3-coloring planar graphs with four triangles

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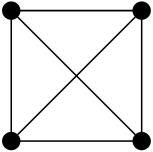
Definitions (4-critical graphs)

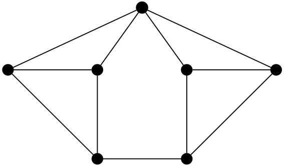
graph $G = (V, E)$

coloring is $\varphi : V \to C$ such that $\varphi(u) \neq \varphi(v)$ if $uv \in E$

G is a *k-colorable* if coloring with $|C| = k$ exists

G is a *4-critical graph* if G is not 3-colorable
but every $H \subset G$ is 3-colorable.





Inspiration

Theorem (Grötzsch ’59)

Every planar triangle-free graph is 3-colorable.

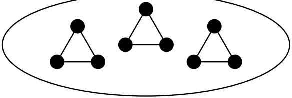
More triangles?

Theorem (Grötzsch ’59)

Every planar triangle-free graph is 3-colorable.

Theorem (Grünbaum ’63; Aksenov ’74; Borodin ’97; Borodin et. al. ’12+)

Let G be a planar graph containing at most three triangles. Then G is 3-colorable.



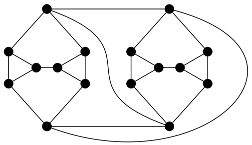
G

Question: What about four triangles?

Call 4-critical planar graph with four triangles a $4, 4$-graph.

3-coloring planar graphs with four triangles?

Havel '69 found



Problem (Sachs '72)

Let G be a $4,4$-graph. Can the triangles be partitioned into two pairs so that in each pair the distance between the triangles is less than two?

3-coloring planar graphs with four triangles?

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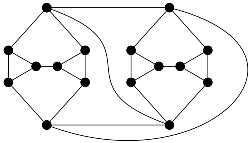
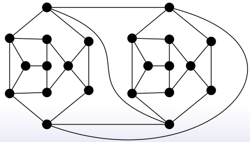


Problem (Sachs '72)

Let G be a $4, 4$-graph. Can the triangles be partitioned into two pairs so that in each pair the distance between the triangles is less than two?

No! Aksenov and Mel'nikov '78,'80, Two infinite series of $4, 4$-graphs.



Problem (Erdős '90)

Describe $4, 4$-graphs.

Borodin '97 - at least 15 infinite families of $4, 4$-graphs (all with 4-faces)

Thomas and Walls '04 Infinite family $\mathcal{TW}$ of $4, 4$-graphs with no 4-faces. ...

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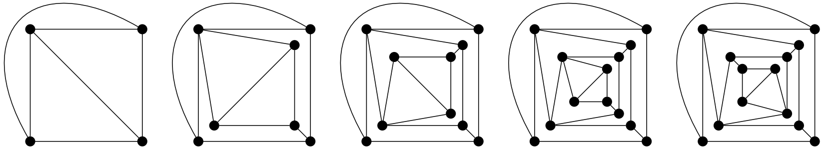
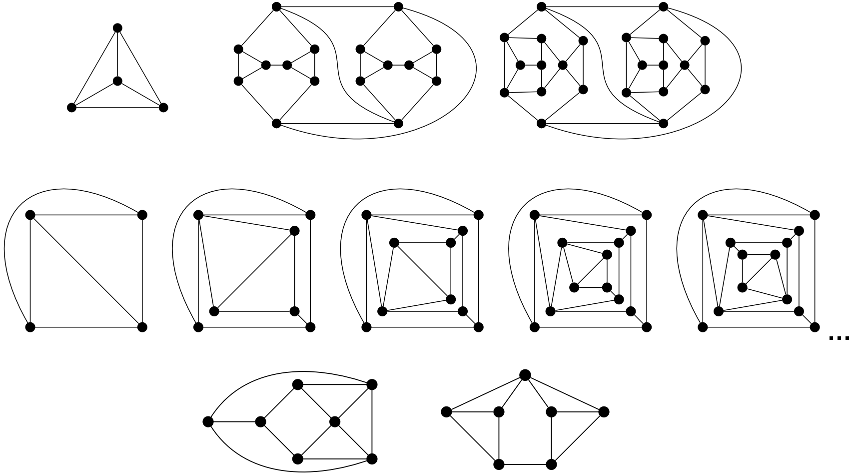


...

Some known $4, 4$-critical planar graphs



Grötzsch's Theorem proof sketch as inspiration

Theorem (Grötzsch '59)

Every planar triangle-free graph G is 3-colorable.

Proof.

CASE1: G has no 4-faces

CASE2: G has a 4-face

□

OUR PLAN:

- characterize $4,4$-graphs without 4-faces
- describe how 4-faces could look like

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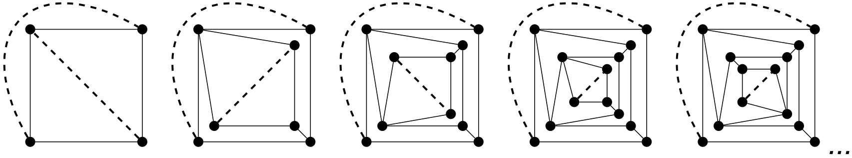
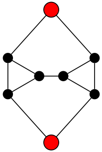
Results

Theorem

All plane 4, 4-graphs with no 4-faces can be obtained from the Thomas-Walls sequence



by replacing dashed edges by edges or

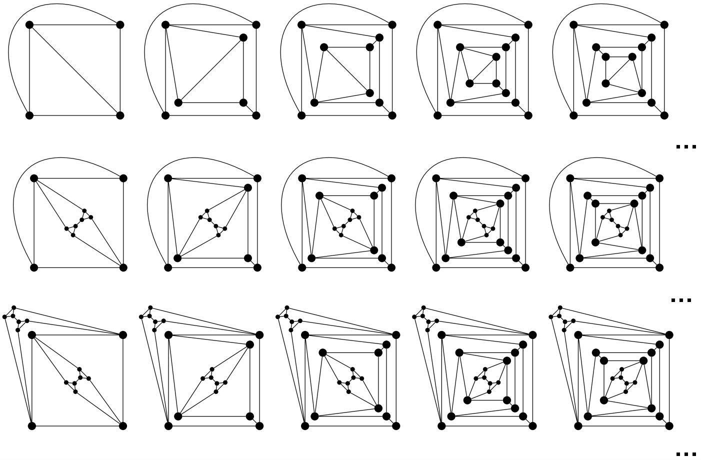


Call them $\mathcal{C}$.

Results

Theorem

All plane $4,4$*-graphs with no* 4*-faces are precisely graphs in*
$\mathcal{C} = \{$



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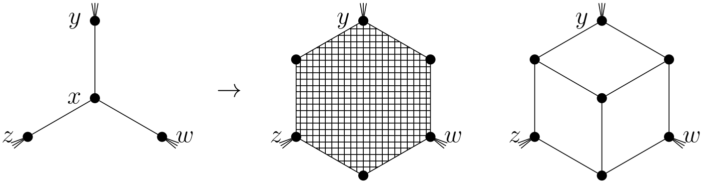
Results

Theorem

All plane $4, 4$-graphs with no 4-faces are precisely graphs in $\mathcal{C}$.

Theorem

Every $4, 4$-graph can be obtained from $G \in \mathcal{C}$ by expanding some vertices of degree 3.



Act 1:

Theorem

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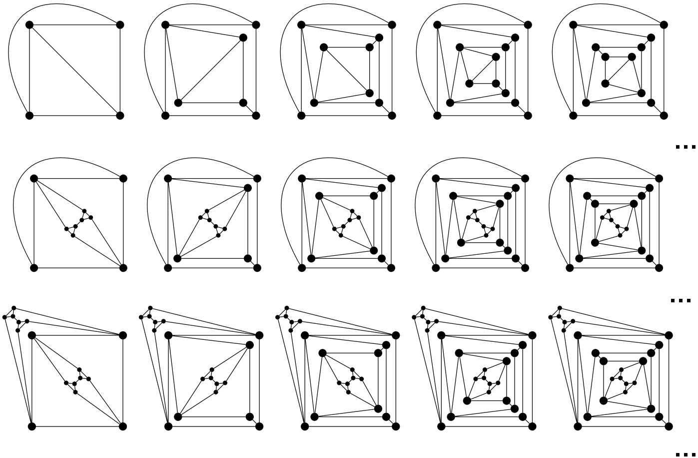

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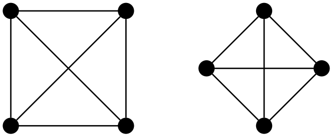
Main tool:

Theorem (Kostochka and Yancey; 12+)

Let G be a 4-critical graph. Then $3e \geq 5v - 2$.
Moreover, $3e = 5v - 2$ iff G is 4-Ore.

G is 4-Ore if $G = K_4$ or G is an Ore composition of two 4-Ore graphs.



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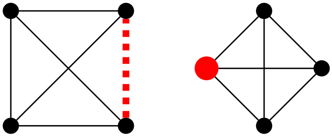

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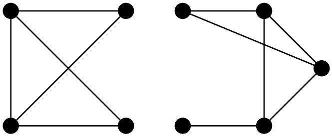

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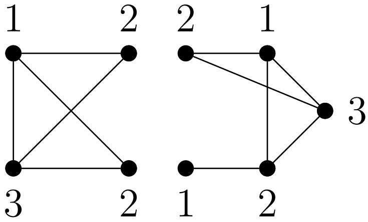

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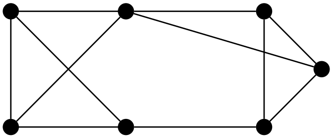

Not 3-colorable.

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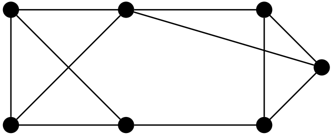

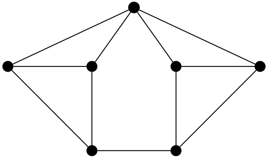

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Observation

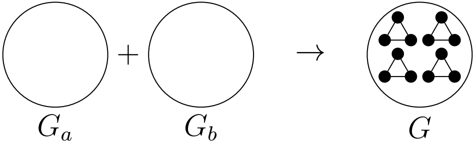
Let G be a plane $4, 4$-graph with no 4-faces. Then $3e \leq 5v - 2$.

Every $4, 4$-graph must be 4-Ore.

Key properties

G is $4, 4$-*Ore* if it is 4-Ore and has 4 triangles.
($4, 4$-graph $\subseteq$ $4, 4$-Ore)

- $4, 4$-Ore is K_4 or Ore composition of two 4-Ore graphs



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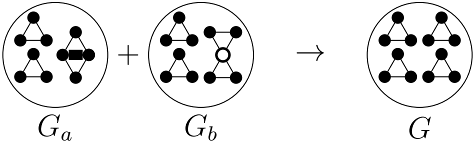
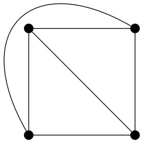


Lemma
G_a and G_b are $4, 4$-Ore.

Description of $4, 4$-Ore (by pictures)



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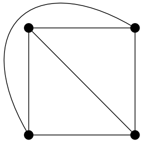

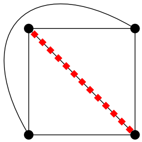

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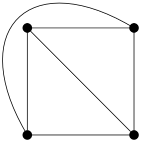

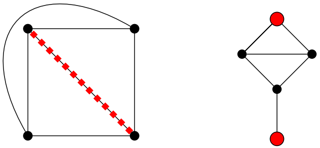

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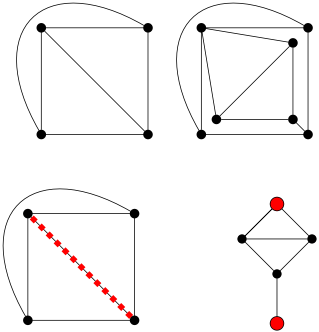

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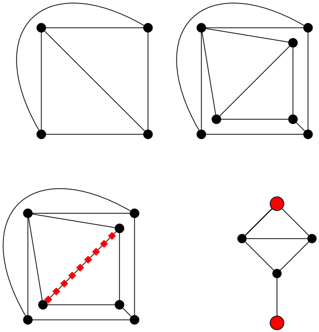

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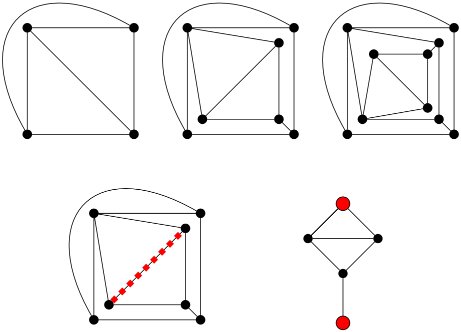

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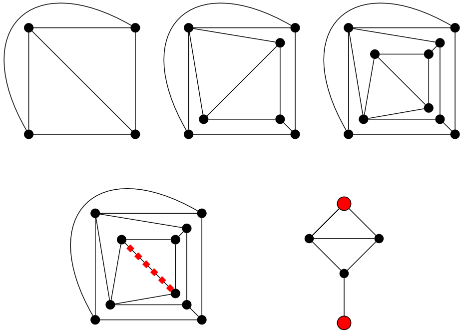

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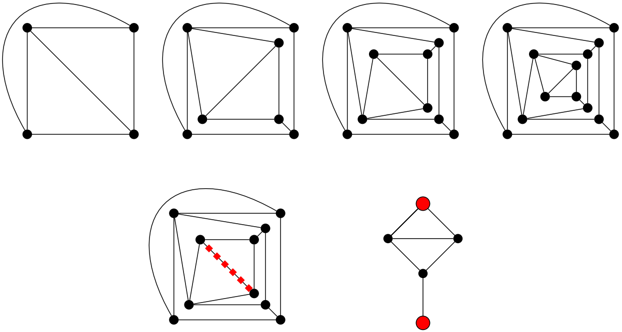

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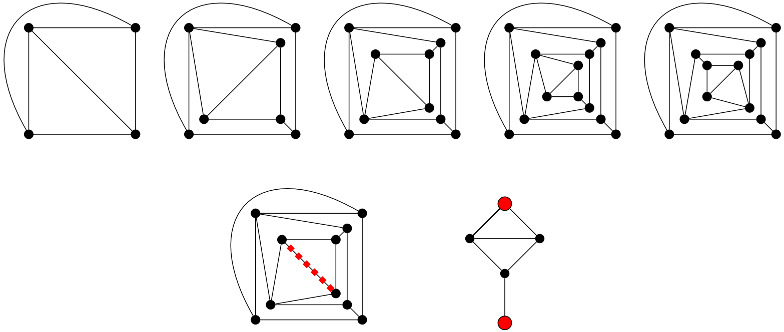

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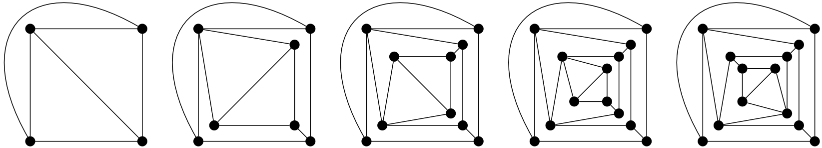
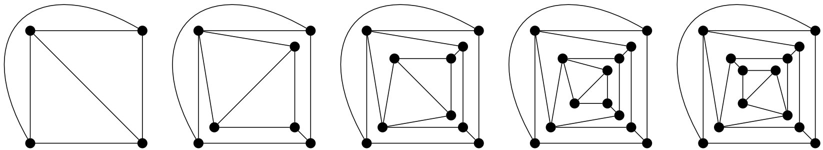
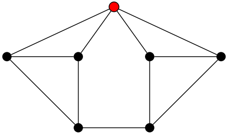


And now few more...

Description of $4, 4$-Ore (by pictures)





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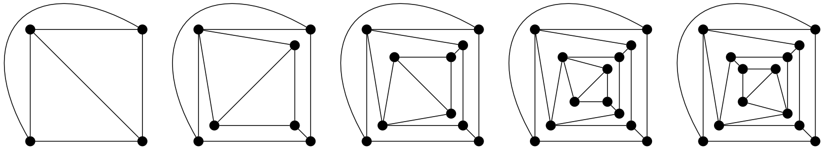

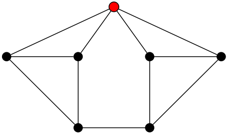

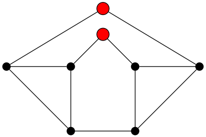

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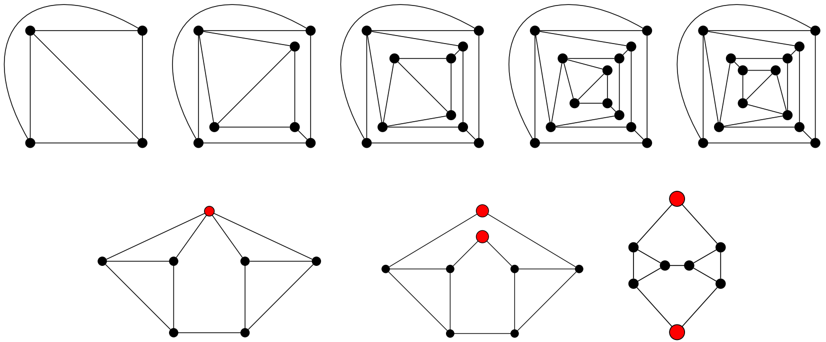
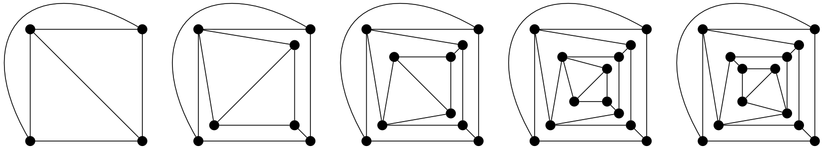
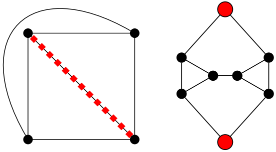


Description of $4,4$-Ore (by picture)





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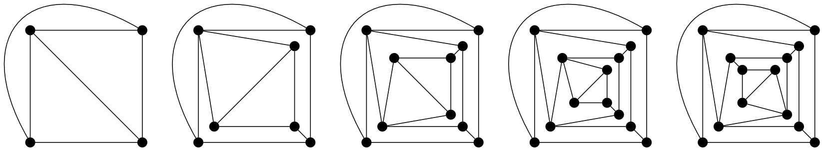

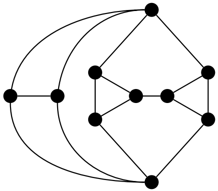

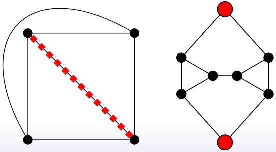

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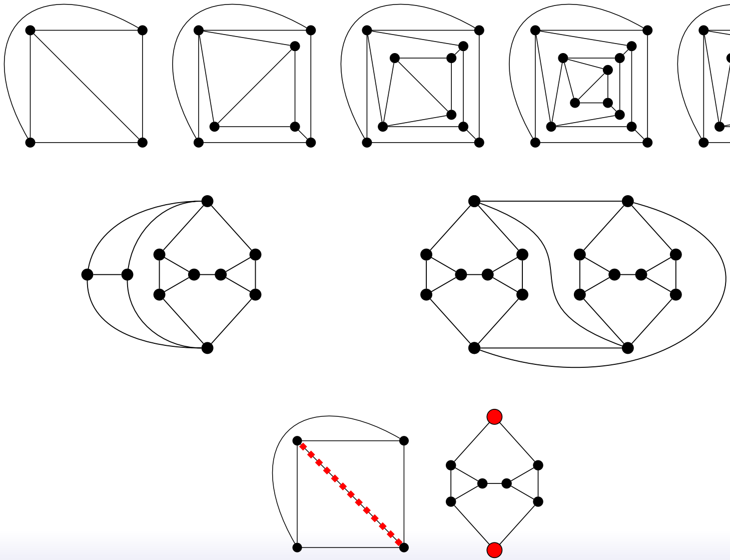

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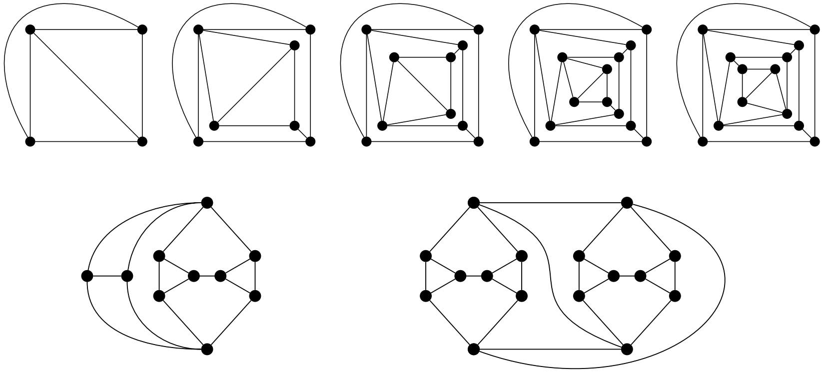


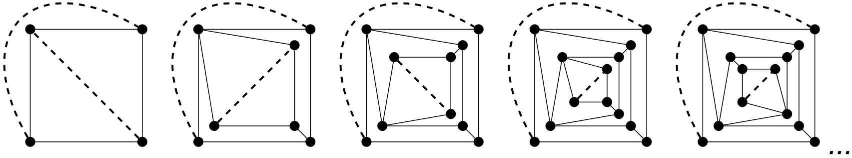
Lemma

Every $4, 4$-Ore graph is planar.
Hence $4, 4$-graphs = $4, 4$-Ore graphs.

Description of $\mathcal{C}$

Theorem

All 4-critical plane graphs with four triangles and no 4-faces can be obtained from the Thomas-Walls sequence



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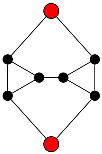
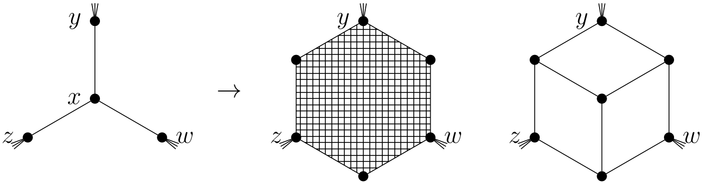


Call them $\mathcal{C}$.

Act 2:

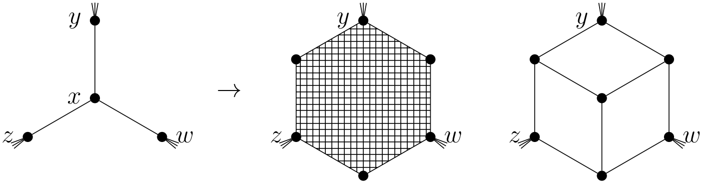
Theorem

Every $4,4$-graph can be obtained from $G \in \mathcal{C}$ by expanding some vertices of degree 3.



(Interior of a 6-cycle is a quadrangulation - only 4-faces)

Why is expansion good?



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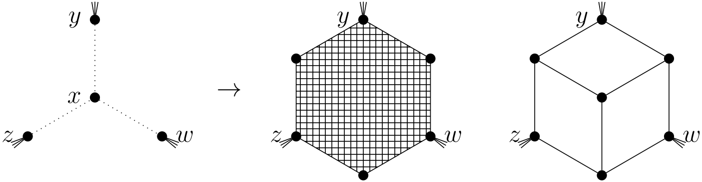

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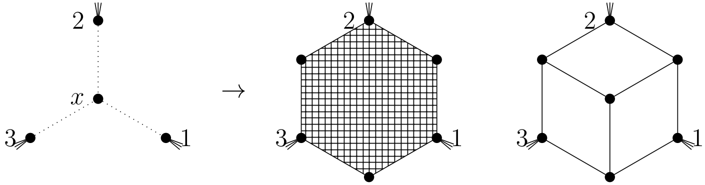
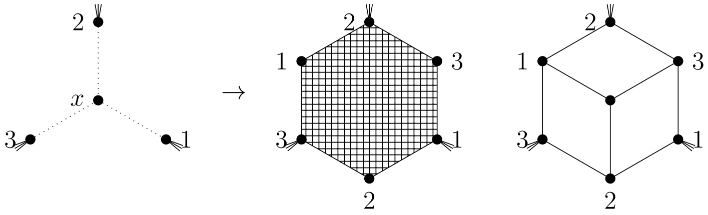


$G - x$ is 3-colorable since G is 4-critical.
Any 3-coloring of $G - x$ gives different colors to y, z, w.

Why is expansion good?

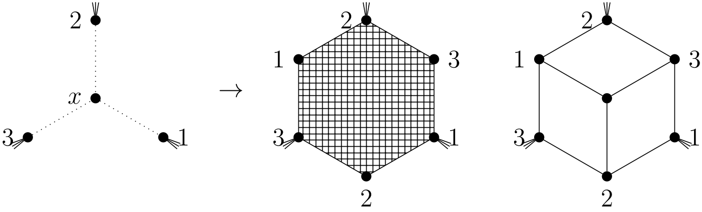


$G - x$ is 3-colorable since G is 4-critical.

Any 3-coloring of $G - x$ gives different colors to y, z, w.

3-coloring extends to a 3-coloring of 6-cycle uniquely.

Why is expansion good?



Theorem (Gimbel and Thomassen '97)

"Quadrangulation is the only possible filling of a 6-cycle in a plane triangle-free 4-critical graph."

Conclusion

Problem (Sachs '72)

Let G be a $4, 4$-graph. Can the triangles be partitioned into two pairs so that in each pair the distance between the triangles is less than two?

Theorem

All plane $4, 4$-graphs with no 4-faces are precisely graphs in $\mathcal{C}$.

Theorem

Every $4, 4$-graph with four triangles can be obtained from $G \in \mathcal{C}$ by expanding some vertices of degree 3.

Corollary

Let G be a $4, 4$-graph. The triangles can be partitioned into two pairs so that in each pair the distance between the triangles is at most two.

Thank you for your attention!