# Large multipartite subgraphs in $H$-free graphs


Ping Hu[1], Bernard Lidický[2], Taísa Martins[3], Sergey Norin[4], and Jan Volec[5]

[1] School of Mathematics, Sun Yat-sen University, Guangzhou, 510275, China. E-mail: huping9@mail.sysu.edu.cn.

[2] Iowa State University, Department of Mathematics, Iowa State University, Ames, Iowa, USA. E-mail: lidicky@iastate.edu

[3] Instituto de Mathemática, Universidade Federal Fluminense, Niterói, Brazil. E-mail: tlmartins@id.uff.br.

[4] Department of Mathematics and Statistics, McGill Univ ersity, Montréal, Canada. E-mail: sergey.norin@mcgill.ca.

[5] Department of Mathematics, Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University in Prague, Trojanova 13, 120 00 Prague, Czech Republic. E-mail: jan@ucw.cz.



**Abstract.** In this extended abstract, we discuss a strengthening of a result of Füredi that every $n$-vertex $K_{r+1}$-free graph can be made $r$-partite by removing at most $\frac{r-1}{2r}n^2 - e(G)$ edges. In addition, we answer a problem of Sudakov showing that $K_6$-free graph can be made bipartite by removing at most $4n^2/25$ edges. We use flag algebras to express local cuts.



**Keywords:** max cut, Turán graph, flag algebras


## 1 Introduction

Let $G$ be a graph. Denote by $\mathrm{del}_r(G)$ the minimum order of a set of edges $X$ such that $G - X$ is $r$-partite. In this extended abstract we explore the problem when $G$ is $K_{r+1}$-free and we focus on $\mathrm{del}_r(G)$ and $\mathrm{del}_2(G)$. In the first case, Füredi [5] provides an upper bound for $\mathrm{del}_r(G)$. Recall that the number of edges in $K_{r+1}$-free graph on $n$ vertices upper bounded by number of edges in $\frac{r-1}{2r} \cdot n^2$.

**Theorem 1 (Füredi [5]).** *Fix $r \geq 2$. If $G$ is an $n$-vertex $K_{r+1}$-free graph, then $\mathrm{del}_r(G) \leq \frac{r-1}{2r} \cdot n^2 - e(G)$.*

When the number of edges of $G$ is close to extremal, Theorem 1 was sharpened in [1,6]. Here we focus on global improvement. We conjecture that Theorem 1 can be strengthened as follows.

*Conjecture 1.* Fix $r \geq 2$. If $G$ is an $n$-vertex $K_{r+1}$-free graph, then $\mathrm{del}_r(G) \leq 0.8\left(\frac{r-1}{2r} \cdot n^2 - e(G)\right)$.

Conjecture 1 would be best possible if true. We describe constructions showing that 0.8 cannot be improved in Section 3. We prove the conjecture for $r \in \{2, 3, 4\}$. As a general step towards Conjecture 1, we show the following improvement of Theorem 1.

**Theorem 2.** *For every $r \geq 2$ there exists $\varepsilon := \varepsilon(r) > 0$ such that the following holds. If $G$ is an $n$-vertex $K_{r+1}$-free graph, then $\mathrm{del}_r(G) \leq (1-\varepsilon)\left(\frac{r-1}{2r} \cdot n^2 - e(G)\right)$.*

Note that $\varepsilon(r) \to 0$ as $r \to \infty$ while Conjecture 1 claims $\varepsilon(r) = 0.2$.

Closely related problem is the following conjecture of Sudakov.

*Conjecture 2 (Sudakov [8]).* Fix $r \geq 5$. For every $K_r$-free graph $G$

$$\mathrm{del}_2(G) \leq \begin{cases} \frac{(r-2)^2}{4(r-1)^2} n^2 & r \text{ even, and} \\ \frac{r-3}{4r} n^2 & r \text{ odd.} \end{cases}$$

Sudakov [8] proved the conjecture for $r = 4$.

**Theorem 3 (Sudakov [8]).** *If $G$ is an $n$-vertex $K_4$-free graph, then $\mathrm{del}_2(G) \leq n^2/9$.*

We prove the conjecture for $r = 6$.

**Theorem 4.** *If $G$ is an $n$-vertex $K_6$-free graph, then $\mathrm{del}_2(G) \leq 4n^2/25$ edges. Moreover, if $\mathrm{del}_2(G) \leq 4n^2/25$, then $G$ is a Turán graph $T(n, 5)$.*

Notice that Conjecture 2 does not cover triangle-free case. The triangle-free case is a conjecture of Erdős [3] stating that every triangle-free graph can be made bipartite by removing at most $n^2/25$ edges. There is a well know bound that removing $n^2/18$ edges is enough by Erdős, Faudree, Pach, and Spencer [4]. An improvement using flag algebras in a similar manner as we do in this paper was announces recently by Balogh, Clemen, and Lidický [2].

The previously mentioned theorems and conjectures can be can be extended from $K_r$-free graph asymptotic results for $H$-free graphs using the regularity lemma.

We extensively use flag algebras, a versatile tool developed by Razborov [7]. We use convention that unlabeled vertices are depicted as black circles, labeled vertices as yellow squares, edges as blue lines. Dashed lines indicate that both edge and non-edge are admissible. We use $[\![.]\!]$ to denote the unlabeling/averaging operator. We omit $\phi(.)$ at most places to improve readability.

The rest of this extended abstract is organized as follows. We describe an alternative proof of Theorem 1 using flag algebras. This demonstrates how the other proofs are constructed. Then we examine the possible extremal constructions for Conjecture 1 and give the main ideas for proving the next case.

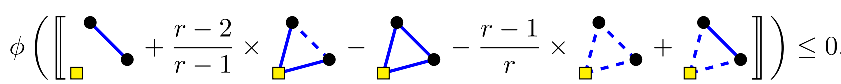
## 2 Theorem 1

As a warm-up to our techniques and arguments, we rephrase the proof to flag algebra language. Suppose Theorem 1 is false, and let $r$ be the smallest integer for which it fails. Clearly, $r \geq 3$. Let $G$ be an $n$-vertex $K_{r+1}$-free graph $G$ such that $\mathrm{del}_r(G) > \frac{r-1}{2r} \cdot n^2 - e(G)$. For a vertex $v \in V(G)$, consider an $r$-partition of $V(G)$ with $A_r := V \setminus N(v)$ being one part, and $(A_1, A_2, \ldots, A_{r-1})$ an $(r-1)$-partition of the neighborhood $N(v)$ given by Theorem 1. It follows that the number of edges inside the parts is at most $e(G[A_r]) + \mathrm{del}_{r-1}(G[N(v)])$, which is ast most

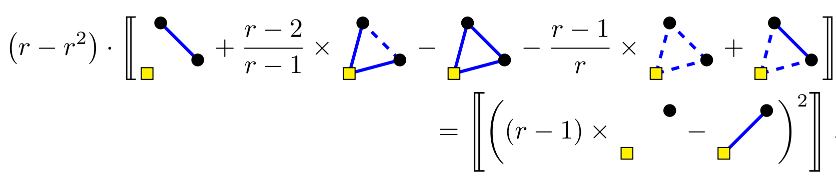
$$e(G[A_r]) + \frac{r-2}{r-1} \cdot \frac{|N(v)|^2}{2} - e(G[N(v)]). \tag{1}$$

On the other hand, this is at least $\mathrm{del}_r(G) > \frac{r-1}{2r} \cdot n^2 - e(G)$. This is in direct contradiction with the following simple flag algebra proposition, which shows that if we choose a vertex $v$ uniformly at random, then the expectation of (1) is at most $\frac{r-1}{2r} \cdot n^2 - e(G)$.

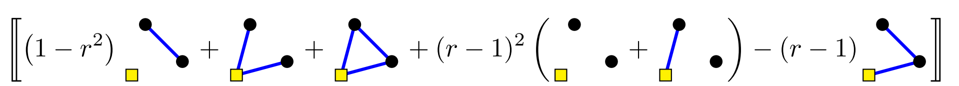
**Proposition 1.** *Fix $r \geq 2$. If $\phi$ is a $K_{r+1}$-free limit, then*



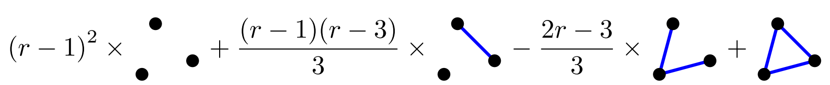
*Proof.* We will show that the following identity holds for every $r > 1$. It immediately proves the statement since the right hand side is non-negative while $r - r^2$ is negative.



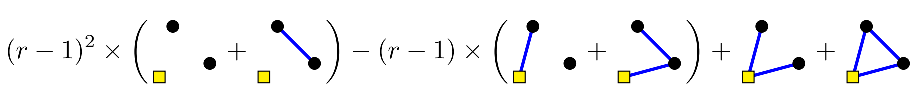
Firstly, observe that the left-hand side is equal to



which averages to


(2)

On the other hand, the right-hand side of the identity is equal to



which again averages to (2). This finished the proof. □

The exact statement of Theorem 1 can be obtained from Proposition 1 by a blow-up argument.

By inspecting the proof above, we can quickly observe that the bound in Theorem 1 is tight only when $G$ is the appropriate Turán graph, i.e., a complete balanced $r$-partite graph. Indeed, in this case Theorem 1 does not allow to remove any edge. However, this is rather a technical "obstacle" and Conjecture 1 improves on it.

## 3 Constructions for Conjecture 1

Conjecture 1 is tight for Turán graphs since it does not allow deletion of any edges in that case. For $r = 3$, the complete balanced bipartite graph and a balanced blow-up of $C_5$. Hence $C_5$ acts similarly as a complete bipartite graph with respect to Conjecture 1 and this propagated to larger $r$.

Given $r \geq 2$, a tight construction for Conjecture 1 can be obtained as follows: Let $H$ be a join of $a$ copies of $K_1$ and $b$ copies of $C_5$, where $a + 2b = r$. Let $G$ be a blow-up of $H$, such that all vertices corresponding to $K_1$s have the weight $1/r$ and all the vertices corresponding to $C_5$s have the weight $2/(5r)$. This captures all tight constructions for $3 \leq r \leq 5$, see Figure 1.

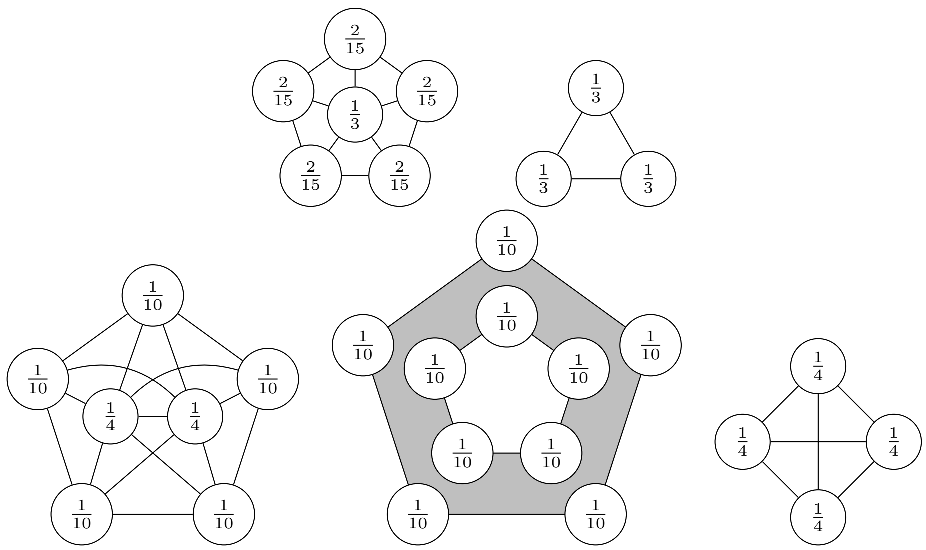
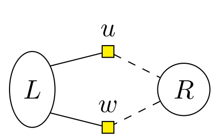


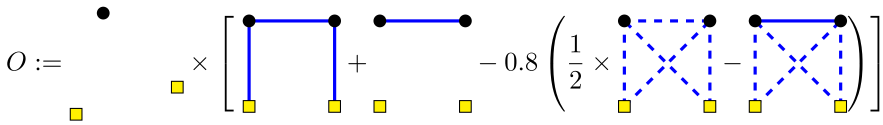
**Fig. 1.** Constructions for Conjecture 1 for $r = 4$ and 5.

### 3.1 Conjecture 1 for $r = 3$

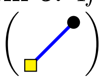
Let $N$ the non-edge type. Let $C$ be the combination of $N$-flags that computes the cut $L := N(u) \cup N(v)$ and $R := V \setminus L$.



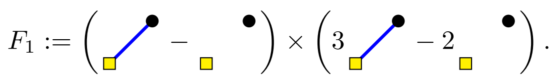
Let $O := \overline{K_3^N} \times (C - 0.8(1/2 - d(G))$, which in written using flag algebras loos like

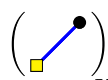


Notice that $\frac{1}{2} - d(G)$ is the density of missing edges to the complete bipartite graph. Then $0.8(\frac{1}{2} - d(G))$ is how many edges we are allowed to delete in Conjecture 1. In order to prove Conjecture 1, we need to show that $O$ is not positive.

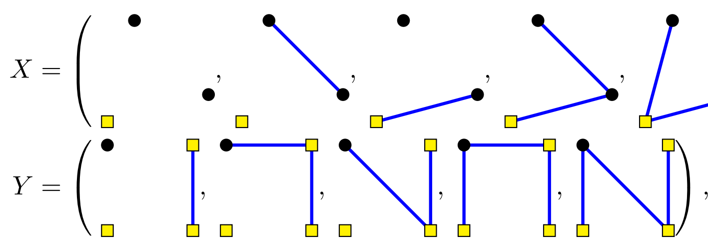
**Theorem 5.** *If $\phi$ is a $K_3$-free limit, then $\phi([\![O]\!]) \leq 0$. Moreover, if $\phi([\![O]\!]) = 0$, then $\phi^1$*  *$\in \{0.4, 0.5\}$ almost surely.*

*Proof.* First, let



Observe that if $\phi([\![F_1^2]\!]) = 0$ then $\phi^1$  $\in \{0.4, 0.5\}$ almost surely.

Next, consider the following two vectors $X$ and $Y$ of $\sigma$-flags, where $\sigma$ is the one-vertex type and the co-cherry type, respectively.



(3)

(4)

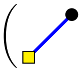
and the following 7 linear combinations of flags using $X$ and $Y$:

$F_2 = X \cdot (6, -9, 0, 0, -6)$, $F_4 = Y \cdot (0, 1, -1, 1, -1)$, $F_6 = Y \cdot (0, 2, -2, 1, -1)$,
$F_3 = X \cdot (4, 0, -3, -4, 4)$, $F_5 = Y \cdot (0, 1, -1, 2, -2)$, $F_7 = Y \cdot (6, 1, 1, -4, -4)$,
$F_8 = Y \cdot (2, -2, -2, 1, 1)$.

Note that using this notation, $F_1 = X \cdot (2, 2, -5/2, -5/2, 3)$.

Using a flag algebra calculation, we can express each term as a linear combination of 5-vertex unlabeled flags and obtain

$$-5065\,[\![O]\!] \geq \frac{554}{5}\,[\![F_1^2]\!] + \frac{133}{10}\,[\![F_2^2]\!] + 69\,[\![F_3^2]\!] + \frac{2471}{2}\,[\![F_4^2]\!] + \frac{7627}{4}\,[\![F_5^2]\!] + \frac{2771}{2}\,[\![F_6^2]\!] + \frac{287}{4}\,[\![F_7^2]\!] + 114\,[\![F_8^2]\!] \geq 0.$$

Therefore, $\phi(\llbracket O \rrbracket) \leq 0$. Moreover, if the equality is attained for $\phi$ then $\phi(\llbracket F_i^2 \rrbracket) = 0$ for all $i \in [8]$, and, in particular, $\phi^1$ (  ) $\in \{0.4, 0.5\}$ almost surely. □

By standard blow-up arguments, we deduce from Theorem 5 that Conjecture 1 holds for $K_3$-free graphs.

## 4 Concluding Remarks

Similar approach to Conjecture 1 as for $r = 3$ works also for $r = 4$ and $r = 5$. Although more ways if finding the partition are more squares are needed in the proof. Here the complexity of the proof grows with $r$ since we need to find good partition of an increasing number of tight constructions. It is not obvious how to generalize this approach to all $r$. However it is possible to obtain an improvement over Theorem 1.

Theorem 4 works along similar lines.

## Acknowledgement

The first author is supported in part by National Natural Science Foundation of China grants 11801593 and 11931002. The second author is supported in part by NSF grant DMS-1855653. The fourth author is supported in part by an NSERC Discovery grant.

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