motivated by the following result of Erdős, Goodman and Pósa [11], we study the problem of covering edges of a given graph by edge-disjoint complete graphs.

Theorem 1 (Erdős, Goodman and Pósa [11]). *The edges of every n-vertex graph can be decomposed into at most $\lfloor n^2/4 \rfloor$ complete graphs.*

In fact, they proved the following stronger statement.

*This work has received funding from the European Research Council (ERC) under the European Unions Horizon 2020 research and innovation programme (grant agreement No 648509). This publication reflects only its authors' view; the European Research Council Executive Agency is not responsible for any use that may be made of the information it contains.

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Theorem 2 (Erdős, Goodman and Pósa [11]). *The edges of every n-vertex graph can be decomposed into at most $\lfloor n^2/4 \rfloor$ copies of K_2 and K_3.*

The bounds given in Theorems 1 and 2 are best possible as witnessed by complete bipartite graphs with parts of equal sizes.

Theorem 1 actually holds in a stronger form that we now present. Chung [7], Győri and Kostochka [19], and Kahn [24], independently, proved a conjecture of Katona and Tarján asserting that the edges of every n-vertex graphs can be covered with complete graphs $C_1, \dots, C_\ell$ such that the sum of their orders is at most $n^2/2$. In fact, the first two proofs yield a stronger statement, which implies Theorem 1 and which we next state as a separate theorem. To state the theorem, we define $\pi_k(G)$ for a graph G to be the minimum integer m such that the edges of G can be decomposed into complete graphs $C_1, \dots, C_\ell$ of order at most k with $|C_1| + \cdots + |C_\ell| = m$, and let $\pi(G) = \min_{k\in\mathbb{N}} \pi_k(G)$.

Theorem 3 (Chung [7]; Győri and Kostochka [19]). *Every n-vertex graph G satisfies that $\pi(G) \le n^2/2$.*

Observe that Theorem 3 indeed implies the existence of a decomposition into at most $\lfloor n^2/4 \rfloor$ complete graphs. McGuinnes [31, 32] extended these results by showing that decompositions from Theorems 1 and 3 can be constructed in the greedy way, which confirmed a conjecture of Winkler of this being the case in the setting of Theorem 1.

In view of Theorem 2, it is natural to ask whether Theorem 3 holds under the additional assumption that all complete graphs in the decomposition are copies of K_2 and K_3, i.e., whether $\pi_3(G) \le n^2/2$. Győri and Tuza [20] provided a partial answer by proving that $\pi_3(G) \le 9n^2/16$ while conjecturing the following.

Conjecture 1 (Győri and Tuza [34, Problem 40]). *Every n-vertex graph G satisfies that $\pi_3(G) \le (1/2 + o(1))n^2$.*

We prove this conjecture. Our result also solves [34, Problem 41], which we state as Corollary 7. We remark that we stated the conjecture in the version given by Győri in several of his talks and by Tuza in [34, Problem 40]; the paper [20] contains a version with a different lower order term.

We would also like to mention a closely related variant of the problem suggested by Erdős, where the cliques in the decomposition have weights one less than their orders. Formally, define $\pi^-(G)$ for a graph to be the minimum m such that the edges of a graph G can be decomposed into complete graphs $C_1, \dots, C_\ell$ with $(|C_1| - 1) + \cdots + (|C_\ell| - 1) = m$. The problem raised by Erdős asserts, see [34, Problem 43], that $\pi^-(G) \le n^2/4$ for every n-vertex graph G. This problem remains open and was proven for K_4-free graphs only recently by Győri and Keszegh [17, 18], who proved that every K_4-free graph with n vertices and $\lfloor n^2/4 \rfloor + k$ edges contains k edge-disjoint triangles.

2 Preliminaries

We follow the standard graph theory terminology; we review here some less standard notation and briefly introduce the flag algebra method. If G is a graph, then $|G|$ denotes the number of vertices of G. Further, if W is a subset of vertices of G, then $G[W]$ is the subgraph of G induced by W, i.e., the subgraph with the vertex set W and all edges with end vertices inside W.

In our arguments, we also consider fractional decompositions. A fractional k-decomposition of a graph G is an assignment of non-negative real weights to complete subgraphs of order at most k such that the sum of the weights of complete subgraphs containing any edge e is equal to one. The weight of a k-decomposition is the sum of the weights of complete subgraphs multiplied by their orders, and the minimum weight of a fractional k-decomposition of a graph G is denoted by $\pi_{k,f}(G)$. Observe that $\pi_{k,f}(G) \leq \pi_k(G)$ for every graph G.

2.1 Flag algebra method

The flag algebra method introduced by Razborov [33] has changed the landscape of extremal combinatorics. It found its applications to many long-standing open problems, e.g. [1–6, 8–10, 12–16, 21, 23, 27–30]. The method is designed to analyze asymptotic behavior of substructure densities and we now briefly describe it.

We start with introducing necessary notation. The family of all finite graphs is denoted by $\mathcal{F}$ and the family of graphs with ℓ vertices by $\mathcal{F}_\ell$. If F and G are two graphs, then $p(F,G)$ is the probability that $|F|$ distinct vertices chosen uniformly at random among the vertices of G induce a graph isomorphic to F; if $|F| > |G|$, we set $p(F,G) = 0$. A type is a graph with its vertices labeled with $1, \dots, |\sigma|$ and a σ-flag is a graph with $|\sigma|$ vertices labeled by $1, \dots, |\sigma|$ such that the labeled vertices induce a copy of σ preserving the vertex labels. In the analogy with the notation for ordinary graphs, the set of all σ-flags is denoted by $\mathcal{F}^\sigma$ and the set of all σ-flags with exactly ℓ vertices by $\mathcal{F}^\sigma_\ell$.

We next extend the definition of $p(F,G)$ to σ-flags and generalize it to pairs of graphs. If F and G are two σ-flags, then $p(F,G)$ is the probability that $|F| - |\sigma|$ distinct vertices chosen uniformly at random among the unlabeled vertices of G induce a σ-flag F; if $|F| > |G|$, we again set $p(F,G) = 0$. Let F and F' be two σ-flags and G a σ-flag with at least $|F| + |F'| - |\sigma|$ vertices. The quantity $p(F,F';G)$ is the probability that two disjoint $|F| - |\sigma|$ and $|F'| - |\sigma|$ subsets of unlabeled vertices of G induce together with the labeled vertices of G the σ-flags F and F', respectively. It holds [33, Lemma 2.3] that

$$p(F,F';G) = p(F,G) \cdot p(F',G) + o(1) \tag{1}$$

where $o(1)$ tends to zero with $|G|$ tending to infinity.

Let $\vec{F} = [F_1, \ldots, F_t]$ be a vector of σ-flags, i.e., $F_i \in \mathcal{F}^\sigma$. If M is a $t \times t$ positive semidefinite matrix, it follows from (1), see [33], that

$$0 \leq \sum_{i,j=1}^{t} M_{ij} p(F_i, G) p(F_j, G) = \sum_{i,j=1}^{t} M_{ij} p(F_i, F_j; G) + o(1). \tag{2}$$

The inequality (2) is usually applied to a large graph G with a randomly chosen labeled vertices in a way that we now describe. Fix σ-flags F and F' and a graph G. We now define a random variable $p(F, F'; G^\sigma)$ as follows: label $|\sigma|$ vertices of G with $1, \ldots, |\sigma|$ and if the resulting graph G' is a σ-flag, then $p(F', F'; G^\sigma) = p(F, F'; G')$; if G' is not a σ-flag, then $p(F_i, F_j; G^\sigma) = 0$. The expected value of $p(F, F'; G^\sigma)$ can be expressed as a linear combination of densities of $(|F| + |F'| - |\sigma|)$-vertex subgraphs of G [33], i.e., there exist coefficients α_H, $H \in \mathcal{F}_{|F|+|F'|-|\sigma|}$, such that

$$\mathbb{E}\, p(F, F'; G^\sigma) = \sum_{H \in \mathcal{F}_{|F|+|F'|-|\sigma|}} \alpha_H \cdot p(H, G) \tag{3}$$

for every graph G. It can be shown that $\alpha_H = \mathbb{E}\, p(F, F'; H^\sigma)$.

Let $\vec{F} = [F_1, \ldots, F_t]$ be a vector of ℓ-vertex σ-flags and let M be a $t \times t$ positive semidefinite matrix. The equality (3) yields that there exist coefficients α_H such that

$$\mathbb{E} \sum_{i,j=1}^{t} M_{ij} p(F_i, F_j; G^\sigma) = \sum_{H \in \mathcal{F}_{2\ell - |\sigma|}} \alpha_H \cdot p(H, G) \tag{4}$$

for every graph G, which combines with (2) to

$$0 \leq \sum_{H \in \mathcal{F}_{2\ell - |\sigma|}} \alpha_H \cdot p(H, G) + o(1) \tag{5}$$

for every graph G, where

$$\alpha_H = \sum_{i,j=1}^{t} M_{ij} \cdot \mathbb{E}\, p(F_i, F_j; H^\sigma)$$

In particular, the coefficients α_H depend only on the choice of $\vec{F}$ and M.

3 Main result

We start with proving the following lemma using the flag algebra method.

Lemma 4. *Let G be a weighted graph with all edges of weight one. It holds that*

$$\mathbb{E}_W \pi_{3,f}(G[W]) \leq 21 + o(1)$$

where W is a uniformly chosen random subset of seven vertices of G.

Proof. We use the flag algebra method to find coefficients c_U, $U \in \mathcal{F}_7$, such that

$$0 \le \sum_{U \in \mathcal{F}_7} c_U \cdot p(U, G) + o(1) \tag{6}$$

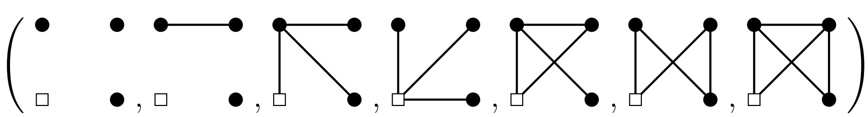
and

$$\pi_{3,f}(U) + c_U \le 21 \tag{7}$$

for every $U \in \mathcal{F}_7$. The statement of the lemma would then follow from (6) and (7) using $\sum_{U \in \mathcal{F}_7} p(U, G) = 1$ as we next show.

$$\begin{aligned}
\mathbb{E}_W \pi_{3,f}(G[W]) &= \sum_{U \in \mathcal{F}_7} \pi_{3,f}(U) \cdot p(U, G) \\
&\le \sum_{U \in \mathcal{F}_7} (\pi_{3,f}(U) + c_U) \cdot p(U, G) + o(1) \\
&\le \sum_{U \in \mathcal{F}_7} 21 \cdot p(U, G) + o(1) = 21 + o(1).
\end{aligned}$$

We now focus on finding the coefficients c_U, $U \in \mathcal{F}_7$, satisfying (6) and (7). Let σ_1 be a flag consisting of a single vertex labeled with 1 and consider the following vector $\vec{F} = (F_1, \dots, F_7)$ of σ_1-flags from $\mathcal{F}_4^{\sigma_1}$ (the single labeled vertex is depicted by a white square and the remaining vertices by black circles).

$\vec{F} =$ 

Let M be the following 7×7-matrix.

$$M = \frac{1}{12 \cdot 10^9} \begin{pmatrix}
1800000000 & 2444365956 & 640188285 & -1524146769 & 1386815580 & -732139362 & -129387078 \\
2444365956 & 4759879134 & 1177441152 & -1783771230 & 2546923788 & -1397639394 & -143552208 \\
640188285 & 1177441152 & 484273772 & -317303211 & 1038156300 & -591902130 & -6783162 \\
-1524146769 & -1783771230 & -317303211 & 1558870290 & -651906630 & 305728704 & 154602378 \\
1386815580 & 2546923788 & 1038156300 & -651906630 & 2285399634 & -1283125950 & -10755036 \\
-732139362 & -1397639394 & -591902130 & 305728704 & -1283125950 & 734039016 & -1621938 \\
-129387078 & -143552208 & -6783162 & 154602378 & -10755036 & -1621938 & 23860164
\end{pmatrix}.$$

The matrix M is a positive semidefinite matrix with rank six; the eigenvector corresponding to the zero eigenvalue is $(1, 0, 3, 1, 0, 3, 0)$. Let

$$c_U = \sum_{i,j=1}^{7} M_{ij} \mathbb{E}\, p(F_i, F_j; U^{\sigma_1}) \, .$$

The inequality (5) implies that

$$0 \le \sum_{U \in \mathcal{F}_7} c_U \cdot p(U, G) + o(1),$$

which establishes (6). The inequality (7) is verified with computer assistance by evaluating the coefficient c_U and the quantity $\pi_{3,f}(U)$ for each $U \in \mathcal{F}_7$. Since $|\mathcal{F}_7| = 1044$, we do not list c_U and $\pi_{3,f}(U)$ here. The computer programs that we used and their outputs have been made available on arXiv as ancillary files and are also available at `http://orion.math.iastate.edu/lidicky/pub/tile23`. □

The following lemma can be derived from the result of Haxell and Rödl [22] on fractional triangle decompositions or from a more general result of Yuster [35].

Lemma 5. *Let G be a graph with n vertices. It holds that $\pi_3(G) \leq \pi_{3,f}(G)+o(n^2)$.*

We now use Lemmas 4 and 5 to prove our main result.

Theorem 6. *Every n-vertex graph G satisfies that $\pi_3(G) \leq (1/2 + o(1))n^2$.*

Proof. Fix an n-vertex graph G. By Lemma 5, it is enough to show that $\pi_{3,f}(G) \leq (1/2 + o(1))n^2$.

Fix an optimal fractional 3-decomposition of $G[W]$ for every 7-vertex subset $W \subseteq V(G)$, and set the weight $w(e)$ of an edge e to the sum of its weights in the optimal fractional 3-decomposition of $G[W]$ with $e \subseteq W$ multiplied by $\binom{n-2}{5}^{-1}$, and the weight $w(t)$ of a triangle t to the sum its weights in the optimal fractional 3-decomposition of $G[W]$ with $t \subseteq W$ also multiplied by $\binom{n-2}{5}^{-1}$. Since each edge e of G is contained in $\binom{n-2}{5}$ subsets W, we have obtained a fractional 3-decomposition of G. The weight of this decomposition is equal to

$$\frac{1}{\binom{n-2}{5}} \sum_{W \in \binom{V(G)}{7}} \pi_{3,f}(G[W]) \leq \frac{\binom{n}{7}}{\binom{n-2}{5}}(21 + o(1)) = n^2/2 + o(n^2) \ ,$$

where the inequality follows from Lemma 4. We conclude that $\pi_{3,f}(G) \leq n^2/2 + o(n^2)$, which completes the proof. □

The next corollary follows directly from Theorem 6.

Corollary 7. *Every n-vertex graph with $n^2/4 + k$ edges contains $2k/3 - o(n^2)$ edge-disjoint triangles.*

4 Concluding remarks

Our first proof of this result, which can be found in [26], combined the flag algebra method and the regularity method arguments. In particular, we proved the fractional relaxation of Conjecture 1 in the setting of weighted graphs and with an additional restriction on its support; this statement was then combined with a blow-up lemma for edge-decompositions recently proven by Kim, Kühn, Osthus and Tyomkyn [25]. It was then brought to our attention that the results from [22, 35] allow obtaining our main result directly from the fractional relaxation, which is the proof that we present here. We believe that the argument using combinatorial designs that we applied in [26] to combine the flag algebra method and the blow-up lemma of Kim et al. [25] can be of independent interest and so we wanted to mention the original proof of our result and its idea here.

We also tried to prove Lemma 4 in the non-fractional setting, i.e., to show that $\mathbb{E}_W \pi_3(G[W]) \leq 21 + o(1)$. Unfortunately, the computation with 7-vertex flags yields only that $\mathbb{E}_W \pi_3(G[W]) \leq 21.588 + o(1)$. We would like to remark that if it were possible to prove Lemma 4 in the non-fractional setting, we would be able to prove Theorem 6 without using additional results as a blackbox: we would consider a random $(n, 7, 2, 1)$-design on the vertex set of an n-vertex graph G as in [26] and apply the non-fractional version of Lemma 4 to this design.

Finally, we would also like to mention two open problems related to our main result. Theorem 6 asserts that $\pi_3(G) \leq n^2/2 + o(n^2)$ for every n-vertex graph G. However, it could be true (cf. the remark after Problem 41 in [34]) that $\pi_3(G) \leq n^2/2+2$ for every n-vertex graph G. The second problem that we would like to mention is a possible generalization of Corollary 7, which is stated in [34] as Problem 42. Fix $r \geq 4$. Does every n-vertex graph with $\frac{r-2}{2r-2}n^2 + k$ edges contain $\frac{2}{r}k - o(n^2)$ edge-disjoint complete graphs of order k?

Acknowledgements

The authors would like to thank Ervin Győri and Katherine Staden for their comments on the problems considered in this paper. The authors would also like to thank Allan Lo for drawing their attention to the paper [22].

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