Short cycle covers of graphs with minimum degree three

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Cycles and Colourings 2008 - Tatranská Štrba

Overview - from the previous talk

- *cycle* is a subgraph with all degrees even
- *circuit* is a connected 2-regular graph
- *cycle cover* is a set of cycles such that each edge is contained in at least one of the cycles
- *cycle double cover* is a set of cycles such that each edge is contained in exactly two cycles

Goal is to find a short cycle cover

Overview - from the previous talk

Conjecture (Alon and Tarsi, 1985)

Every m-edge bridgeless graph has a cycle cover of length at most $7m/5 = 1.4\ m$ *(SCC)*

Conjecture (Seymour, 1979 and Szekeres, 1973)

Every bridgeless graph has a cycle double cover (CDC)

Theorem (Jamshy, Raspaud and Tarsi, 1989)

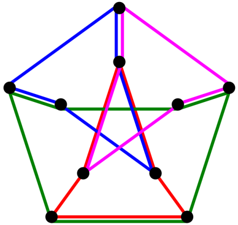
Every m-edge graph which admits 5-flow has a cycle cover of length at most $8m/5 = 1.6\ m$

Theorem (Kráľ, Nejedlý and Šámal, 2007+)

Every m-edge cubic graph has a cycle cover of length at most $34m/21 \approx 1.619\ m$

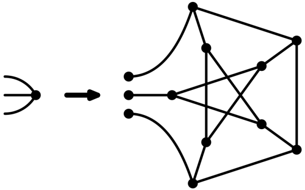
Short Cycle Cover implies Cycle Double Cover 1/2

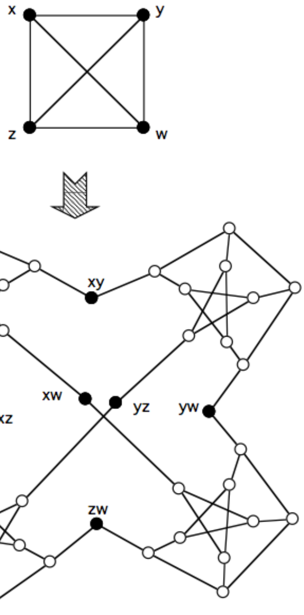
- reduction due to Jamshy and Tarsi (1992)
- CDC is enough to prove for cubic bridgeless graphs (splitting vertices, contracting 2-vertices)
- the Petersen graph has SCC of length $7m/5$



Short Cycle Cover implies Cycle Double Cover 2/2

- replace every vertex of a cubic graph G by part of Petersen

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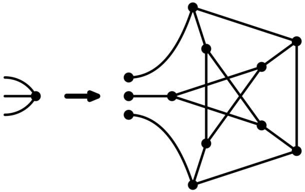

- find SCC - necessary behaves like on Petersen
- convert SCC back to G, edges covered by 1 or 2 cycles
- remove edges covered twice, the resulting graph is another cycle

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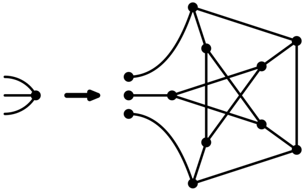

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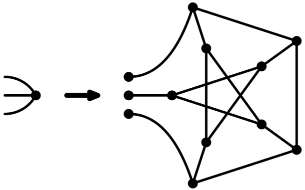

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Bridgeless graphs with mindegree three

Theorem (Kaiser, Král, L., Nejedlý, 2007+)

Bridgeless graph $G = (V, E)$ with mindegree three has a cycle cover of length at most $44m/27 \approx 1.630\ m$.

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Bridgeless graphs

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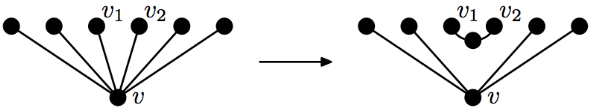

- suppress 2-vertices and add weights to edges (cubic graph), w is sum of all weigths
- create rainbow 2-factor
  - find a matching of weight $\leq w/3$ and 2-factor F
  - contract F and obtain nowhere-zero-4-flow
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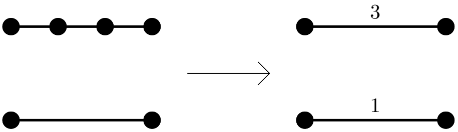

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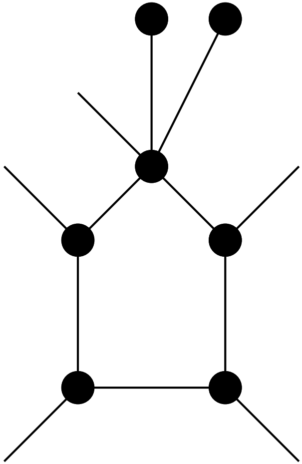

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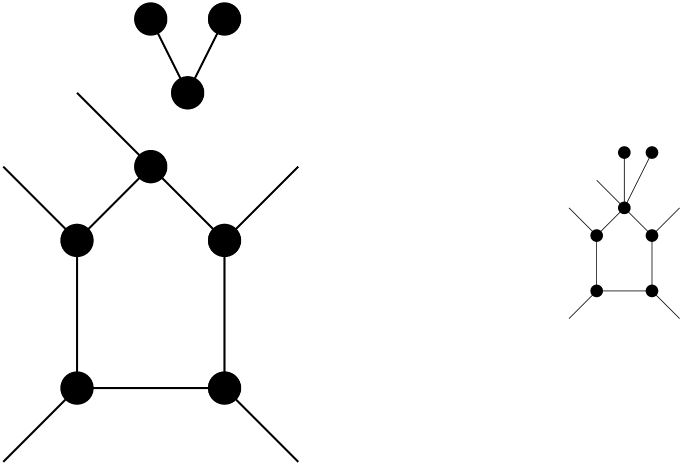

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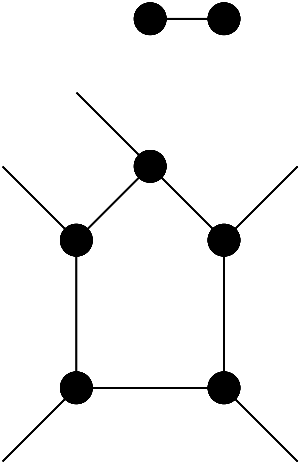

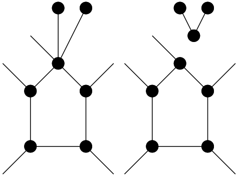

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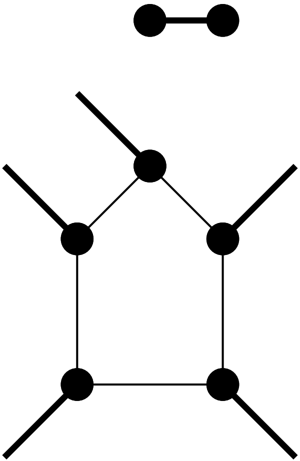

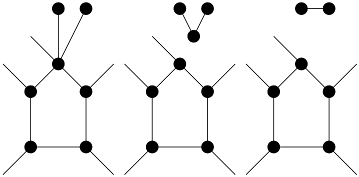

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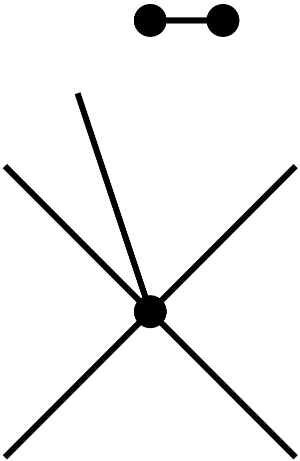

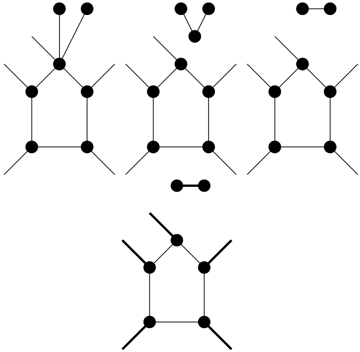

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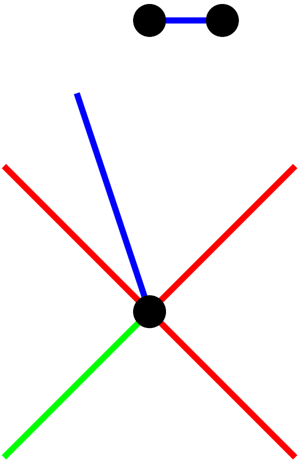

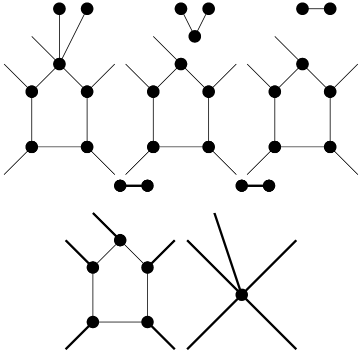

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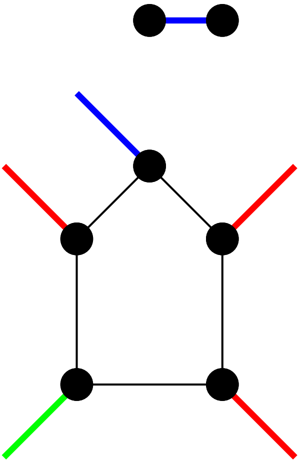

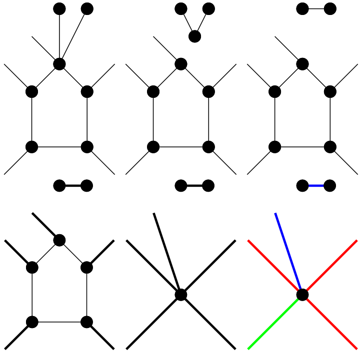

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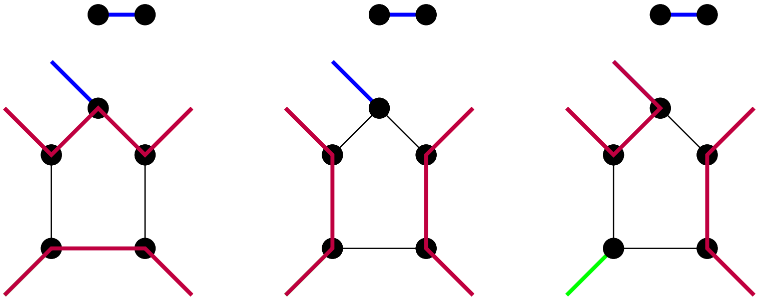

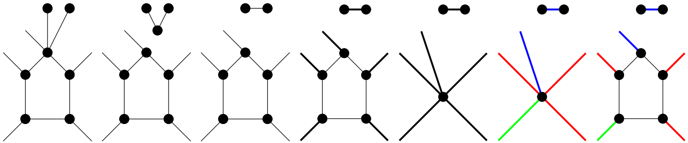

A little bit of computation

- assume $r \leq g \leq b$
- assume $r + g + b \leq m/3$
- usage $3r, 2g, b$ and $3/2F$.
- size of the cover:

$$3r+2g+b+3/2F = 2(r+g+b)+3/2F = 3m/2+m/6 = 5m/3$$

Bridgeless graphs with mindegree three - improvements

- combination of two cycle covers
- little bit unfriendly to vertices of degree two
- unable to split 4-vertices (expading)
- improve the nowhere-zero-4-flow

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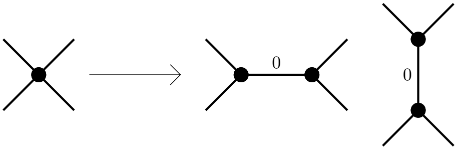

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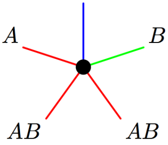

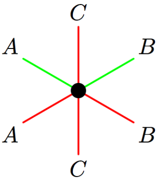

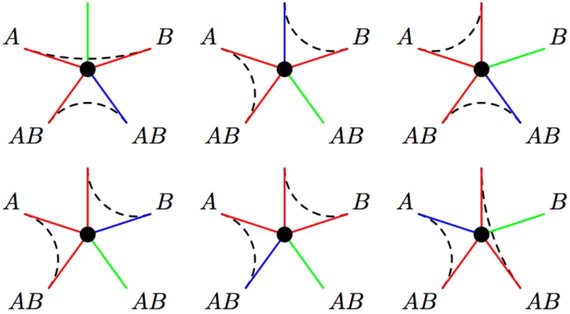
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Thank you for your attention