$L(2,1,1)$-Labeling Is NP-Complete for Trees

Petr A. Golovach [1], Bernard Lidický [2], and Daniël Paulusma [3]

[1]University of Bergen, Bergen, Norway

[2]University of Illinois, Urbana, USA

[3]University of Durham, Durham, UK

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Basic definitions

Definition ($L(p_1, \ldots, p_k)$-labelings)

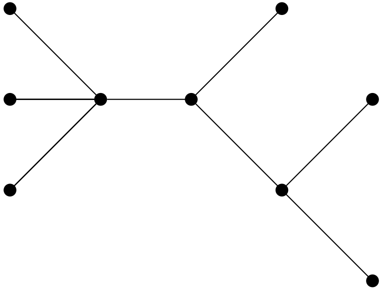
For positive integers $p_1, \ldots, p_k$, $p_1 \geq \ldots \geq p_k$, and λ, an $L(p_1, \ldots, p_k)$-labeling of a graph G with the span λ is a mapping $f\colon V(G) \to \{0, 1, \ldots, \lambda\}$ such that for any vertices u, v, $|f(u) - f(v)| \geq p_i$ if $\mathrm{dist}_G(u, v) \leq i$, $i \in \{1, \ldots, k\}$.

The minimum span for which an $L(p_1, \ldots, p_k)$-labeling exists is denoted by $\lambda_{p_1,\ldots,p_k}(G)$.

$L(1)$-labeling is classical coloring.

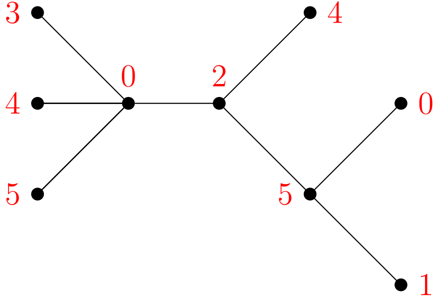
Basic definitions

Examples of $L(2,1)$ and $L(2,1,1)$-labelings



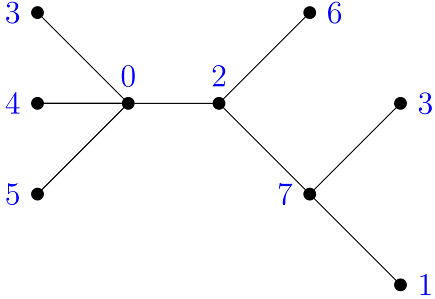
Basic definitions

$L(2,1)$-labeling of span 5



Basic definitions

$L(2,1,1)$-labeling of span 7



Basic definitions

Problem ($L(p_1, \ldots, p_k)$-labeling)

Parameters: *positive integers* $p_1, \ldots, p_k$.

Instance: *a graph* G *and a positive integer* λ.

Question: *does* G *have an* $L(p_1, \ldots, p_k)$*-labeling with span* λ?

Known results

General graphs:

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- $L(2,1,1)$-labeling can be solved in polynomial time for $\lambda < 5$ and is NP-complete otherwise (Fiala, Golovach and Kratochvíl; '04).
- Exact algorithms for $L(2,1)$-labeling of graphs (Král'; '05 and Havet, Klazar, Kratochvíl, Kratsch, Liedloff; '11).

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- $L(2,1)$-labeling is NP-complete for graphs $\textbf{tw} \leq 2$ (Fiala, Golovach and Kratochvíl).
- $L(1,1,\ldots,1)$-labeling is solvable in polynomial time (Zhou, Kanari and Nishizeki; '00).

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- $L(p_1,1)$-labeling is NP-complete if p_1 is part of the input (Golovach; '06).

Known results

Theorem

Every tree T satisfies

$$\Delta(T) + 1 \leq \lambda_{2,1}(T) \leq \Delta(T) + 2.$$

Theorem (King, Ras, Zhou; '10 and indep. Fiala, Golovach, Kratochvíl; '04)

Every tree T satisfies

$$\omega(T^3) - 1 \leq \lambda_{2,1,1}(T) \leq \omega(T^3).$$

Main result

Theorem (Golovach, L., Paulusma)

The $L(2,1,1)$-labeling problem is NP-complete for the class of trees.

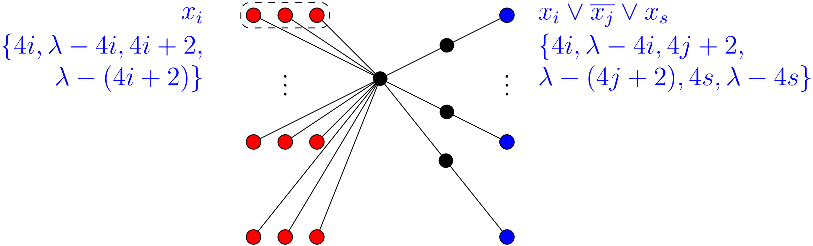
Sketch of the proof

Problem (3-Satisfiability)

Instance: *variables* $x_1, \ldots, x_x$ *and clauses* $C_1, \ldots, C_m$.

Question: *can* $\phi = C_1 \wedge \ldots \wedge C_m$ *be satisfied?*

Sketch of the proof - Idea of the reduction



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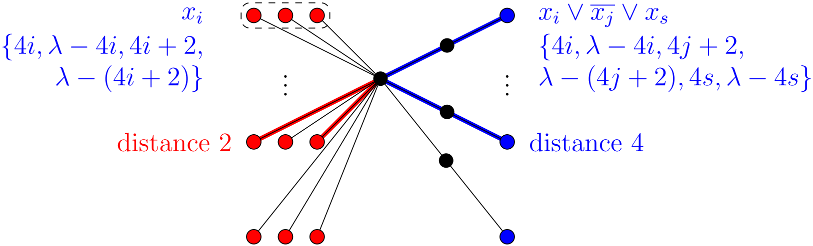

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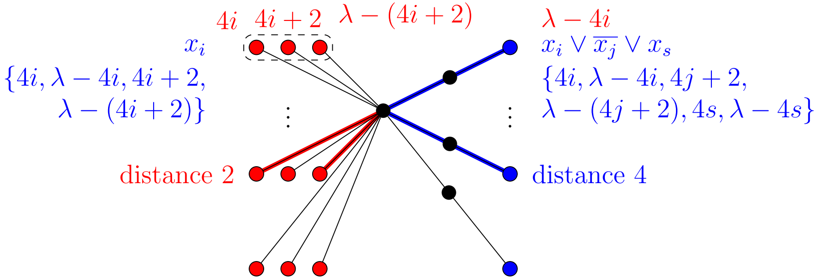

$$x_i = \begin{cases} \textit{true}, & \text{if } 4i \text{ or } \lambda - 4i \text{ is not used,} \\ \textit{false}, & \text{if } 4i+2 \text{ or } \lambda-(4i+2) \text{ is not used.} \end{cases}$$

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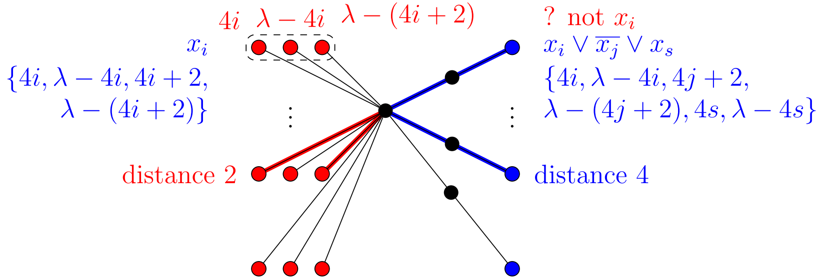

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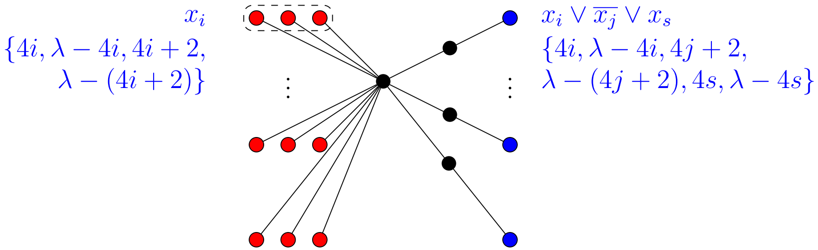

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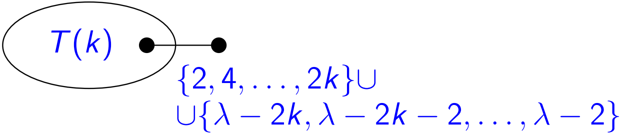

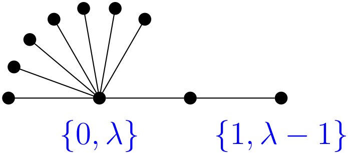

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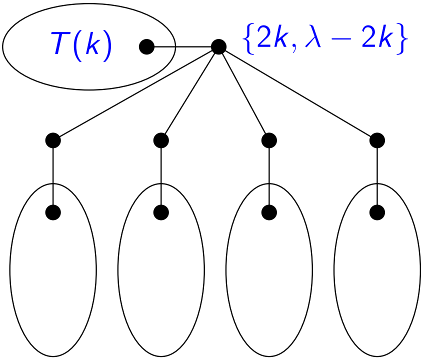

$2k-2$ copies of $T(k-1)$

Trees $F(k)$

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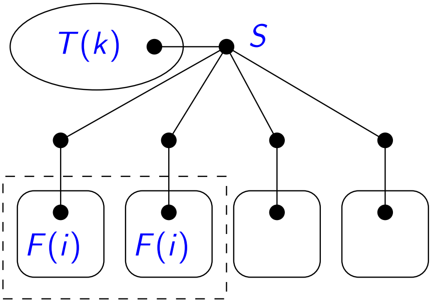

For each $i \in \{1, \ldots, k\}$,
s.t. $2i \notin S$

Forcing of a list
$S \subseteq \{2, 4, \ldots, 2k\} \cup \{\lambda - 2k, \lambda - 2k - 2, \ldots, \lambda - 2\}$ s.t. $\forall x \in S$, $\lambda - x \in S$.

Cyclic labelings

Definition (Cyclic metric (modulo $\lambda + 1$))

For positive integers $a, b \in \{0, \ldots, \lambda\}$,
$|a - b|_c = \min\{|a - b|, \lambda + 1 - |a - b|\}$.

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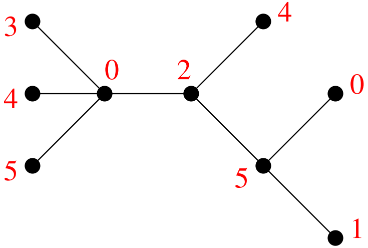
Cyclic labelings

$C(2,1)$-labeling of span 6



Cyclic labelings

What is the computational complexity of $C(2,1,1)$-Labeling on trees?

Thank you for your attention!