5-Coloring Graphs with 4 Crossings

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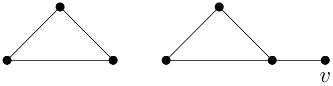
Basic definitions - quick reminder

Let $G = (V, E)$ be a graph and C a set of colors.

- *coloring* is a mapping $c : V \to C$.
- *chromatic number* $\chi(G)$ is minimum k such that G can be properly colored using k colors.
- G is *k-critical* if $\chi(G) = k$ and for every subgraph H of G holds $\chi(H) < k$.



What are k-critical graphs good for?

If $\chi(G) = k$ then G contains a k-critical subgraph
Algorithm for k colorability of G

- let K be all $(k+1)$-critical graphs
- test if any $H \in K$ is a subgraph of G
    - YES - G is not k-colorable
    - NO - G is k-colorable

is polynomial time if K is finite.

k-critical graphs on surfaces

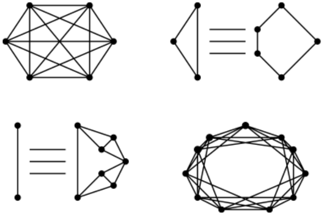
How many k-critical graphs are on a given surface?

k	number	author	year
≥ 8	finite	Dirac	1956
7	finite	Thomassen	1994
6	finite	Thomassen	1997
5	infinite	Fisk	1978
4	infinite	Fisk	1978

Do we know some of the lists?

6-critical graphs on surfaces

1. projective plane Dirac, 1956
   K_6
2. torus Thomassen, 1994



3. Klein bottle Chenette, Postle, Streib, Thomas and Yerger, independently Kawarabayashi, Král', Kynčl and L., 2008

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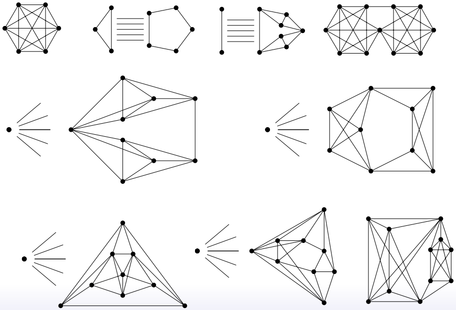


Crossings

Let G be embedded in the plane

- minimum number of crossings - $\mathrm{cr}(G)$
- crossing is defined by two edges
- *cluster of a crossing C* are endpoints of C

What raises $\chi(G)$? Clusters far apart or close?

Distant or close clusters?

Observation

If all clusters have a common vertex, then $\chi(G) \leq 5$*.*

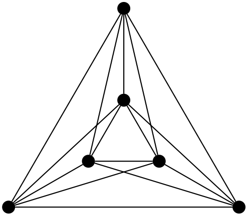
Theorem (Král' and Stacho, 2008)

If clusters of all crossings are disjoint, then $\chi(G) \leq 5$*.*

Let $G = (V, E)$ be a graph. An independent set $I \subseteq V$ is a stable crossing cover if $G - I$ is planar.

Theorem (Oporowski and Zhao, 2008)

If $\mathrm{cr}(G) \leq 3$ *and* $\omega(G) \leq 5$ *then G is 5 colorable.*
The only 6-critical graph with $\mathrm{cr}(G) \leq 3$ *is* K_6*.*



Conjecture (Oporowski and Zhao, 2008)

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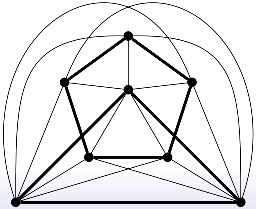

Improvements

Theorem (Oporowski and Zhao, 2008)

The only 6-critical graph with $\mathrm{cr}(G) \le 3$ *is* K_6*.*

Theorem

The only 6-critical graph with $\mathrm{cr}(G) \le 4$ *is* K_6*.*
If $\mathrm{cr}(G) \le 4$ *and* $\omega(G) \le 5$ *then* G *is 5 colorable.*

Theorem

The only 6-critical graph which is planar after removing three edges is K_6*.*
If G *is planar after removing three edges and* $\omega(G) \le 5$ *then* G *is 5 colorable.*

Theorem (+ Z. Dvořák)

There exists a 6-critical graph with $\mathrm{cr}(G) = 5$ *different from* K_6*.*

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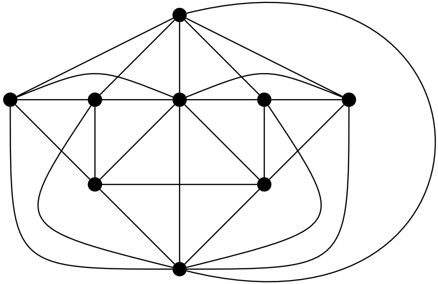

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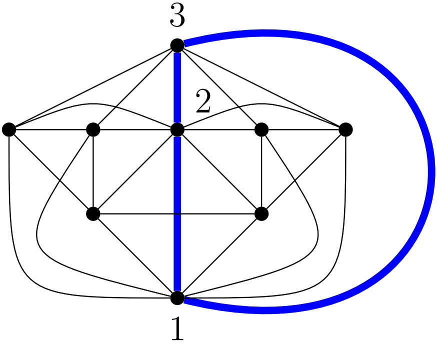

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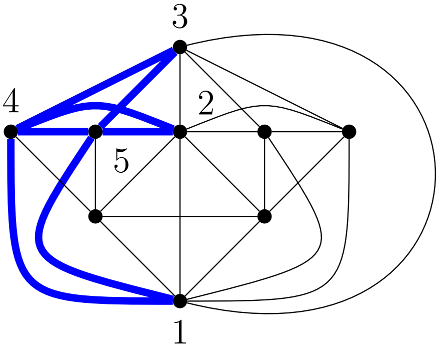

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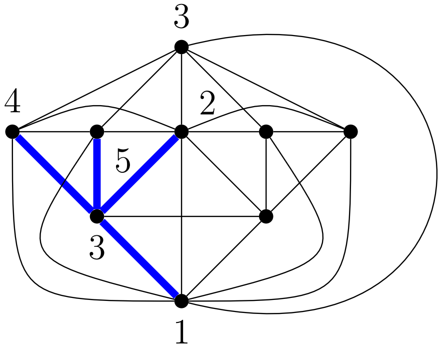

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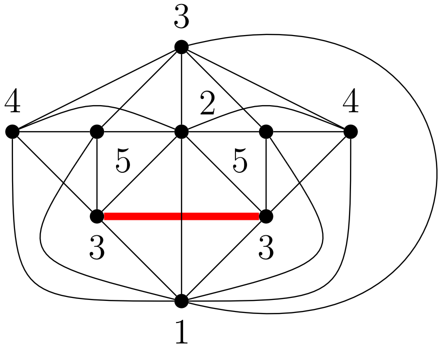
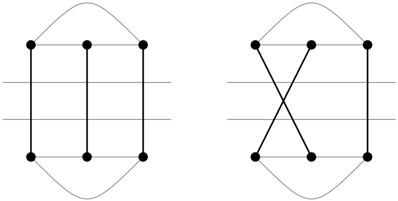


Theorem

The only 6-critical graph which is planar after removing three edges is K_6.
If G is planar after removing three edges F and $\omega(G) \leq 5$ then G is 5 colorable.

- edges in F share vertices
- endpoints of edges in F are a lot adjacent
- small adjacency of the edges



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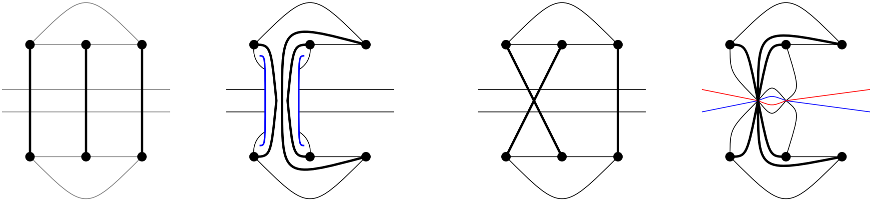

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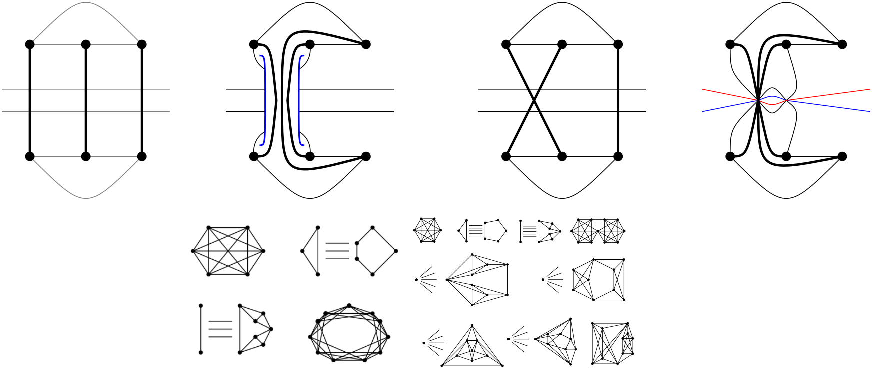
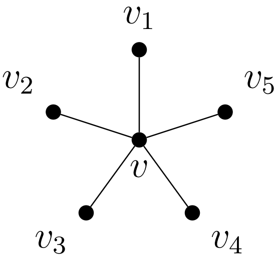


Theorem

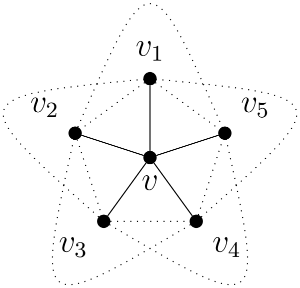
The only 6-critical graph with $\mathrm{cr}(G) \leq 4$ *is* K_6*.*
If $\mathrm{cr}(G) \leq 4$ *and* $\omega(G) \leq 5$ *then G is 5 colorable.*

- take the smallest counterexample
- each edge crossed once
- find a 5-vertex



The only 6-critical graph with $\mathrm{cr}(G) \leq 4$ is K_6.

- find a 5-vertex
- try Kempe chains
- try to identify neighbours of v



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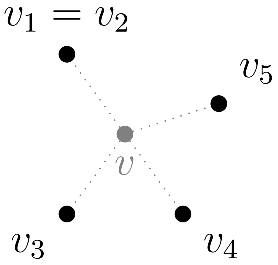

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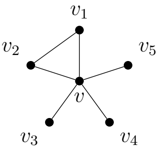
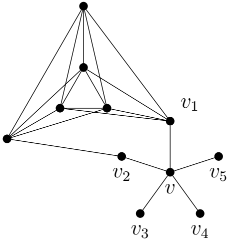
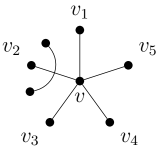
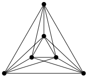
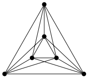
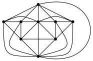






What next?

$\mathrm{cr}(G)$	list
0,1,2	-
3,4	
5	,  , . . .

Problem

List all 6-critical graphs with 5 crossings.