# 3-coloring planar graphs with four triangles

Oleg V. Borodin, Zdeněk Dvořák, Alexandr V. Kostochka, Bernard Lidický, Matthew Yancey

Sobolev Institute of Mathematics and Novosibirsk State University
Charles University in Prague
University of Illinois at Urbana-Champaign

54th Midwest Graph Theory Conference
Miami University in Oxford, OH
April 6, 2013

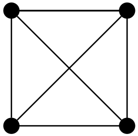
# Definitions (4-critical graphs)

graph $G = (V, E)$

*coloring* is $\varphi : V \to C$ such that $\varphi(u) \neq \varphi(v)$ if $uv \in E$

$G$ is a *$k$-colorable* if coloring with $|C| = k$ exists

$G$ is a *4-critical graph* if $G$ is not 3-colorable
but every $H \subset G$ is 3-colorable.



# Inspiration

Theorem (Grötzsch '59)

*Every planar triangle-free graph is 3-colorable.*

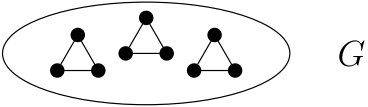
# More triangles?

Theorem (Grötzsch '59)

*Every planar triangle-free graph is 3-colorable.*

Theorem (Grünbaum '63; Aksenov '74; Borodin '97; Borodin et. al. '12+)

*Let $G$ be a planar graph containing at most three triangles. Then $G$ is 3-colorable.*



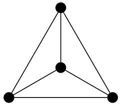

Question: What about four triangles?

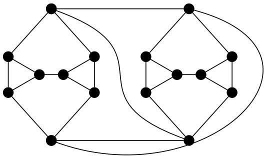
# 3-coloring planar graphs with four triangles?

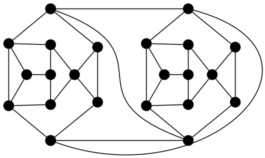
First studied by Aksenov in 70's

Problem (Erdős '92)

*Are the following three graphs all* 3*-critical planar graphs with four triangles?*





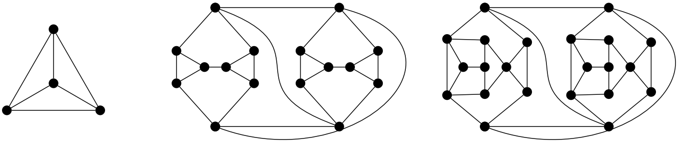


Some (partial) results announced by Borodin '97.

# 3-coloring planar graphs with four triangles?

Problem (Erdős '92)

*Are the following three graphs all 3-critical planar graphs with four triangles?*

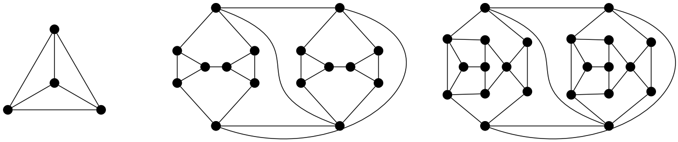


Not true...

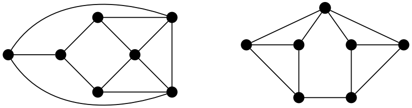
# 3-coloring planar graphs with four triangles?

## Problem (Erdős '92)

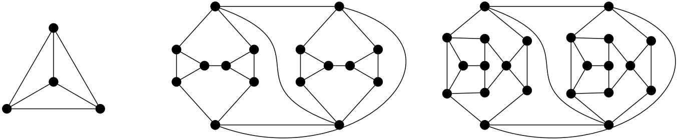
*Are the following three graphs all 3-critical planar graphs with four triangles?*



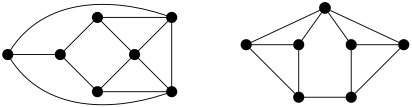
Not true...



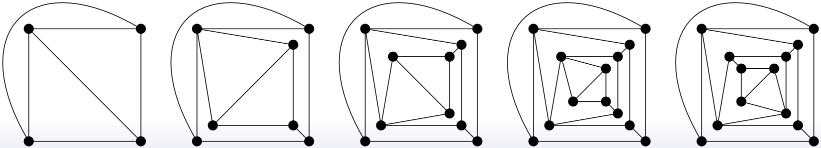
# 3-coloring planar graphs with four triangles?



Not true...



Even infinitely many more!



...

# How to describe?

### Observation

*In every 3-coloring of a 4-face, two non-adjacent vertices have the same color.*

PLAN:

- characterize 4-critical plane graph with four triangles and no 4-faces
- describe how 4-faces could look like

# Results

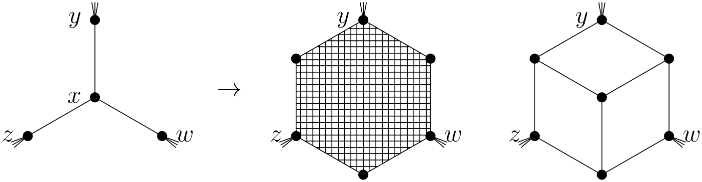
### Theorem

*4-critical plane graphs without 4-faces are precisely graphs in $\mathcal{C}$.*

$\mathcal{C}$ is described later...

### Theorem

*Every 4-critical plane graph can be obtained from $G \in \mathcal{C}$ by expanding some vertices of degree 3.*



# Act 1: no 4-faces

**Theorem**

*4-critical plane graphs without 4-faces are precisely graphs in $\mathcal{C}$.*

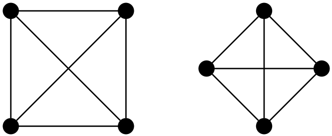
# (no 4-faces) Main tool:

Theorem (Kostochka and Yancey; 12+)

*Let $G$ be a 4-critical graph. Then $3|E(G)| = 5|V(G)| - 2$ iff $G$ is 4-Ore.*

$3|E(G)| = 5|V(G)| - 2$ holds for plane graphs with four triangles and without 4-faces (and all other faces 5-faces).

$G$ is 4-*Ore* if $G = K_4$ or $G$ is an Ore composition of two 4-Ore graphs.



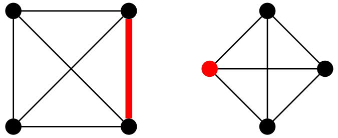
# (no 4-faces) Main tool:

Theorem (Kostochka and Yancey; 12+)

*Let $G$ be a 4-critical graph. Then $3|E(G)| = 5|V(G)| - 2$ iff $G$ is 4-Ore.*

$3|E(G)| = 5|V(G)| - 2$ holds for plane graphs with four triangles and without 4-faces (and all other faces 5-faces).

$G$ is 4-*Ore* if $G = K_4$ or $G$ is an Ore composition of two 4-Ore graphs.



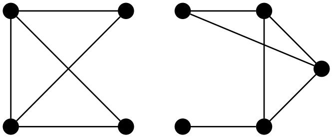
# (no 4-faces) Main tool:

Theorem (Kostochka and Yancey; 12+)

*Let $G$ be a 4-critical graph. Then $3|E(G)| = 5|V(G)| - 2$ iff $G$ is 4-Ore.*

$3|E(G)| = 5|V(G)| - 2$ holds for plane graphs with four triangles and without 4-faces (and all other faces 5-faces).

$G$ is *4-Ore* if $G = K_4$ or $G$ is an Ore composition of two 4-Ore graphs.



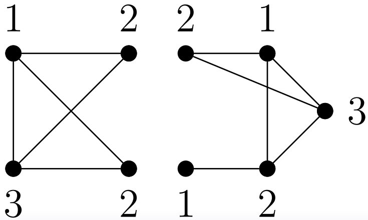
# (no 4-faces) Main tool:

Theorem (Kostochka and Yancey; 12+)

*Let $G$ be a $4$-critical graph. Then $3|E(G)| = 5|V(G)| - 2$ iff $G$ is $4$-Ore.*

$3|E(G)| = 5|V(G)| - 2$ holds for plane graphs with four triangles and without $4$-faces (and all other faces $5$-faces).

$G$ is $4$-*Ore* if $G = K_4$ or $G$ is an Ore composition of two $4$-Ore graphs.



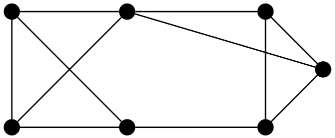
# (no 4-faces) Main tool:

Theorem (Kostochka and Yancey; 12+)

*Let $G$ be a 4-critical graph. Then $3|E(G)| = 5|V(G)| - 2$ iff $G$ is 4-Ore.*

$3|E(G)| = 5|V(G)| - 2$ holds for plane graphs with four triangles and without 4-faces (and all other faces 5-faces).

$G$ is *4-Ore* if $G = K_4$ or $G$ is an Ore composition of two 4-Ore graphs.



Not 3-colorable.

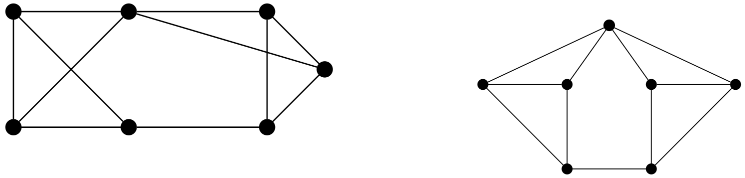
# (no 4-faces) Main tool:

Theorem (Kostochka and Yancey; 12+)

*Let $G$ be a $4$-critical graph. Then $3|E(G)| = 5|V(G)| - 2$ iff $G$ is $4$-Ore.*

$3|E(G)| = 5|V(G)| - 2$ holds for plane graphs with four triangles and without $4$-faces (and all other faces $5$-faces).

$G$ is $4$-*Ore* if $G = K_4$ or $G$ is an Ore composition of two $4$-Ore graphs.



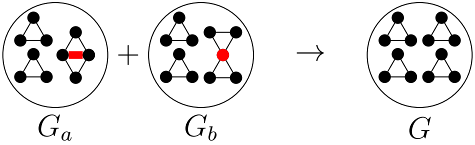
Not $3$-colorable.

# (no 4-faces) Key property

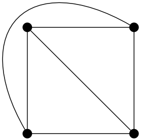
$G$ is $4,4$-*graph* if it is $4$-Ore and has 4 triangles

### Lemma

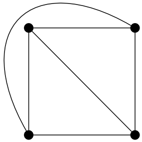
*$4,4$-graph $G$ is $K_4$ or Ore composition of two $4,4$-graphs $G_a$ and $G_b$.*

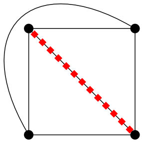


# Description of $4, 4$-graphs (by pictures)

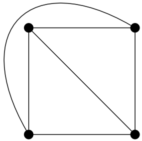


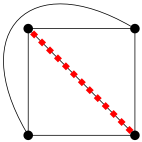
# Description of $4, 4$-graphs (by pictures)

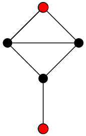




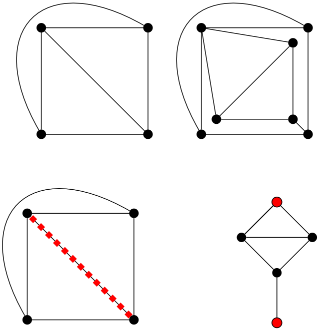
# Description of $4, 4$-graphs (by pictures)



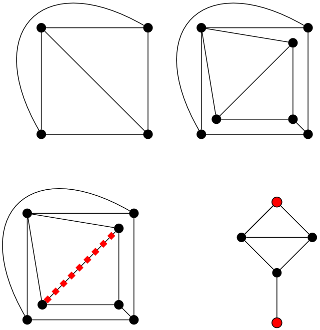




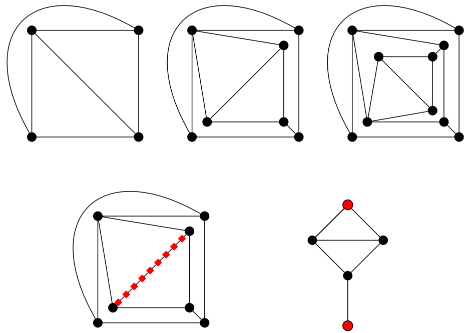
# Description of $4, 4$-graphs (by pictures)



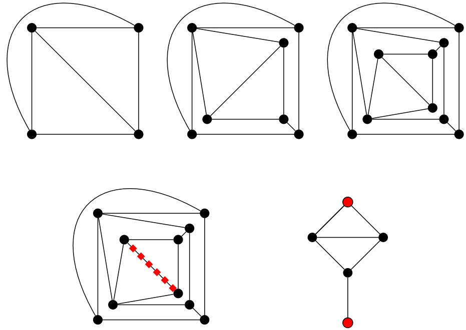
# Description of $4, 4$-graphs (by pictures)



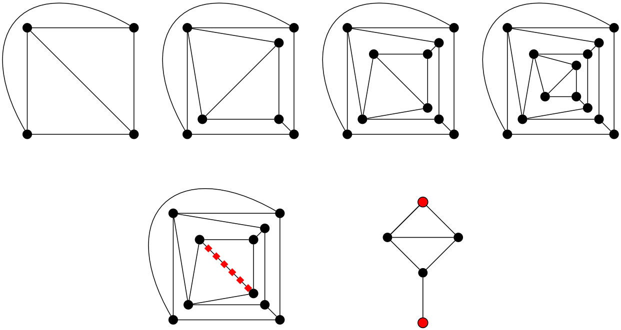
# Description of $4, 4$-graphs (by pictures)



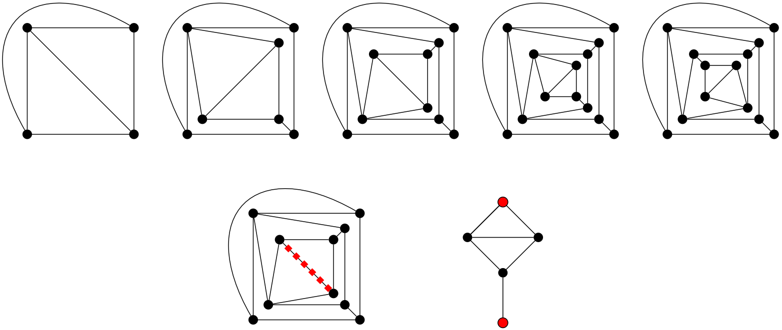
# Description of $4, 4$-graphs (by pictures)



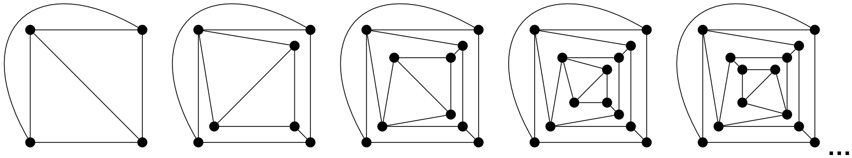
# Description of $4, 4$-graphs (by pictures)



# Description of $4, 4$-graphs (by pictures)



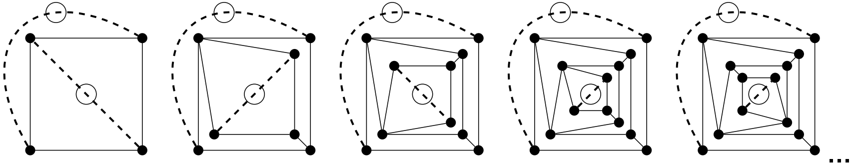
# Description of $4, 4$-graphs (by pictures)



Infinite class - same as Thomas-Walls for the Klein bottle without contractible 3- and 4-cycles.

And now few more...

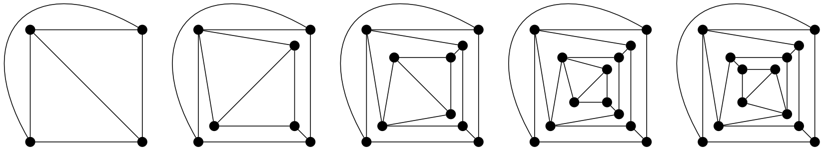
# Description of $4, 4$-graphs (by pictures)



Infinite class - same as Thomas-Walls for the Klein bottle without contractible 3- and 4-cycles.

And now few more...

# Description of $4, 4$-graphs (by pictures)

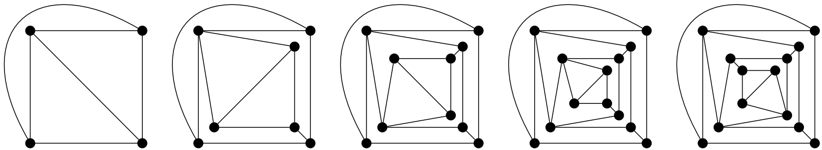


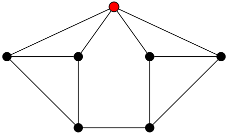
...

Infinite class - same as Thomas-Walls for the Klein bottle without contractible $3$- and $4$-cycles.

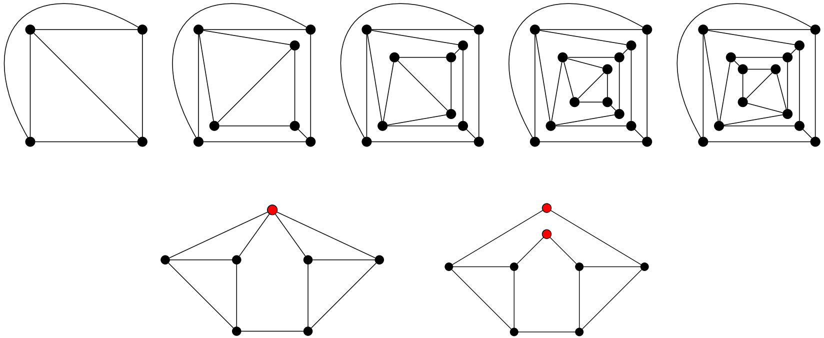
And now few more...

# Description of $4, 4$-graphs (by pictures)

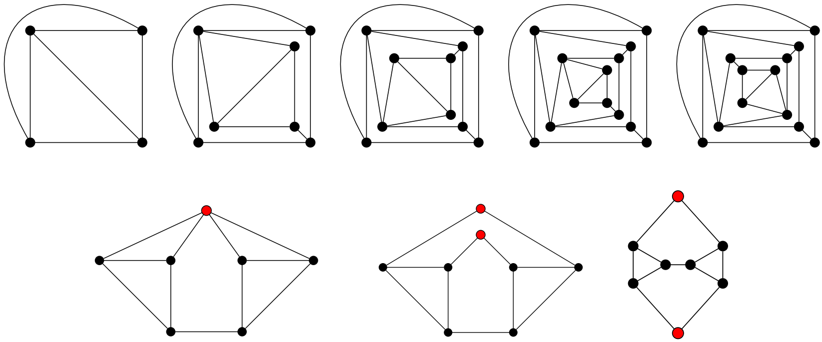




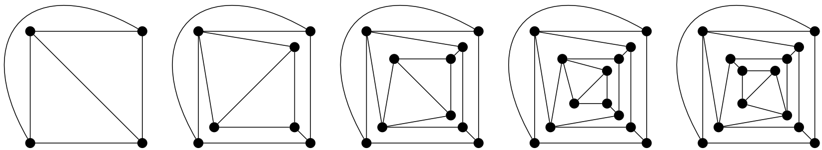
# Description of $4, 4$-graphs (by pictures)

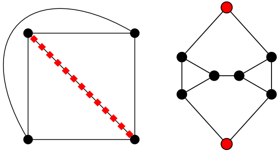


# Description of $4, 4$-graphs (by pictures)

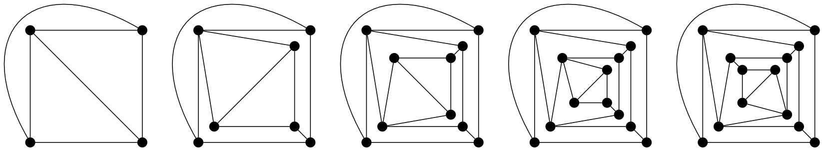


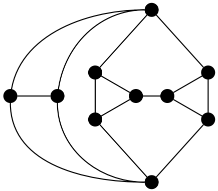
# Description of $4, 4$-graphs (by picture)

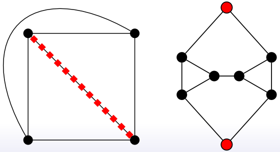




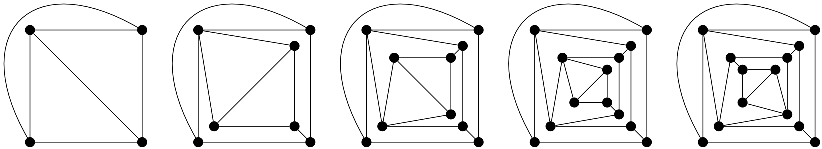
# Description of $4, 4$-graphs (by picture)

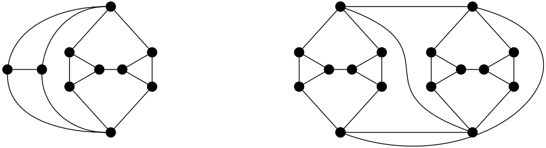


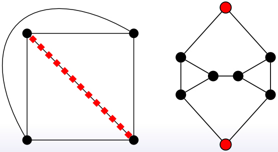




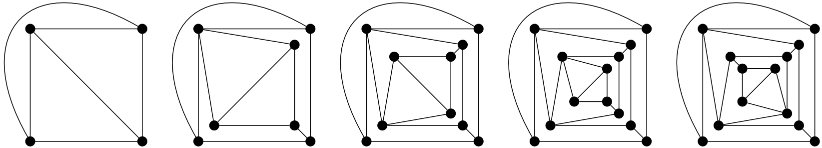
# Description of $4, 4$-graphs (by picture)

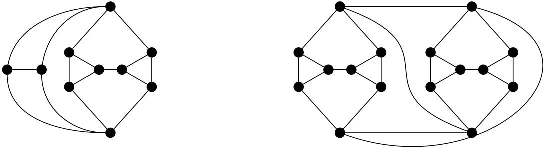






# Description of $4, 4$-graphs (by picture)



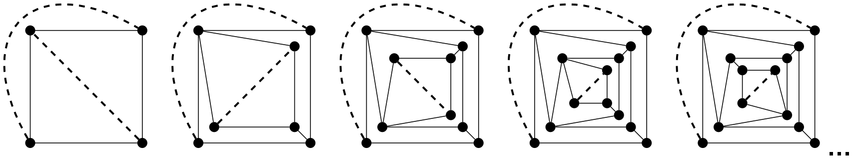


## Lemma

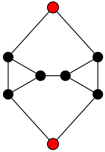
*Every $4, 4$-graph is planar.*

# Description of $\mathcal{C}$

All 4-critical plane graphs with four triangles and no 4-faces can be obtained from the Thomas-Walls sequence



by replacing dashed edges by edges or

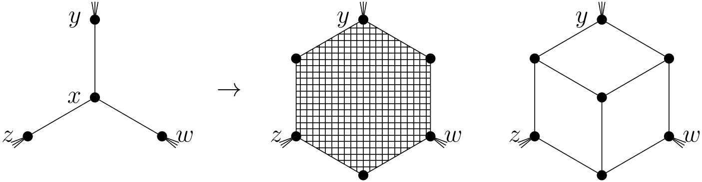


.

# Act 2: 4-faces

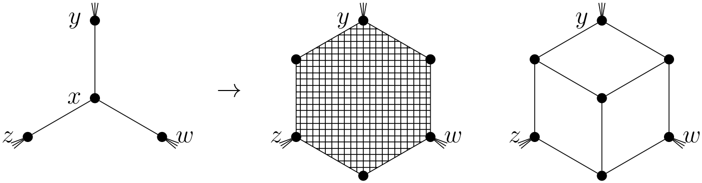
**Theorem**

*Every 4-critical plane graph can be obtained from $G \in \mathcal{C}$ by expanding some vertices of degree 3.*

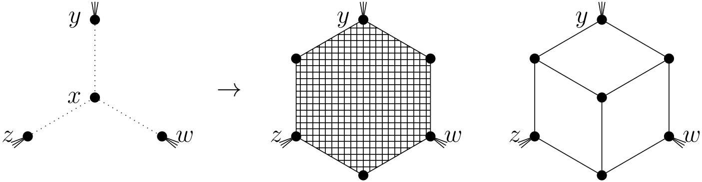



(Interior of a 6-cycle is a quadrangulation - only 4-faces)

# Why is expansion good?

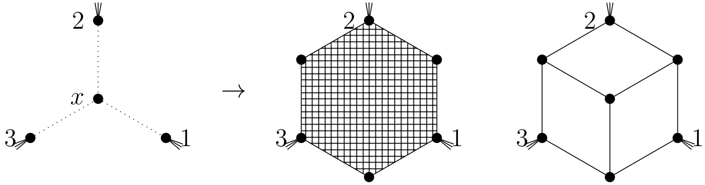


# Why is expansion good?



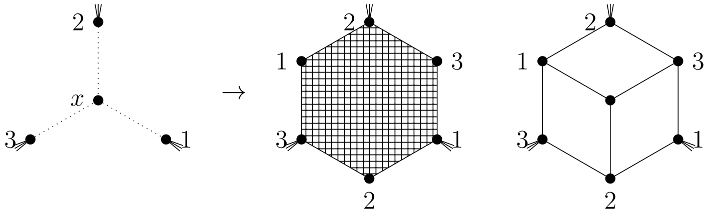

$G - x$ is 3-colorable since $G$ is 4-critical.

# Why is expansion good?



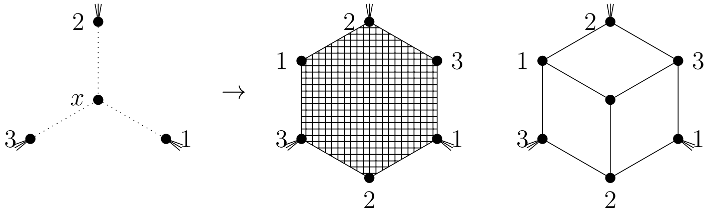

$G - x$ is 3-colorable since $G$ is 4-critical.
Any 3-coloring of $G - x$ gives different colors to $y, z, w$.

# Why is expansion good?




$G - x$ is 3-colorable since $G$ is 4-critical.
Any 3-coloring of $G - x$ gives different colors to $y, z, w$.
3-coloring extends to a 3-coloring of 6-cycle uniquely.

# Why is expansion good?




### Theorem (Gimbel and Thomassen ’97)

*Let $G$ be a planar triangle-free graph with chordless outer 6-cycle $C$. Let $c$ be a coloring of $C$ by colors 1,2,3. Then $c$ cannot be extended to a 3-coloring of $G$ if and only if $G$ interior of $C$ contains a quadrangulation and opposite vertices of $C$ have the same color.*

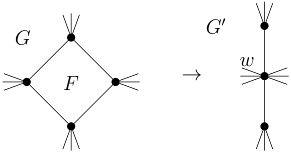
# Proof idea

## Theorem

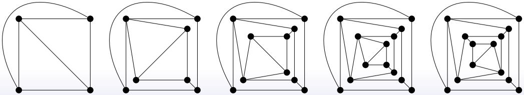
*Every 4-critical plane graph can be obtained from $G \in \mathcal{C}$ by expanding some vertices of degree three.*

Let $G$ be a minimal counterexample.

- obtain $G'$ from $G$ by identifying opposite vertices of a 4-face




- obtain 4-critical subgraph $G''$ of $G'$
- $G''$ has no 4-faces (hence described in Act 1!)

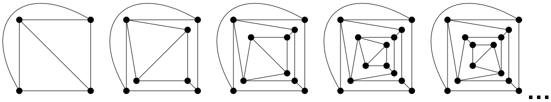


...

# Proof idea

Let $G$ be a minimal counterexample.

- obtain $G'$ from $G$ by identifying opposite vertices of a 4-face
- obtain 4-critical subgraph $G''$ of $G'$
- $G''$ has no 4-faces (hence described in Act 1!)



- Reconstruct $G$ from $G''$ by guessing $w$, decontractig $w$ and adding other vertices that were removed.

$$G \xrightarrow{\text{identification}} G' \xrightarrow{\text{critical subgraph}} G''$$

$$G \xleftarrow{\text{adding vertices}} G_1 \xleftarrow{\text{decontraction}} G''$$

Thank you for your attention!