Packing chromatic number for lattices

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Packing Chormatic Number

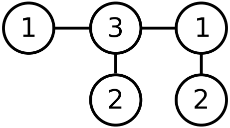
Definition

Graph $G = (V, E)$, $P_d \subseteq V$ is *d-packing* if $\forall u, v \in P_d : \mathrm{distance}(u, v) > d$.

1-packing is an independent set

Definition

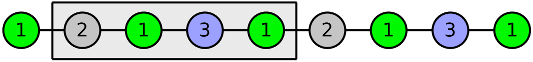
Packing chromatic number is the minimum k such that $V = P_1 \cup P_2 \cup ... \cup P_k$; denoted by $\chi_\rho(G)$.



About $\chi_\rho(G)$

- We study bounds for infinite lattices / graphs.
- Example infinite path P_∞

$$\chi_\rho(P_\infty) \leq 3$$



d-packing	ρ_d
1	$1/2$
2	$1/4$
3	$1/4$

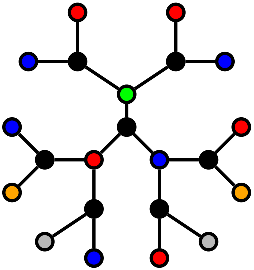
ρ_d is density of d-packing

Tree

Theorem (Sloper '02)

3-regular infinite tree T_3: $\chi_\rho(T_3) \leq 7$

d-packing	ρ_d
1	$1/2$
2	$1/6$
3	$1/6$
4	$1/18$
5	$1/18$
6	$1/36$
7	$1/36$



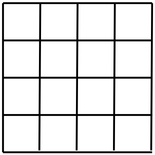
Theorem (Sloper '02)

4-regular infinite tree T_4: no bound on $\chi_\rho(T_4)$

Square lattice

Theorem (Goddard et al. ’02)

For infinite planar square lattice $\mathcal{R}_2$:
$9 \leq \chi_\rho(\mathcal{R}_2) \leq 23$



Theorem (Schwenk ’02)

$\chi_\rho(\mathcal{R}_2) \leq 22$

Theorem (Finbow and Rall ’07)

3-dimensional square lattice $\mathcal{R}_3$: no bound on $\chi_\rho(\mathcal{R}_3)$.

Hexagonal Lattice

Theorem (Brešar, Klavžar and Rall '07)

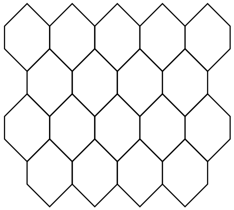
Hexagonal lattice $\mathcal{H}$*:* $6 \leq \chi_\rho(\mathcal{H}) \leq 8$

Theorem (Vesel '07)

$7 \leq \chi_\rho(\mathcal{H})$

Theorem

$\chi_\rho(\mathcal{H}) \leq 7$



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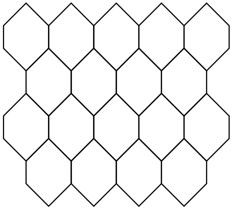

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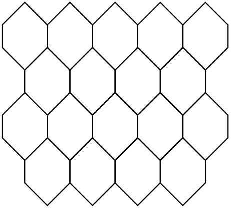

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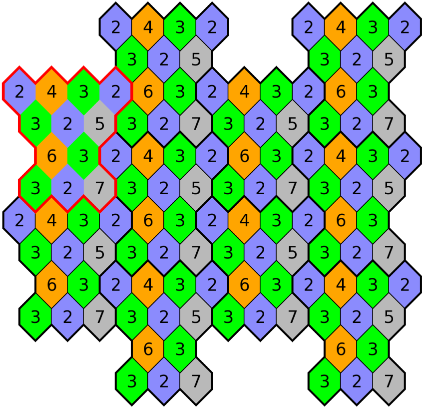
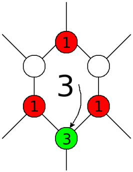


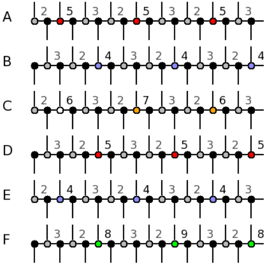


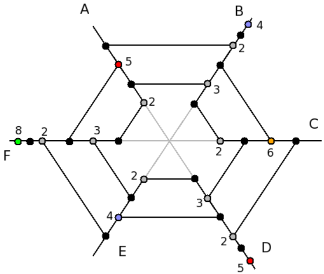
d-packing	ρ_d
1	1/2
2	1/6
3	1/6
4	1/24
5	1/24
6	1/24
7	1/24

Spider web (Hex lattice on cylinder)

Theorem

Spider web $\mathcal{W}$: $\chi_\rho(\mathcal{W}) \leq 9$



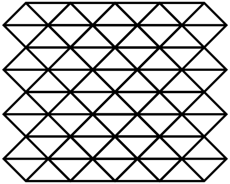


P_d	ρ_d
1	1/2
2	1/6
3	1/6
4	1/18
5	1/18
6	1/72
7	1/72
8	1/72
9	1/72

Triangular lattice $\mathcal{T}$

Theorem (F. and L. and independently Finbow and Rall)

Infinite triangular lattice $\mathcal{T}$ has unbounded packing chromatic number.



Proof.

Count the density of d-packings.

Triangular lattice $\mathcal{T}$

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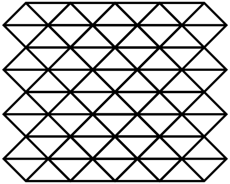
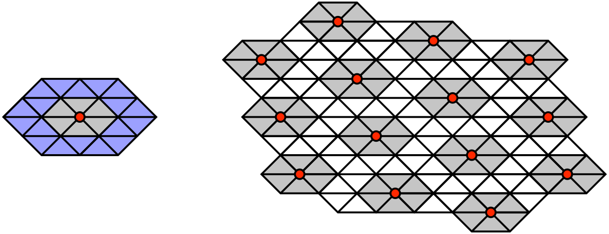


Proof.

Count the density of d-packings.

Idea of counting density of 2-packing.



Resize hex to $1/2$ and fill the lattice.
$\rho_2 \leq 1/7$

Sum of ρ_d for $\mathcal{T}$

d-packing	radius	upper bound on ρ_d
1		$1/3$
2	2	$1/7$
3	2	$1/7$
4	3	$1/19$
5	3	$1/19$
6	4	$1/37$
$2x-2$	x	$1/3x^2-3x+1$
$2x-1$	x	$1/3x^2-3x+1$

$$\sum_{d=1}^{\infty} \rho_d \leq \frac{1}{3} + \frac{2}{7} + \frac{2}{19} + 2\int_{3}^{\infty} \frac{1}{3x^2-3x+1}\,\mathrm{d}x \leq \frac{1977}{1995} < 1$$

Open problems

- What is the maximum packing chromatic number for a cubic graph?
- What is $\chi_\rho(\mathcal{R}_2)$ for the infinite planar square lattice $\mathcal{R}_2$?
- Is there a polynomial time alogrithm for deciding $\chi_\rho(G)$ for trees?
  ($\chi_\rho(G) \leq 3$ is in *P* and $\chi_\rho(G) \leq 4$ is NP-hard for general *G*)