4-critical graphs on surfaces without contractible cycles of length at most 4

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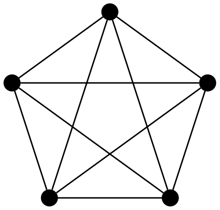
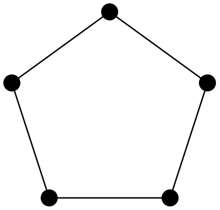
Definitions (k-critical graphs)

graph $G = (V, E)$

G is a *k-critical graph* if G is not $(k-1)$-colorable
but every $H \subset G$ is $(k-1)$-colorable.



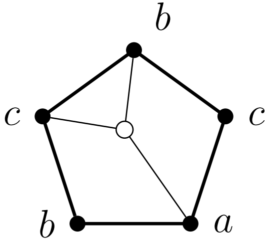


We are interested in 4-critical graphs.

Definitions (S-critical graphs)

graph $G = (V, E)$, $S \subset G$

G is *S-critical graph* if for every $S \subset H \subset G$ exists a 3-coloring of S that extends to a 3-coloring of H but does not extend to a 3-coloring of G.



Note that $\emptyset$-critical graph is 4-critical.

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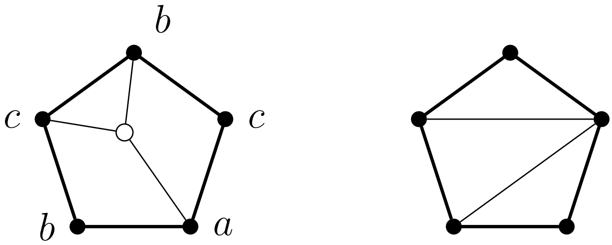

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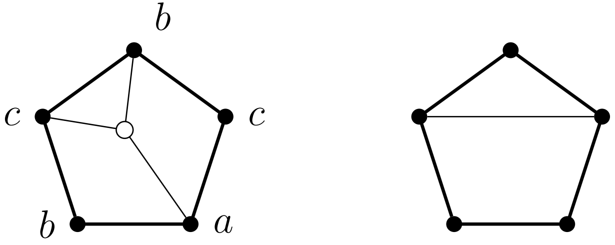

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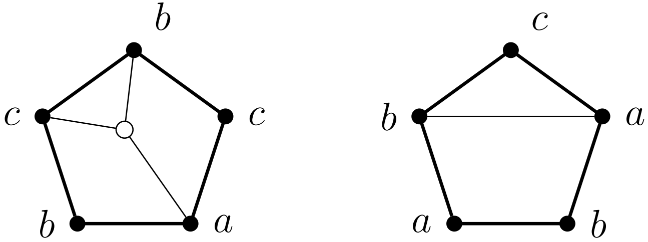

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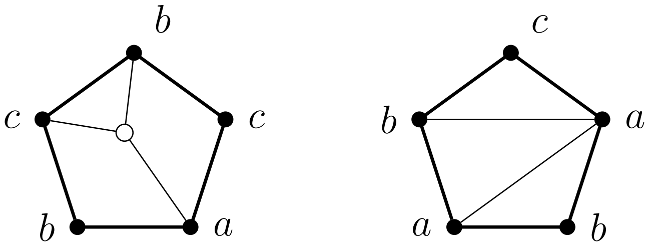

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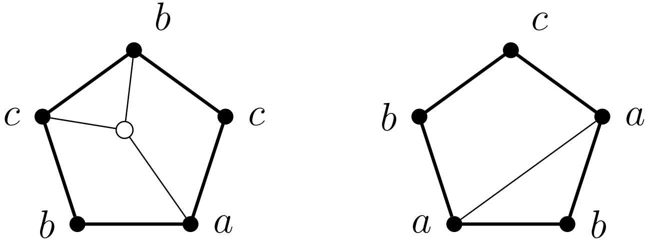

Note that $\emptyset$-critical graph is 4-critical.

Theorem (Grötzsch; 59)

Every planar triangle-free graph is 3-colorable.

Does it extend to graphs of higher genus?

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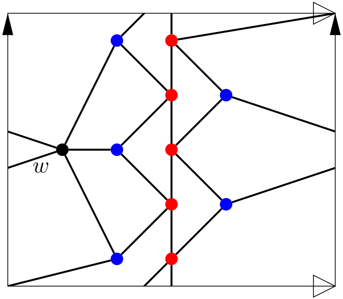
Theorem (Youngs; 96)

There are non 3-colorable triangle-free projective planar graphs.

Torus example:



Constraint on 4-cycles is needed.

Theorem (Thomassen; 03)

For every surface Σ there are finitely many 4-critical graph of girth 5 embeddable on Σ.

Theorem (Dvořák, Král', Thomas; 12+)

For every surface Σ of genus g the 4-critical graphs of girth 5 embeddable on Σ have at most Kg vertices.

Theorem (Thomassen, 94)

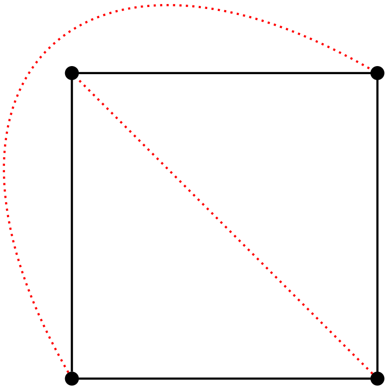
Every graph in the projective plane without contractible 3-cycle or 4-cycle is 3-colorable.

Theorem (Thomassen; 94)

Every graph in the torus without contractible 3-cycle or 4-cycle is 3-colorable.

Theorem (Thomas, Walls; 04)

There are infinitely many 4-critical graphs embedded in the Klein bottle without contractible 3-cycle or 4-cycle.



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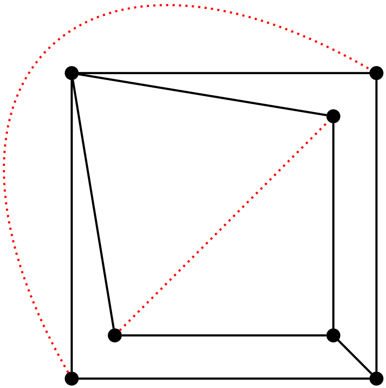

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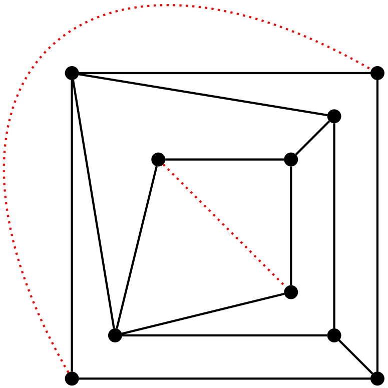

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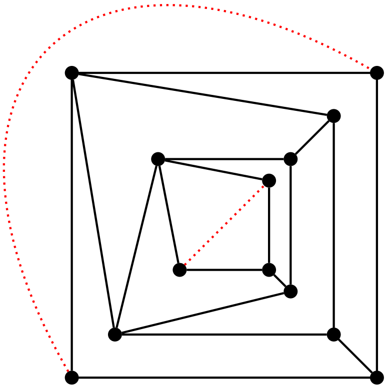

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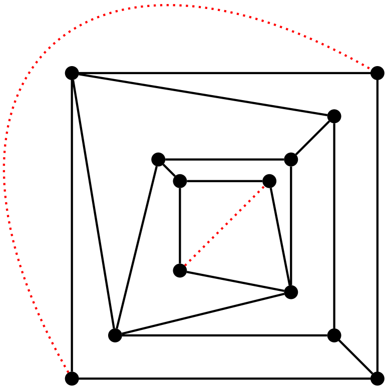
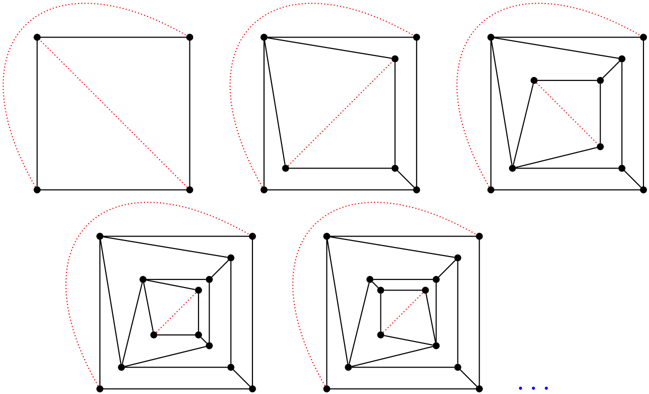


Theorem (Thomas, Walls; 04)

Every $\mathcal{F}$-free graph embeddable in the Klein bottle without contractible 3-cycle and 4-cycle is 3-colorable.

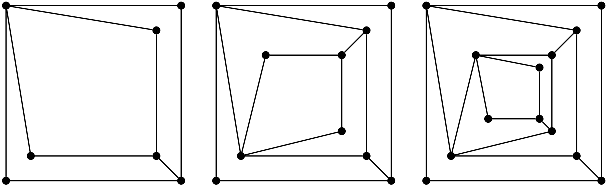
$\mathcal{F} =$



$\ldots$

Remember

$\mathcal{C} =$



$\cdots$

(will make 4-critical graphs of arbitrary size)

Theorem (Dvořák, Král', Thomas; 12+)

For every surface Σ *of genus* g *the* 4*-critical graphs of girth* 5 *embeddable on* Σ *have at most* Kg *vertices.*

Theorem (Dvořák, Král, Thomas; 12+)

Let $K = 10^{28}$*. Let* G *be a graph embedded in a surface* Σ *of genus* g *and let* $\{F_1, F_2, \ldots, F_k\}$ *be a set of faces of* G *such that the open region corresponding to* F_i *is homeomorpic to the open disk for* $1 \leq i \leq k$*. If* G *is* $(F_1 \cup F_2 \ldots \cup F_k)$*-critical and every cycle of length of at most* 4 *in* G *is equal to* F_i *for some* $1 \leq i \leq k$*, then*

$$|V(G)| \leq \ell(F_1) + \ldots + \ell(F_k) + K(g + k).$$

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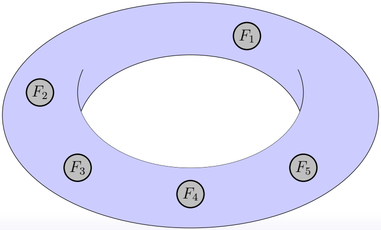
Let $K = 10^{28}$. Let G be a graph embedded in a surface Σ of genus g and let $\{F_1, F_2, \ldots, F_k\}$ be a set of faces of G such that the open region corresponding to F_i is homeomorpic to the open disk for $1 \leq i \leq k$. If G is $(F_1 \cup F_2 \ldots \cup F_k)$-critical and every cycle of length of at most 4 in G is equal to F_i for some $1 \leq i \leq k$, then

$$|V(G)| \leq \ell(F_1) + \ldots + \ell(F_k) + K(g + k).$$

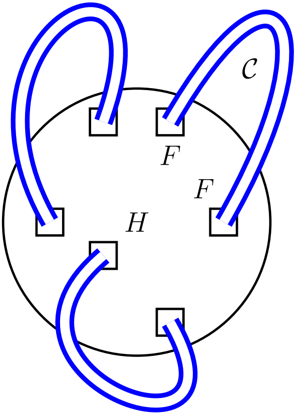


Theorem (Dvořák, L.)

There exists a function $f(g) = O(g)$ with the following property. Let G be a 4-critical graph embedded in a surface Σ of genus g so that every contractible cycle has length at least 5. Then G contains a subgraph H such that

- $|V(H)| \leq f(g)$*, and*
- *if F is a face of H that is not equal to a face of G, then F has exactly two boundary walks, each of the walks has length 4, and the subgraph of G drawn in the closed region corresponding to F belongs to $\mathcal{C}$.*

4-critical graph is a small graph H with members of $\mathcal{C}$



4-critical graph is a small graph H with members of $\mathcal{C}$

Proof idea

- allow faces $\mathcal{H} = \{F_1, F_2, \ldots, F_k\}$ like (Dvořák, Král', Thomas; 12+)
- split non-contractible 3-cycle or 4-cycle C into (two) new face(s) in $\mathcal{H}$
  - genus decreases
  - $|\mathcal{H}|$ decreases
  - C separates one F_i (process all such C last together)
- all 3-cycles and 4-cycles precolored
  - use (Dvořák, Král', Thomas; 12+)

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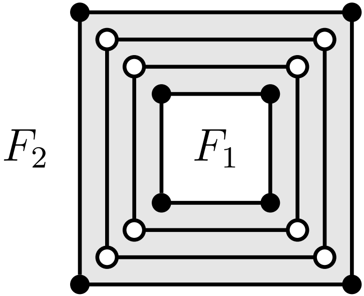
Left to check: graphs in the plane (cylinder) with $\{F_1, F_2\}$

Lemma (Dvořák, L.)

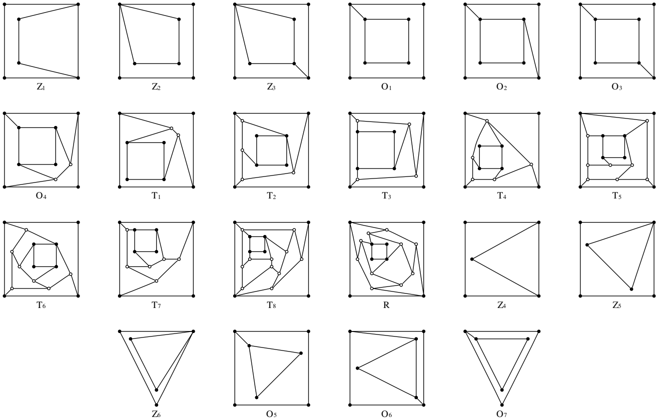
Let G be a plane graph and F_1 and F_2 faces of G. If G is $(F_1 \cup F_2)$-critical and every cycle of length at most 4 separates F_1 from F_2, then $G \in \mathcal{C}$ or G has at most 20 vertices.



- distance between two consecutive separating ≤ 4-cycles is > 4 (discharging)
- distance between every two consecutive separating ≤ 4-cycles is ≤ 4 (on next slides)

Lemma (Dvořák, L.)

Let G be a plane graph and F_1 and F_2 faces of G. If G is $(F_1 \cup F_2)$-critical and F_1 and F_2 are the only ≤ 4 cycles and the distance between F_1 and F_2 is at most 4 then G is one of 22 graphs.



Glue them together, the only infinite sequence is $\mathcal{C}$.
Others have ≤ 20 vertices.

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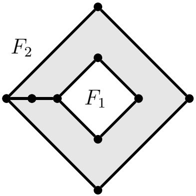

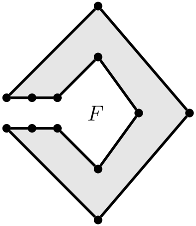

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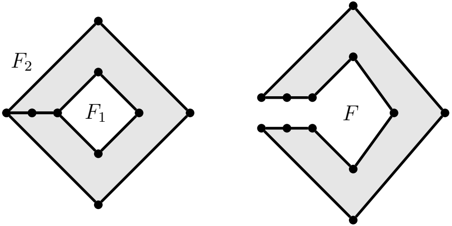
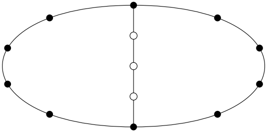


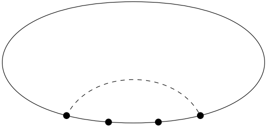
Results in a planar F-critical graph of girth 5 (with the outer face F).

Theorem (Dvořák, Kawarabayashi; 12+)

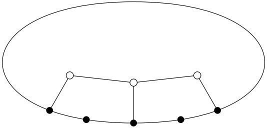
Every F-critical planar graph of girth 5 with the outer face F contains one of



(a)



(b)



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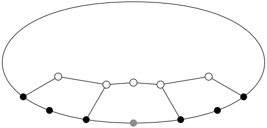

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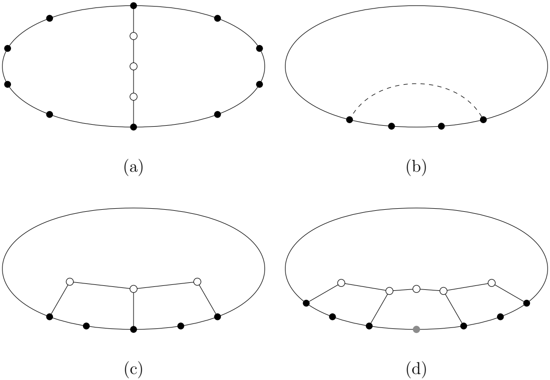

Can be used for generating F-critical graphs (on a computer)

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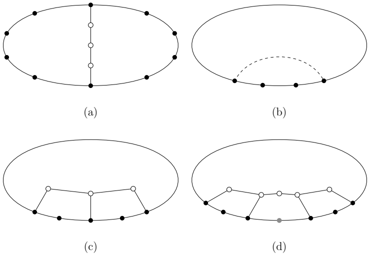

Can be used for generating F-critical graphs (on a computer)
List known for the outer face of size ≤ 12, we extended to ≤ 16.
Maybe up to 20 computable.

Summary

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Theorem (Dvořák, L.)

There are 7969 F*-critical planar graphs of girth 5 with outer face* F *of size 16.*

Thank you for your attention!