Inducibility of 4-vertex tournaments

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Abstract

We determine the inducibility of all tournaments with at most 4 vertices together with the extremal constructions. The 4-vertex tournament containing an oriented C_3 and one source vertex has a particularly interesting extremal construction, first conjectured by Bożyk, Grzesik and Kielak. It is an unbalanced blow-up of an edge, where the sink vertex is replaced by a quasi-random tournament and the source vertex is iteratively replaced by a copy of the construction itself.

1 Introduction

One of the central questions in extremal graph theory is to maximize the number of induced copies of a given graph H in a larger host graph on a fixed number of vertices. Denoting the number of vertices by of a graph G by $|G|$, let $I(H,G)$ be the number of vertex subsets of G which induce a graph isomorphic to H, and let

$$I(H,n) = \max_{|G|=n} I(H,G).$$

We normalize these definitions and write $i(H,G) = \frac{I(H,G)}{\binom{|G|}{|H|}}$ and $i(H,n) = \frac{I(H,n)}{\binom{n}{|H|}}$. This implies that $0 \leq i(H,G) \leq 1$, and we can think of $i(H,G)$ as a subgraph density. An easy averaging argument shows that $i(H,n)$ is monotone decreasing and thus converges for $n \to \infty$. Pippenger and Golumbic [25] define the *inducibility of* H as the limit of this quantity,

$$i(H) = \lim_{n\to\infty} i(H,n).$$

Determining inducibilities is notoriously difficult, and the answer is known only for very few explicit graphs H. A major breakthrough for the problem was the introduction of the flag algebra

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method by Razborov [26] in 2007, and since then the inducibility of a good number of small graphs has been determined with the help of this method [1, 3, 10]. While we are using this method as well in this paper, we will not thoroughly explain it here but rather direct the reader to earlier papers [2,8,21]. In a nutshell, the method uses semidefinite programming to solve an optimization problem on subgraph densities which can be set up in a very structured and easily computer assisted way, almost to the point where one may call it fully automated. We can add any number of linear constraints on the subgraph densities to the semidefinite program. Nevertheless, we do not even know $i(P_4)$, i.e. the inducibility of the path on four vertices, and we do not even have a conjecture for the answer.

On the other end of the spectrum, Fox, Huang, and Lee [12], and independently Yuster [27], have determined exact values for $i(H,n)$ and thus $i(H)$ for all n and almost all large enough graphs H by studying random graphs. They show that the extremal construction is an iterated blow-up of the given graph, a fractal like structure. This iterated blow-up construction was already established by Pippinger and Golumbic as a general lower bound for inducibilities, and they asked which graphs meet this lower bound. There are numerous other results on inducibility [5,9,13–15,19,20, 22,23].

All of these questions can be studied for directed graphs as well, the definitions naturally transfer. Falgas-Ravry and Vaughan [11] studied inducibility of small outstars using flag algebras. Huang [18] extended the result to all outstars. This was further generalized to other stars by Hu, Ma, Norin, and Wu [17]. Short paths with further restrictions were considered in [6] and orientations of a 4-cycle in [16]. In an REU (Research Experience for Undergraduates) in 2018, Burgher and Burke studied and conjectured extremal constructions for most oriented graphs (directed graphs without 2-cycles) of up to 4 vertices using the flag algebra method. In a similar and independent project around the same time, Bożyk, Grzesik and Kielak [4] established the same and more bounds and constructions for oriented graphs.

In this paper, we look closer at the tournaments in this list, i.e. oriented complete graphs. The number of non-isomorphic tournaments on k vertices is slightly smaller than the number of graphs, and flag algebra computations tend to have similar power. The two projects mentioned in the previous paragraph both found inducibility bounds and closely matching lower bound constructions for all tournaments on up to 4 vertices, where the results are easy or trivial for all but three of these 8 small tournaments. These last three tournaments on 4 vertices have very interesting constructions, and in this paper we prove that these constructions are indeed optimal for large n.

In a somewhat related question, Mubayi and Razborov [24] considered edge colored tournaments and showed that for every tournament T on $k \geq 4$ vertices whose edges are colored by $\binom{k}{2}$ distinct colors, the iterated blow-up of T achieves $i(T,n)$. This implies that $i(T) = \frac{k!}{k^k - k}$ in this rainbow setting.

2 Results

We discuss tournaments on at most four vertices. For the tournaments T_1 and T_2 on one and two vertices, respectively, any tournament T has $i(T_k,T) = 1$, and thus $i(T_k) = i(T_k,n) = 1$. Similarly, for all transitive tournaments TT_k on $k \geq 3$ vertices, the transitive tournament TT_n on $n \geq k$ vertices is the unique tournament on n vertices with $i(TT_k,T) = 1$, and thus $i(TT_k) = i(TT_k,n) = 1$. On the other hand, $i(TT_3,T)$ is minimized exactly if T has all out-degrees in $\{\frac{n-2}{2}, \frac{n-1}{2}, \frac{n}{2}\}$. This easily follows from counting TT_3 by first choosing the source vertex, and then any two out-neighbors. As a consequence, one gets for the only other tournament on three vertices C_3:

Proposition 1 (Folklore)**.** *The number of induced copies of C_3 is maximized if and only if every vertex of a tournament has out-degree in $\{\frac{n-2}{2}, \frac{n-1}{2}, \frac{n}{2}\}$.*

This implies $i(C_3) = 1/4$ and leaves us with three 4-vertex tournaments to consider, see Figure 1: the tournaments we get from C_3 by adding a source vertex (C_3^+), a sink vertex (C_3^-), and by adding a vertex of out-degree 1 or 2 (this choice results in isomorphic outcomes, the carousel C_4 defined in the next paragraph).

Figure 1: The four 4-vertex tournaments.

Let us now define the class $\mathscr{C}_n$ of *carousels* on $n \geq 3$ vertices. A tournament T is in $\mathscr{C}_n$ if its vertices can be labeled $\{v_1, v_2, \ldots, v_n\}$ such that $v_iv_j \in E(T)$ if $0 < j-i < \frac{n}{2}$ or if $-n < j-i < -\frac{n}{2}$. An easy exercise shows that a tournament T is in $\mathscr{C}_n$ if and only if for every $x \in V(T)$, the in- and out-neighborhoods induce transitive tournaments (T is *locally transitive*) and are as balanced as possible (T is *balanced* when $|V(T)|$ is odd, or *nearly balanced* when $|V(T)|$ is even). See Figure 2 for an illustration.

Observe that for odd n and for $n = 4$ (up to isomorphism), $\mathscr{C}_n$ contains exactly one tournament, and we will call this unique carousel C_n. For even $n \geq 6$, $\mathscr{C}_n$ contains more than one tournament, depending on the directions of the arcs $v_iv_{i+\frac{n}{2}}$. For even n, we denote by $C_n \in \mathscr{C}_n$ the unique tournament we get from deleting one vertex in C_{n+1}. Note that one can alternatively construct C_n from C_{n-1} by duplicating one vertex and adding the edge between the two otherwise identical vertices in either direction.

Our first result describes precisely all extremal constructions for $I(C_4, n)$ for large enough n.

Theorem 2. *For $n \geq 4$, the tournaments maximizing $I(C_4, T)$ are precisely the tournaments in $\mathscr{C}_n$. Consequently, $i(C_4) = \frac{1}{2}$, and for every n, we have*

$$I(C_4, n) = \begin{cases} \frac{n(n^2-1)(n-3)}{48} & \text{if } n \text{ is odd,} \\ \frac{n(n^2-4)(n-3)}{48} & \text{if } n \text{ is even.} \end{cases}$$

Note that the asymptotic statement that $i(C_4) = \frac{1}{2}$ is also proved in [4], with a proof very similar to the one we provide in the next section. Numeric bounds from flag algebra computations indicate that a similar statement may also be true for C_5, C_6, C_7 and C_8, and we conjecture it is true for all k. See the discussion at the end of this paper for a few more details on this. Observe that for $k \geq 5$ and even $n \geq k$, C_n contains more copies of C_k than the other members of $\mathscr{C}_n$, so our conjectured extremal tournament is unique for $k \geq 5$.

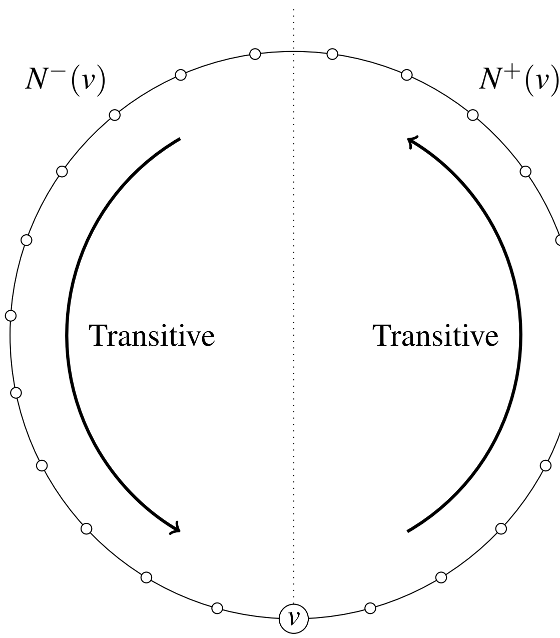


Figure 2: For odd n, the carousel $C \in \mathscr{C}_n$ is unique and vertex transitive. For even n, the directions of the diagonals can be chosen arbitrarily, resulting in several non-isomorphic tournaments.

Conjecture 3. *For all $k \geq 5$ and $n \geq k$, the unique n-vertex tournaments maximizing $I(C_k, T)$ are the tournaments C_n.*

The only tournaments on 4 vertices left to consider are the two tournaments C_3^- and C_3^+. As one gets C_3^- from C_3^+ by reversal of all arcs, the tournaments extremal for C_3^- are precisely the reversals of the tournaments extremal for C_3^+, so it suffices to only study C_3^+. Consider the following probabilistic construction of a tournament $\tilde{T}_n$ on n vertices which was discovered independently by Burgher and Burke, and in [4] with an almost matching upper bound via the flag algebra method. For some fixed $\alpha \in (0,1)$, partition the vertices into two sets H_n (for high out-degree) and L_n (for low out-degree) of size $\lceil \alpha n \rceil$ and $\lfloor (1-\alpha)n \rfloor$, respectively. On the set L_n, direct the edges uniformly at random, i.e. insert a random tournament R on $\lfloor (1-\alpha)n \rfloor$ vertices. All arcs between the sets are directed from H_n to L_n. On the set H_n, iterate the construction, i.e. insert the tournament $\tilde{T}_{\lceil \alpha n \rceil}$ inductively. See Figure 3 for a sketch of the iterated construction.

Note that with probability approaching 1 for large n, we have $i(H,R) = \mathbb{E}(i(H,R)) + o(1)$ for every tournament H in a random tournament R on n vertices. We may thus choose a (quasi-random) sequence of tournaments R_n on n vertices with $i(H,R_n) = \mathbb{E}(i(H,R)) + o(1)$, and use this sequence in place of the probabilistic construction described above.

In this construction, all copies of C_3^+ lie completely in H_n, completely in L_n, or have exactly one vertex in H_n and three vertices forming a C_3 in L_n. Notice that $i(C_3, R_{\lfloor (1-\alpha)n \rfloor}) = 1/4 + o(1)$ and $i(C_3^+, R_{\lfloor (1-\alpha)n \rfloor}) = 1/8 + o(1)$. As $i(C_3^+, T_{\lceil \alpha n \rceil}) = i(C_3^+, T_n) + o(1)$, we have

$$\begin{aligned} i(C_3^+, T_n) &= \alpha^4 i(C_3^+, T_n) + 4\alpha(1-\alpha)^3 i(C_3, R_{\lfloor (1-\alpha)n \rfloor}) + (1-\alpha)^4 i(C_3^+, R_{\lfloor (1-\alpha)n \rfloor}) + o(1) \\ &= \alpha^4 i(C_3^+, T_n) + 4\alpha(1-\alpha)^3 \tfrac{1}{4} + (1-\alpha)^4 \tfrac{1}{8} + o(1), \end{aligned}$$

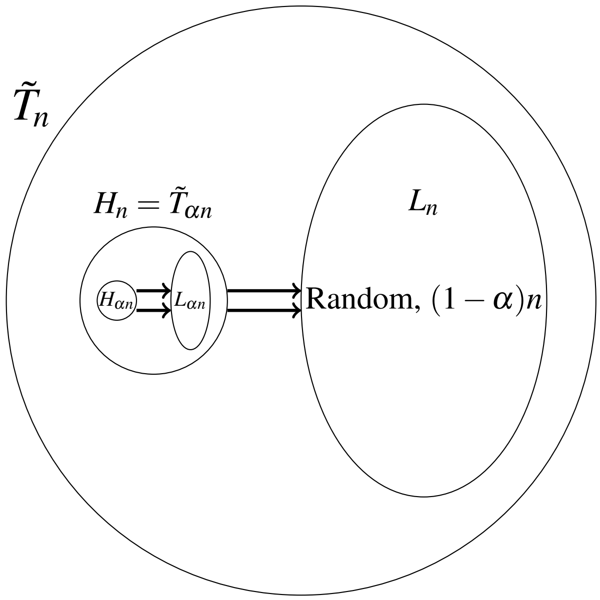


Figure 3: A construction asymptotically maximizing the number of copies of C_3^+. For $\alpha \in [0,1]$ and n sufficiently large, this construction $\tilde{T}_n$ can be decomposed into subtournaments L_n, of size about $(1-\alpha)n$, and H_n, of size about αn with the properties shown above.

so

$$i(C_3^+, T_n) = \frac{\alpha(1-\alpha)^3 + \frac{1}{8}(1-\alpha)^4}{1-\alpha^4} + o(1).$$

Maximizing this quantity gives us $\alpha = \frac{1}{5}(2\sqrt[3]{9} - 2 - \sqrt[3]{3}) \approx 0.1435836$, and

$$i(C_3^+, T_n) + o(1) = \frac{1}{8}\left(8 - 9\sqrt[3]{3} + 3\sqrt[3]{9}\right) \approx 0.157500667.$$

We show in Section 4 that all large extremal tournaments for C_3^+ essentially look this way. While we can not determine exactly the extremal tournaments T_n, we can at least say that the limit object is a unique graphon (when the original definition of graphons is transferred to the tournament setting).

Theorem 4. *Let $(T_n)_{n=1}^{\infty}$ be a sequence of tournaments on n vertices with $I(C_3^+, T_n) = I(C_3^+, n)$. Let $\alpha = \frac{1}{5}(2\sqrt[3]{9} - 2 - \sqrt[3]{3})$. For sufficiently large n, the vertex set of T_n can be partitioned into sets L_n and H_n so that $|H_n| = \alpha n + o(1)$, all arcs between these sets are from H_n to L_n, the sequence of tournaments $(T_n[L_n])_{n=1}^{\infty}$ is quasi-random, and $I(C_3^+, T_n[H_n]) = I(C_3^+, |H_n|)$. Hence*

$$i(C_3^+) = \frac{1}{8}\left(8 - 9\sqrt[3]{3} + 3\sqrt[3]{9}\right) \approx 0.157500667.$$

While C_3^+ may not be the most interesting tournament to consider at first, we find this extremal construction fascinating. It combines quasi-random parts with iterated blow-ups, and is thus more complex than most known extremal constructions for other problems.

3 Proof of Theorem 2

Proof of Theorem 2. We begin by observing the following identity for all tournaments T on at least 4 vertices:

$$i(C_3,T) = \tfrac{1}{2}i(C_4,T) + \tfrac{1}{4}i(C_3^+,T) + \tfrac{1}{4}i(C_3^-,T). \tag{1}$$

This follows from the fact that the probability to find a C_3 when picking three vertices at random is equal to the probability to first find C_4, C_3^+, or C_3^- when picking four vertices, times the appropriate probability that removing one of these vertices leaves a C_3.

Multiplying both sides by $\binom{n}{4}$, we can express this relationship in terms of a direct count of induced C_4 for any tournament T:

$$I(C_3,T)\cdot\tfrac{n-3}{4} = \tfrac{1}{2}I(C_4,T) + \tfrac{1}{4}(I(C_3^+,T)+I(C_3^-,T)),$$

implying that

$$I(C_4,T) = I(C_3,T)\cdot\tfrac{n-3}{2} - \tfrac{1}{2}(I(C_3^+,T)+I(C_3^-,T)).$$

Let $T \in \mathscr{C}_n$. Then every vertex in T has out-degree in $\{\frac{n-2}{2},\frac{n-1}{2},\frac{n}{2}\}$, so by Proposition 1, $I(C_3,T)$ is maximized. On the other hand, the out-neighborhoods and in-neighborhoods of all vertices in T induce transitive tournaments, so $I(C_3^+,T) = I(C_3^-,T) = 0$. This shows that T maximizes $I(C_4,T)$. The ideas up to this point are very similar to the proof in [4].

To extend their result to Theorem 2, it remains to show that no other tournament shares this property. For this, let T be any $\{C_3^+,C_3^-\}$-free, (near) regular tournament, and let $v_1 \in V(T)$ with $d^+(v_1) = k \in \{\frac{n-2}{2},\frac{n-1}{2},\frac{n}{2}\}$. As T is C_3^+-free, the out-neighborhood of v_1 is C_3-free and therefore transitive, and we may relabel the out-neighbors in this induced order as $\{v_2,v_3,\dots,v_{k+1}\}$. Similarly, the in-neighborhood is transitive, and we may relabel it in the induced order as $\{v_{k+2},\dots,v_n\}$.

Now suppose, for the sake of contradiction, that $T \notin \mathscr{C}_n$, and thus there exists an arc v_iv_j with $0 < i-j < \frac{n}{2}$ or if $-n < i-j < -\frac{n}{2}$. Let us first assume that $0 < i-j < \frac{n}{2}$. As $\{v_2,v_3,\dots,v_{k+1}\}$ and $\{v_{k+2},\dots,v_n\}$ are transitively ordered, we have $j \le k$ and $i \ge k+1$. As v_j has out-degree at least $\frac{n-1}{2}$, v_j has an out-neighbor $v_{i'}$ with $i' > i$, implying that $v_iv_{i'} \in E(T)$. But now $T[v_1,v_j,v_i,v_{i'}] \simeq C_3^+$, a contradiction.

Let us now assume that $-n < i-j < -\frac{n}{2}$, and so $i \le k$ and $j > k$. Similarly as before, there now exists a j' with $k < j' < j$ and $v_{j'}v_i \in E$, which again implies that $T[v_1,v_i,v_{j'},v_j] \simeq C_3^+$, a contradiction proving the theorem. □

4 Proof of Theorem 4

Proof of Theorem 4. We start with an upper bound for the inducibility of C_3^+ using standard flag algebra methods. Notice that the upper bound is not sharp, which is common for extremal constructions involving iterations. We will always assume that n is large enough that we are allowed to suppress lower order terms in our computations.

Claim 4.1. $i(C_3^+,n) \in (0.157500667, 0.157500672)$.

Proof. We know that $i(C_3^+, n) > 0.157500667$ by our construction. Using standard plain flag algebra techniques, we find that

$$i(C_3^+) \le \frac{2205009402351719189231861082004597041252856}{14000000000000000000000000000000000000000000} < 0.157500672.$$

We computed with flags of size 8, and the computation, including the subsequent claims, ran for about 6 hours on a standard desktop. Certificates are too large to be presented here, and do not add much insight. They can be found at http://lidicky.name/pub/tournaments. □

In the next claim, a symmetrization argument gives that every vertex is in roughly the same number of C_3^+. Recall that in the theorem statement, $(T_n)_{n=1}^{\infty}$ is a sequence of tournaments on n vertices with $I(C_3^+, T_n) = I(C_3^+, n)$.

Claim 4.2. *Every vertex is in $i(C_3^+, T_n)\binom{n-1}{3} + O(n^2)$ many copies of C_3^+.*

Proof. By definition, the average number of copies a vertex is in is $i(C_3^+, T_n)\binom{n-1}{3}$. Let v be a vertex which is in the fewest copies $C_3^+(v)$, and let w be a vertex which is in the most copies $C_3^+(w)$. Let $C_3^+(vw)$ be the number of copies containing both v and w. If we delete v, and add a copy of w, we gain

$$C_3^+(w) - C_3^+(v) - C_3^+(vw)$$

copies of C_3^+. As T_n is extremal, this quantity must be non-positive. Observing that $C_3^+(vw) = O(n^2)$ shows the claim. □

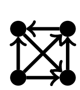
The traditional way to extract structure from flag algebra computations is to look for subgraphs for which the computations tell you that they have zero ($+o(1)$) density in every extremal construction. But this only works if the bounds from the computation are sharp. If the computations do not give sharp bounds like in our case, another approach is to do the opposite, and to compute bounds on subgraphs which occur with high density to find a general structure of the extremal example, and then use stability methods to establish the finer structure. Neither of these approaches has much promise in this problem without new ideas. As a large part of the conjectured extremal tournament is quasi-random, all subgraphs appear with a frequency similar to a random tournament, and structural differences to a random tournament are difficult to extract.

Inspired by the conjectured extremal tournament, we are looking for other features. A first observation is that the degree distribution is concentrated around a few discrete values. All vertices in L_n have about the same fairly small out-degree, and all vertices in H_n have very large out-degree, where the corresponding discrete values are a sequence converging to 1 when normalized. A second observation is that all arcs between L_n and H_n are directed from H_n to L_n. We use flag algebra computations to prove that these two observations are true in every extremal tournament, and from this we are able to prove the theorem.

Let $deg(x)$ be the normalized out-degree distribution function for an extremal tournament T_n:

$$deg(x) = \tfrac{1}{n}\left|\{v \in V(T) : d^+(v) = xn\}\right|.$$

For the remainder of the proof, the word "normalized" will be suppressed for simplicity. To make our computations more intuitive to follow, we will often denote the quantity $i(H, T_n)$ by a picture of the graph H, so we might write

 $= i(C_3^+, T_n).$

We now show that T_n has a degree distribution similar to the conjectured example, i.e. almost all vertices have degrees in small intervals around the degrees appearing in the construction. While we might be able to separate the high degree vertices into more degree bands with more effort, this will not be needed later, so we collect them all in one interval. We state these bounds up to a precision useful later in the proof.

Claim 4.3. *For all $x \in [0, 0.416] \cup [0.44057, 0.8849]$, $deg(x) = 0$. In other words, all but $o(n)$ vertices have degrees either in $(0.416, 0.44057)$ or in $(0.8849, 1]$.*

Proof. We prove this claim by showing three bounds. First we investigate vertices v with $d^+(v) \leq 0.85n$ and obtain lower and upper bounds on $d^+(v)$, namely that $d^+(v) \in (0.416, 0.44057)$. For the third bound, we switch to vertices v with $d^+(v) \geq 0.85n$ and show that actually $d^+(v) > 0.8849n$.

We begin with the lower bound of the support of $deg(x)$. Fix some vertex $v \in V(T_n)$ and color all vertices in $N^+(v)$ black and color $N^-(v)$ white. We will use flag algebras to bound the proportion of black vertices in $T_n - v$, and to this end we begin setting up a program that can be bounded by the plain flag algebra method. Since $i(C_3^+, T_n - v) > 0.157500667$, we know the sum of the densities of all 2-colorings of C_3^+ is at least 0.157500667. We reduce our search space with the constraint that $\bullet \leq 0.85$, interpreted as v having normalized out-degree at most 0.85.

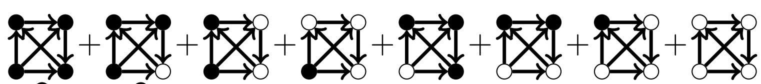
Ignoring lower order terms, we also know that every vertex is in the same number of C_3^+ (see Claim 4.2), so we can add an additional constraint to reflect this fact. If v plays the role of the source vertex in the C_3^+, then the remaining three vertices are all in $N^+(v)$ and induce a C_3. Otherwise, v plays the role of one of the vertices in the C_3, and the other three vertices induce a transitive triangle where the source and sink are in $N^-(v)$ and the last vertex is in $N^+(x)$. Our coloring scheme thus allows us to include the final bound in the following program:

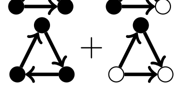
Objective:

minimize $\bullet$

Constraints:

$\bullet \leq 0.85$

$0.157500667 \leq$ 

$0.157500667 \leq$ 

From this program, we find that $\bullet > 0.416$. More precisely,

$$\bullet \geq \frac{3745132053776853970635292684882343419187823353945331}{9000000000000000000000000000000000000000000000000000} > 0.416.$$

Similarly, we obtain $\bullet < 0.44057$, or more precisely that

$$\bullet \leq \frac{3965045776267019146300369491791058488457641019475359}{9000000000000000000000000000000000000000000000000000} < 0.44057,$$

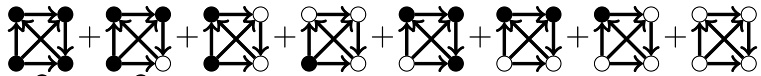
from the following program:

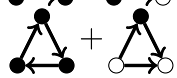
Objective:

maximize ●

Constraints:

● ≤ 0.85

$0.157500667 \leq$ 

$0.157500667 \leq$ 

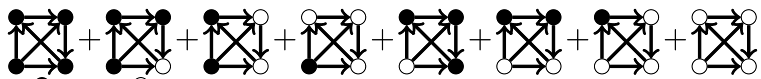
These two results imply that for large enough n, no vertices have normalized out-degree in $[0, 0.416] \cup [0.44057, 0.85]$. We extend this result with the following program restricting the degree of large out-degree vertices:

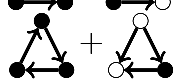
Objective:

minimize ●

Constraints:

● ≥ 0.85

$0.157500667 \leq$ 

$0.157500667 \leq$ 

This program outputs the lower bound ● > 0.8849, completing the proof of this claim. More precisely, it gives

$$\bullet \geq \frac{79643492792204114952035119469621860304229032162826 24}{9000000000000000000000000000000000000000000000000} > 0.88492769769.$$

All certificates can be found at http://lidicky.name/pub/tournaments. □

Let H_n be the set of vertices in T_n with normalized out-degree in $(0.8849, 1]$, and L_n be the set of vertices with normalized out-degree in $(0.416, 0.44057)$. The above claim implies that $H_n \cup L_n = V(T_n)$. We now show that no arcs in T_n are directed from L_n to H_n, once again using a coloring-scheme to acquire localized information in an extremal construction.

Claim 4.4. *For every $x \in L_n$ and $y \in H_n$, $yx \in E(T_n)$.*

Proof. Let $x \in L_n$, $y \in H_n$, so x has normalized out degree in $[0.415, 0.441]$ and y has normalized out-degree at least 0.8849. We color $V(T_n) - \{x, y\}$ with the following scheme, in which the top color represents the relation to x, and the bottom color represents the relation to y (see also Figure 4):

- Assign color black-black ● to $N^+(x) \cap N^+(y)$,
- Assign color black-white ◓ to $N^+(x) \cap N^-(y)$,
- Assign color white-black ◒ to $N^-(x) \cap N^+(y)$,

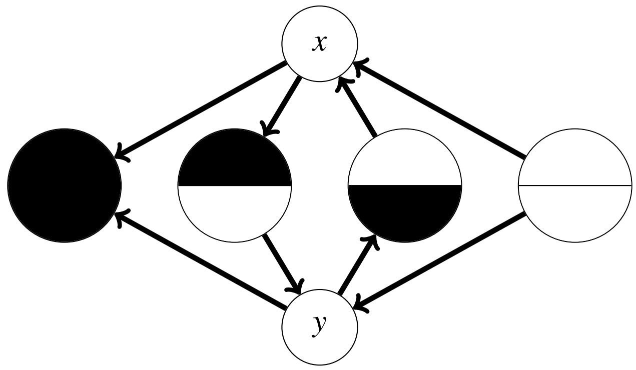


Figure 4: Four-coloring scheme for $T_n - \{x, y\}$

.

- Assign color white-white ⊖ to $N^-(x) \cap N^-(y)$.

In order to model the out-degree assumptions, we will use the following constraints:

$$0.416 \leq ● + ◓ \leq 0.44057 \quad \text{and} \quad 0.8849 \leq ● + ◒.$$

As in the proof of Claim 4.3, any programs involving this color scheme can include a constraint to ensure that x is in the right number of C_3^+ with vertices in $V(T_n)\backslash y$, and that y is in the right number of C_3^+ with vertices in $V(T_n)\backslash x$.

The purpose of this set up is to show that $x \to y$ results in fewer C_3^+ than $y \to x$, so we need to count C_3^+ which include both of these vertices, with the edge between the vertices in either direction. For this, we look again at Figure 4. If $x \to y$, we create a C_3^+ with each arc ⊖→◒ and with each arc ●→◓ . On the other hand, if $y \to x$, we create a C_3^+ with each arc ⊖→◓ and with each arc ●→◒ .

Similarly as above, we can now pose the following program bounding the difference between C_3^+ containing $x \to y$ and containing $y \to x$. Note that there are up to 96 different C_3^+ in $T_n - \{x, y\}$ with 4 colors. Also, when counting the C_3^+ in $T_n - y$ containing x, we have to account for the colors induced by the arcs with y.

Objective: maximize

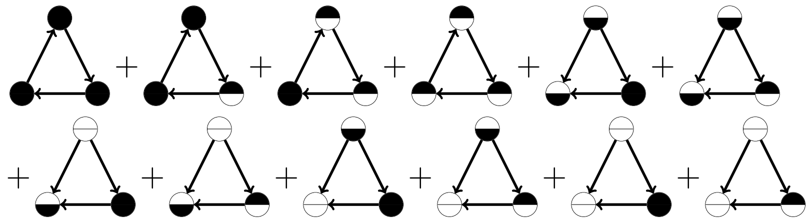
$$\left((⊖ \to ◒) + (● \to ◓) \right) - \left((⊖ \to ◓) + (● \to ◒) \right).$$

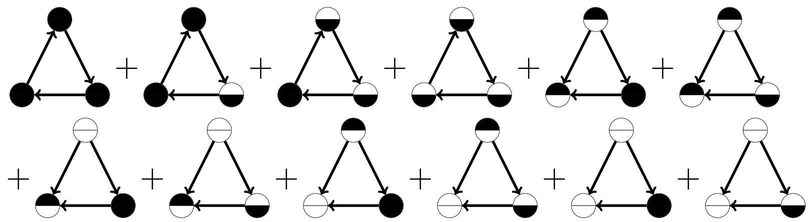
Constraints:

$$0.416 \le ● + ◓ \le 0.44057$$

$$0.8849 \le ● + ◒$$

$$0.157500667 \le \text{ sum of all 4-colorings of } C_3^+$$

0.157500667 ≤ 

0.157500667 ≤ 

We find that the solution to this program is bounded above by -0.077:

$$\frac{-94611767053769387132533069422799745047344297}{1220703125000000000000000000000000000000000000} \approx -0.077506,$$

implying that $(y \to x)$ results in at least $0.077 \cdot \binom{n}{2}$ more copies of C_3^+ than $(x \to y)$ in T_n for sufficiently large n, proving our claim. Certificates can be found at http://lidicky.name/pub/tournaments. □

Having determined the behavior of the relationship between H_n and L_n, we now focus on the internal behavior of H_n. The following claim implies that, for large enough n, the overall structure of T_n iterates into H_n.

Claim 4.5. $I(C_3^+, T_n[H_n]) = I(C_3^+, |H_n|)$.

Proof. The only copies of C_3^+ in T_n are those chosen completely in H_n, completely in L_n, or with precisely 1 vertex chosen from H_n. The arcs in $T_n[H]$ impact neither the second nor third type of C_3^+. Therefore, $T_n[H_n]$ is extremal and the claim follows. □

We next focus on showing that the sizes of H_n and L_n are correct. While we could prove a slightly stronger bound here with the same method, we only need $|L_n| < (\frac{6}{7} - \varepsilon)n$ later in Claim 4.9.

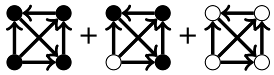
Claim 4.6. $|L_n| < \frac{6}{7}n - 0.00001n \approx 0.85713n$.

Proof. First, since there are no arcs from L_n to H_n, the average out-degree of vertices in L_n is $\frac{|L_n|-1}{2}$, so by Claim 4.3

$$\frac{1}{n}|L_n| \in (0.832, 0.88114).$$

We would like a tighter upper bound, so we pose the following program wherein we color the vertices in L_n black and the vertices in H_n white. In this program, we assume that $|L_n| \ge \frac{6}{7}n$ and show that the density of C_3^+ is then bounded above by a bound smaller than in our construction, implying the claim. We note as well that $\bullet\!\!\longrightarrow\!\!\circ$ is a forbidden subgraph by Claim 4.4, so we include this as a constraint in the program as well.

Objective:

maximize 

Constraints:

$$6/7 - 0.00001 \leq \bullet \leq 0.88114$$

Forbidden Subgraph:

$$\bullet\!\!\longrightarrow\!\!\circ$$

This program is bounded above by
$$\frac{141750035467643236211349281693358548643157719744000}{900000000000000000000000000000000000000000000000000} < 0.15750004 < i(C_3^+, T_n),$$
thus cannot be extremal, and so implies that T_n must satisfy $|L_n| < \frac{6}{7}n$. Certificates can be found at http://lidicky.name/pub/tournaments. ☐

We next aim to prove that the sequence $(T_n[L_n])_{n=1}^{\infty}$ is quasi-random. To do so we prove Claim 4.7, a consequence of the characterization of quasi-random tournaments in (Chung and Graham [7]). They list 11 different equivalent properties characterizing quasi-random sequences G_n of tournaments on n vertices, but we will only use the first two.

- P_1: For every fixed tournament H, $i(H, G_n)$ converges to the expected value in the random tournament on n vertices.
- P_2: $\lim_{n\to\infty} i(C_4, G_n) = \frac{3}{8}$.

Since graphons are completely determined by subgraph densities, P_1 implies that every quasi-random sequence of tournaments has the same limit graphon. The following claim is an addition to the 11 properties listed in [7], tailored to our problem.

Claim 4.7. *A sequence of tournaments $(G_n)_{n=1}^{\infty}$ with $|G| = n$ is quasi-random if and only if*
$$\lim_{n\to\infty} i(C_3, G_n) = \tfrac{1}{4} \ \textit{and} \ \lim_{n\to\infty} i(C_3^+, G_n) = \tfrac{1}{8}.$$

Proof. The "only if" statement follows immediately from property P_1, so we concern ourselves with proving the "if" statement. Let $(G_n)_{n=1}^{\infty}$ be a sequence of tournaments so that $|G_n| = n$, and recall from Proposition 1 that (near) regular tournaments are the tournaments which maximize the number of induced copies of C_3.

So, assume that
$$\lim_{n\to\infty} i(C_3, G_n) = \tfrac{1}{4} \ \text{and} \ \lim_{n\to\infty} i(C_3^+, G_n) = \tfrac{1}{8},$$

and observe that this implies that the degrees in the tournaments are concentrated around $\frac{n}{2}$, i.e. all but $o(n)$ vertices have out-degree $\frac{n}{2}+o(n)$. Now observe that

$$\begin{aligned}
\frac{1}{8}\binom{n}{4}+o(n^4) &= I(C_3^+,G_n)\\
&= \sum_{v\in V(G_n)} I(C_3,G_n[N^+(v)])\\
&= \sum_{v\in V(G_n)} i(C_3,G_n[N^+(v)])\binom{n/2}{3}+o(n^3), \text{ by degree concentration}\\
&\le \sum_{v\in V(G_n)} \left(\frac{1}{4}+o(1)\right)\binom{n/2}{3}+o(n^3), \text{ by the inducibilty of } C_3\\
&= \frac{1}{8}\binom{n}{4}+o(n^4).
\end{aligned}$$

This implies that $i(C_3,G_n[N^+(v)]) = \frac{1}{4}+o(1)$ for all but at most $o(n)$ vertices $v\in V(G_n)$. This equality also implies that $i(TT_4,G_n)=\frac{3}{8}+o(1)$. Now

$$\begin{aligned}
&\frac{1}{4}+o(1) = i(C_3,G_n) = \frac{1}{2}i(C_4,G_n)+\frac{1}{4}i(C_3^+,G_n)+\frac{1}{4}i(C_3^-,G_n), \text{ and}\\
&\frac{1}{4}+o(1) = \frac{1}{3}i(TT_3,G_n) = \frac{1}{6}i(C_4,G_n)+\frac{1}{4}i(C_3^+,G_n)+\frac{1}{4}i(C_3^-,G_n)+\frac{1}{3}i(TT_4,G_n),
\end{aligned}$$

so

$$o(1) = i(C_3,G_n)-\frac{1}{3}i(TT_3,G_n) = \frac{1}{3}i(C_4,G_n)-\frac{1}{8}+o(1),$$

and thus $i(C_4,G_n)=\frac{3}{8}+o(1)$. This last statement is equivalent to property P_2. □

Claim 4.8. *In any tournament T on n vertices, $i(C_3^+,T)\le\frac{1}{8}+\frac{2}{3}(\frac{1}{4}-i(C_3,T))+o(1)$.*

Proof. Using the plain flag algebra method, we show

$$3i(C_3^+,T)+2i(C_3,T)\le\frac{7}{8}+o(1).$$

The claim follows by rearranging the inequality. Certificates can be found at http://lidicky.name/pub/tournaments. □

Claim 4.9. *The sequence $(T_n[L_n])$ is quasi-random.*

Proof. Let $L=\frac{1}{n}|L_n|$. By Claim 4.5, $i(C_3^+,T_n[H_n])=i(C_3^+,T_n)+o(1)$. Thus, the density of the C_3^+ which are not completely contained in H_n is $(1-(1-L)^4)i(C_3^+,T_n)+o(1)$. We have,

$$\begin{aligned}
L^4\tfrac{1}{8}+4(1-L)L^3\tfrac{1}{4} &\le (1-(1-L)^4)i(C_3^+,T_n)+o(1)\\
&= L^4\cdot i(C_3^+,T_n[L_n])+4(1-L)L^3\cdot i(C_3,T_n[L_n])+o(1)\\
&\le L^4\cdot\left(\tfrac{1}{8}+\tfrac{2}{3}(\tfrac{1}{4}-i(C_3,T_n[L_n]))\right)+4(1-L)L^3\cdot i(C_3,T_n[L_n])+o(1)\\
&= \tfrac{7}{24}L^4+L^3 i(C_3,T_n[L_n])(4-\tfrac{14}{3}L)+o(1)\\
&\le \tfrac{7}{24}L^4+L^3\tfrac{1}{4}(4-\tfrac{14}{3}L)+o(1)\\
&= L^4\tfrac{1}{8}+4(1-L)L^3\tfrac{1}{4}+o(1).
\end{aligned}$$

The first inequality is true as the left side is the value of the next term we would expect if we replaced $T_n[L_n]$ by a random tournament on the same vertices. The second inequality follows from Claim 4.8. For the last inequality, note that $0 < L < \frac{6}{7} - 0.00001 + o(1)$ implying $4 - \frac{14}{3}L > 0.00004 + o(1)$. Thus, the left side is maximized if and only if C_3 is maximized at $\frac{1}{4}$. As the first and the last term in this chain of inequalities are equal up to $o(1)$, we have equality throughout. Thus $i(C_3, T_n[L_n]) = \frac{1}{4} + o(1)$ and $i(C_3^+, T_n[L_n]) = \frac{1}{8} + o(1)$, proving the claim using Claim 4.7. $\square$

Claim 4.10. *The normalized size of L_n is $L = \frac{1}{5}\left(7 + \sqrt[3]{3} - 2\sqrt[3]{9}\right) + o(1)$, and our construction converges in the graphon language to the limit object for the inducibility of C_3^+.*

Proof. We know that $L < 6/7 + o(1)$, that $T_n[L_n]$ is quasi-random, that all arcs between H_n and L_n point towards L_n, and that $i(C_3^+, T_n[H_n]) = i(C_3^+, T_n) + o(1)$ since $T_n[H_n]$ is extremal for C_3^+. Thus,

$$i(C_3^+, T_n) = L^4 \cdot \frac{1}{8} + \binom{4}{1} L^3(1-L) \cdot \frac{1}{4} + (1-L)^4(i(C_3, T_n) + o(1)).$$

This is maximized when $i(C_3^+, T_n) = \frac{1}{8}\left(8 - 9\sqrt[3]{3} + \sqrt[3]{3^5}\right) + o(1)$ and $L = 1 - \alpha + o(1)$. $\square$

We have thus shown that every extremal tournament matches our construction up to the choice of the sequence of quasi-random tournaments, completing the proof of this theorem. $\square$

5 Discussion

In this section, we discuss some of the peculiarities of this problem and its solutions, including the novel strategies introduced in this paper. First and foremost, we know of no other inducibility problem for which all extremal constructions include a quasi-random component as in the case of C_3^+ and C_3^- and ask the following question:

Problem 1. *For what classes of graphs (undirected or directed) do the extremal constructions for the corresponding inducibility problem involve non-trivial quasi-random components?*

For C_3^+, the extremal construction was conjectured by noting that our tournament can be decomposed into a source vertex and a C_3; described another way, we begin with an arc and blow up the head into a C_3. Essentially, we ask the following: for a digraph $G = (V, E)$ with cut $C = (S, T)$ and cut-set of size $|S| \cdot |T|$, for what structures $G[S]$ and $G[T]$ does the resulting inducibility problem have as extremal solutions constructions for which $\alpha \cdot 100\%$ of the vertices induce a "typical random graph structure" for some $\alpha \in (0, 1)$? Natural candidates for consideration would include $G[T] \cong C_3$ and $G[S]$ isomorphic to any 2-vertex digraph or 3-vertex tournament.

Historically, flag algebra techniques have been leveraged to determine bounds on global graph densities. The models developed in Claims 4.3 and 4.4, however, resulted in bounds on localized information. In the case of Claim 4.3, we were able to determine something very powerful regarding the distribution of out-degrees in extremal constructions, namely that all vertices have normalized out-degrees in a very specific set. In the case of Claim 4.4, we were able to determine the direction of an arc between any pair of vertices which satisfy basic constraints related to their out-degrees.

Finally, we want to make an observation about Conjecture 3. Let $k \geq 5$ be odd, and let $n > k$. Let $X \subset V(C_n)$ be a set of k vertices such that $C_n[X] \cong C_k$. Observe that for every vertex $v \in V(C_n) \setminus X$, we have $C_n[X \cup \{x\}] \cong C_{k+1}$. If we now express $i(C_k, T)$ in a tournament T in terms of densities of $(k+1)$-vertex graphs similarly to (1), we can easily conclude that Conjecture 3 is true for $k+1$ if it is true for k, so it suffices to prove it for all odd k. Standard plain flag algebra computations give sharp bounds for $i(C_5)$ and $i(C_7)$, and further show that C_n is $o(n^2)$ arc flips away from every extremal tournament for C_5 and C_7 (and thus for C_6 and C_8 by this observation), but we have not seriously tried to show the full conjecture for these cases, which would require to find the exact extremal tournaments.

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