List coloring and crossings

Zdeněk Dvořák, Bernard Lidický and Riste Škrekovski

Charles University
University of Ljubljana

CanaDAM 2011 - Victoria

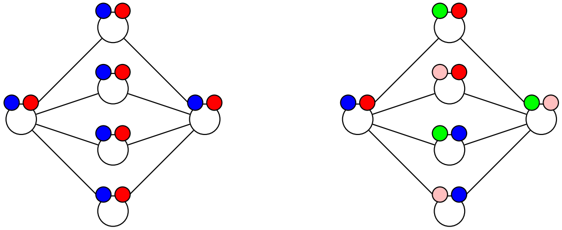
List coloring - quick reminder

Let G be a graph and C set of colors.

- *coloring* is a mapping $c : V(G) \to C$.
- coloring is *proper* if adjacent vertices have distinct colors
- *chromatic number* $\chi(G)$ is minimum k such that G can be properly colored using k colors.
- *list assignment* is a mapping $L : V(G) \to 2^C$
- *list coloring* (L-coloring) is a coloring c such that $c(v) \in L(v)$ for all $v \in V(G)$
- *choosability* $\text{ch}(G)$ is minimum k such that if $|L(v)| \geq k$ for all $v \in V(G)$ then G can be properly L-colored

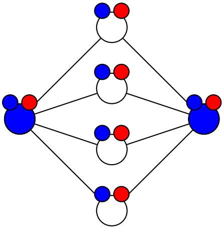
Chromatic number vs. Choosability

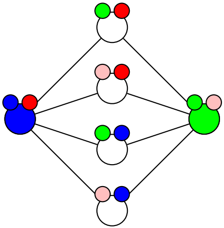
- $\chi(G) \leq \mathrm{ch}(G)$
- $\chi(G) \leq \Delta(G) + 1$ and also $\mathrm{ch}(G) \leq \Delta(G) + 1$
- Exists graph G: $\chi(G) < \mathrm{ch}(G)$



Chromatic number vs. Choosability

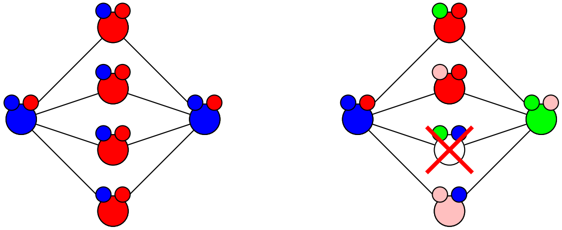
- $\chi(G) \leq \mathrm{ch}(G)$
- $\chi(G) \leq \Delta(G) + 1$ and also $\mathrm{ch}(G) \leq \Delta(G) + 1$
- Exists graph G: $\chi(G) < \mathrm{ch}(G)$





Chromatic number vs. Choosability

- $\chi(G) \leq \text{ch}(G)$
- $\chi(G) \leq \Delta(G) + 1$ and also $\text{ch}(G) \leq \Delta(G) + 1$
- Exists graph G: $\chi(G) < \text{ch}(G)$



List coloring - motivation to our problem

Theorem (Thomassen, 1994)

Every planar graph is 5-choosable.

Theorem (Voigt, 1994)

There exists a planar graph which is not 4-choosable

Is it possible to strengthen the theorem of Thomassen to allow some crossings?

List coloring - Thomassen's details

Corollary (Thomassen, 1994)

Every planar graph is 5-choosable.

Theorem (Thomassen, 1994)

Let G be a plane graph, F vertices of the outer face and $u_1, u_2 \in V(F)$ adjacent. Let L be a list assignment such that for every $v \in V(G)$:

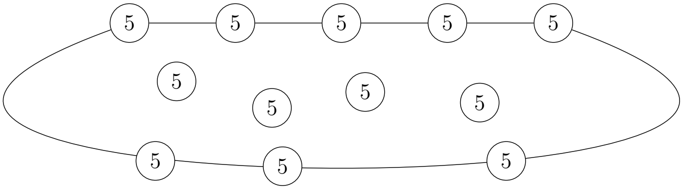
$$|L(v)| \geq \begin{cases} 1 & v \in \{u_1, u_2\} \\ 3 & v \in V(F) \setminus \{u_1, u_2\} \\ 5 & \textit{otherwise} \end{cases}$$

If $|L(u_1) \cup L(u_2)| \geq 2$ then G is L-colorable.

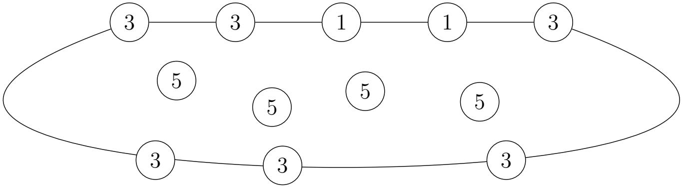
u_1, u_2 are *precolored*

List coloring - Thomassen's details

Corollary



Theorem



Crossings and 5-coloring graphs

Crossing number of G, $\mathrm{cr}(G)$ is the minimum number of crossings edges in a drawing of G.

Theorem (Oporowski and Zhao, 2005)

Every graph with crossing number at most two is 5-colorable.

Crossings and 5-coloring graphs

Crossing number of G, $\mathrm{cr}(G)$ is the minimum number of crossings edges in a drawing of G.

Theorem (Oporowski and Zhao, 2005)

Every graph with crossing number at most two is 5-colorable.

Observation (Erman et al., 2010)

Every graph with crossing number at most one is 5-choosable.

Our result

Theorem (Oporowski and Zhao, 2005)

Every graph with crossing number at most two is 5-colorable.

Observation (Erman et al., 2010)

Every graph with crossing number at most one is 5-choosable.

Theorem

Every graph with crossing number at most two is 5-choosable.

Independently obtained by Campos and Havet.

What we really proved

Theorem (original)

Let G be a graph and L a list assignment such that

- $\mathrm{cr}(G) \leq 2$ *and* $|L(v)| \geq 5$ *for every* $v \in V(G)$.

Then G is L-choosable.

Theorem (stronger)

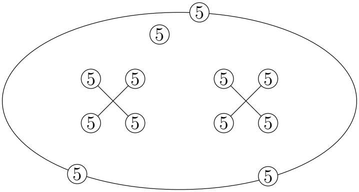
Let G be a graph and L a list assignment such that either

- $\mathrm{cr}(G) \leq 2$ *and* $|L(v)| \geq 5$ *for every* $v \in V(G)$, *or*
- $\mathrm{cr}(G) \leq 1$, *G contains a triangle T,* $L(v) = 1$ *for all* $v \in V(T)$, $L(u) \neq L(v)$ *if u and v are two distinct vertices of T and* $|L(v)| \geq 5$ *for all* $v \in V(G) \setminus V(T)$.

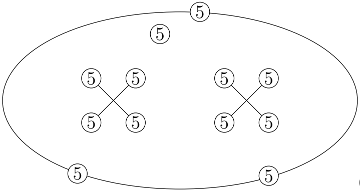
Then G is L-choosable.

What we really proved

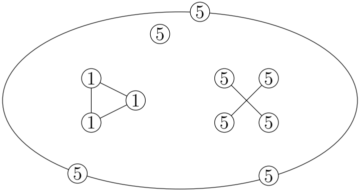
Original



Stronger

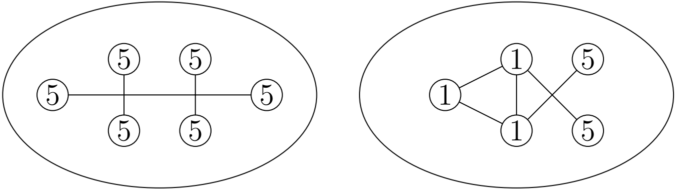


or



Proof idea

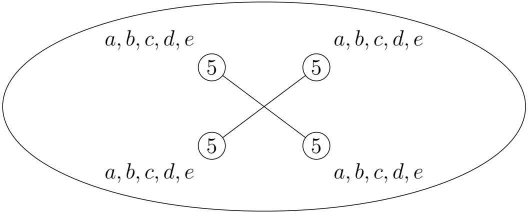
- **deal with small cases (one edge crossed twice,...)**



- restrict to the case with precolored triangle
- use Thomassen's result

Proof idea

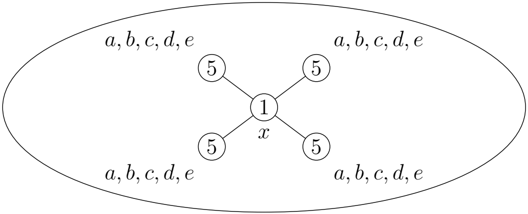
- deal with small cases (one edge crossed twice,...)
- **restrict to the case with precolored triangle**



- use Thomassen's result

Proof idea

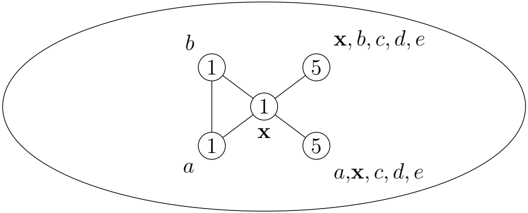
- deal with small cases (one edge crossed twice,...)
- **restrict to the case with precolored triangle**



- use Thomassen's result

Proof idea

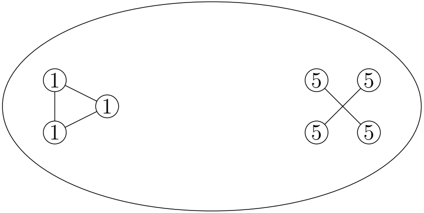
- deal with small cases (one edge crossed twice,...)
- **restrict to the case with precolored triangle**



- use Thomassen's result

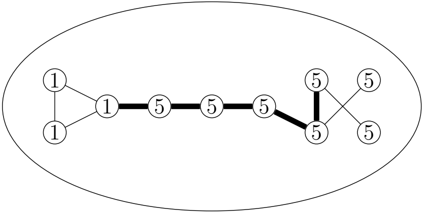
Proof idea

- deal with small cases (one edge crossed twice,...)
- restrict to the case with precolored triangle
- **use Thomassen's result**



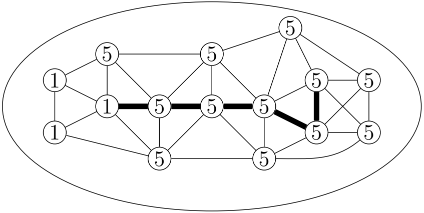
Proof idea

- deal with small cases (one edge crossed twice,...)
- restrict to the case with precolored triangle
- **use Thomassen's result**



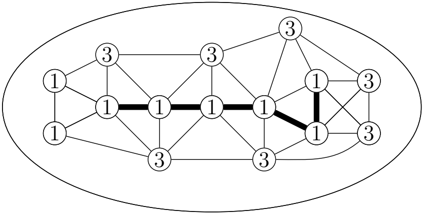
Proof idea

- deal with small cases (one edge crossed twice,...)
- restrict to the case with precolored triangle
- **use Thomassen's result**



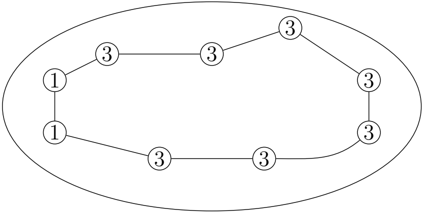
Proof idea

- deal with small cases (one edge crossed twice,...)
- restrict to the case with precolored triangle
- **use Thomassen's result**



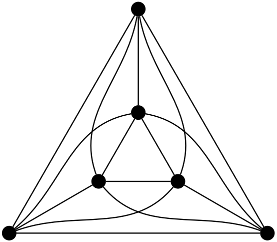
Proof idea

- deal with small cases (one edge crossed twice,...)
- restrict to the case with precolored triangle
- **use Thomassen's result**



What about more crossings?

Not for three crossings

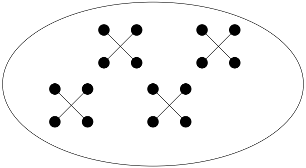


$$6 = \chi(K_6) \leq \mathrm{ch}(K_6)$$

What about more crossings and 5-coloring?

Theorem (Kráľ and Stacho, 2008)

If a graph G has a drawing in the plane in which no two crossings are dependent, then $\chi(G) \leq 5$

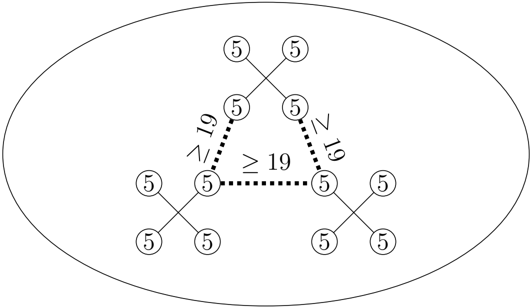


Crossings are not too close to each other.

More crossings and list coloring

Theorem (Dvořák, L. and Mohar)

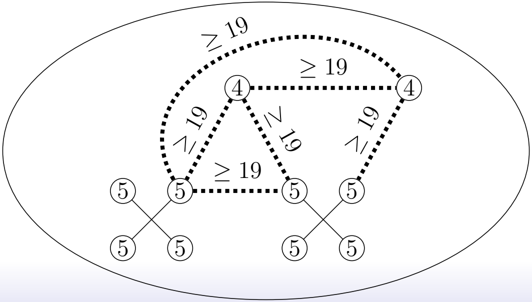
If a graph G has a drawing in the plane in which distance between every two crossings is at least 19, then $\mathrm{ch}(G) \leq 5$*.*



More crossings and list coloring

Theorem (Dvořák, L. and Mohar)

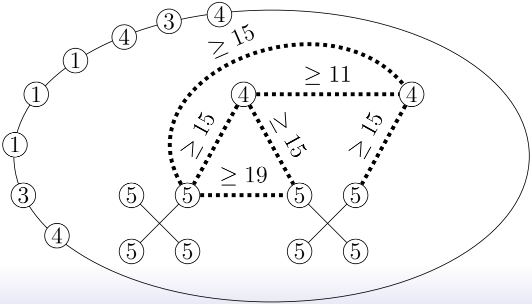
Let G be a graph, $N \subset V(G)$ and L a list assignment such that $L(v) \geq 4$ for $v \in N$ and $L(v) \geq 5$ otherwise. If G has a drawing in the plane in which distance between every two crossings, crossing and a vertex of N and two vertices of N is at least 19, then G is L-colorable.



More crossings and list coloring

Theorem (Dvořák, L. and Mohar)

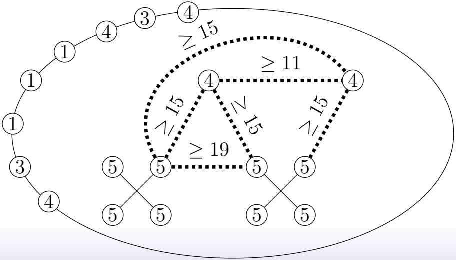
Let G be a graph, $N \subset V(G)$ and L a list assignment such that $L(v) \geq 4$ for $v \in N$ and $L(v) \geq 5$ otherwise. Let G has a drawing in the plane in which distance between every crossings and vertices of N is large. Let L be more restricted for the outer face. If G is not one of 16 exceptions then G is L-colorable.

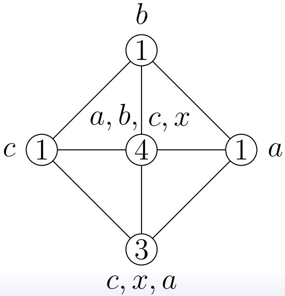


More crossings and list coloring

Theorem (Dvořák, L. and Mohar)

Let G be a graph, $N \subset V(G)$ and L a list assignment such that $L(v) \geq 4$ for $v \in N$ and $L(v) \geq 5$ otherwise. Let G has a drawing in the plane in which distance between every crossings and vertices of N is large. Let L be more restricted for the outer face. If G is not one of 16 exceptions then G is L-colorable.





Thank you for your attention

Special thanks to Robert Šámal for all the chocolate yesterday.