Number of monotone subsequences of length four in permutations

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Problem

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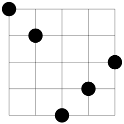
k = 3
n = 5



$(5,4,1,2,3)$

(5,4,1),(5,4,2),(5,4,3)
(1,2,3)

Problem

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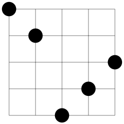

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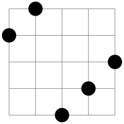

(4,5,1,2,3)

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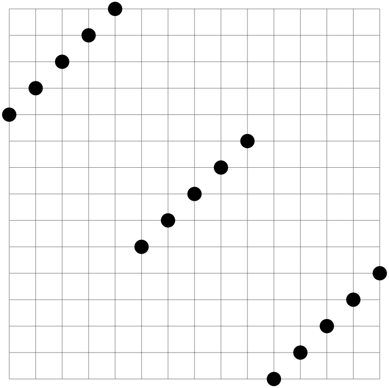
Conjecture

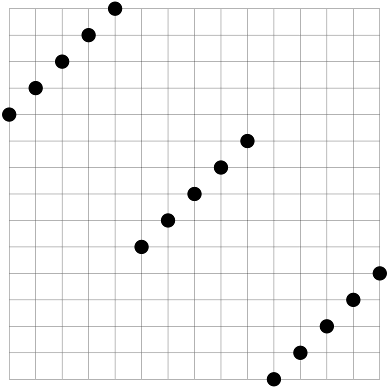
Conjecture (Myers 2002)

The number of monotone subsequences of length k is minimized by a permutation on $[n]$ with $k-1$ increasing runs of as equal lengths as possible.



$k = 4, n = 15$

Extremal case is not unique



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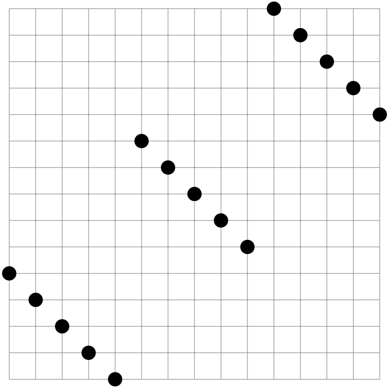

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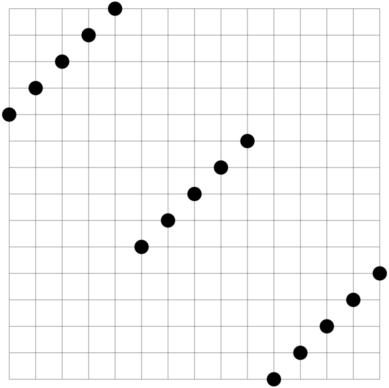

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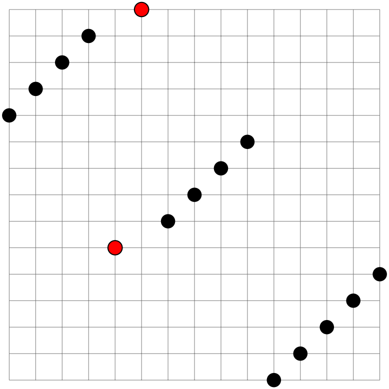

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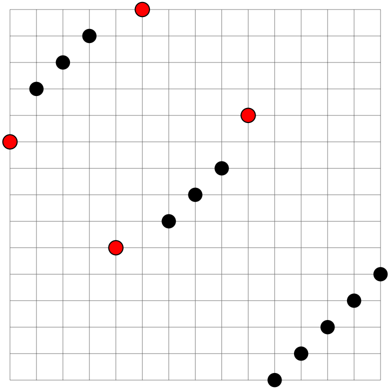

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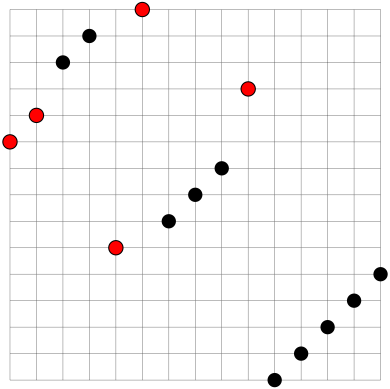

Theorem (Balogh, Hu, L., Pikhurko, Udvari, Volec '14+)
Myers' conjecture is true for $k = 4$ and n sufficiently large.

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We translate the problem to graphs and use flag algebras.

From permutations to permutation graphs

(1,2)

(2,1)

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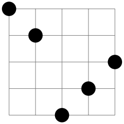

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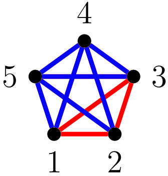

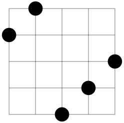

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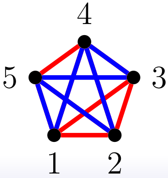
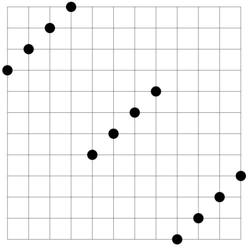
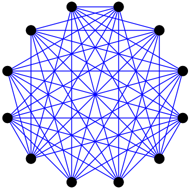
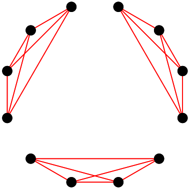
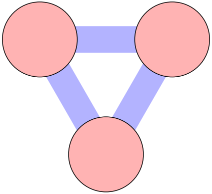


EXTREMAL EXAMPLE ($k = 4$)

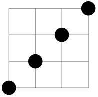




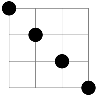




As flag algebra question ($k = 4$)



(1,2,3,4)



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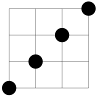

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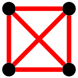

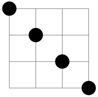

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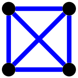

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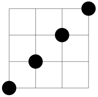

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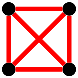

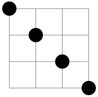

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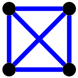
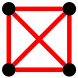
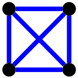


minimize  $+$ 

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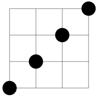

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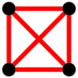

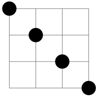

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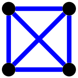
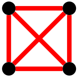
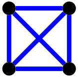
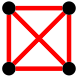
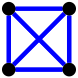


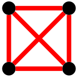
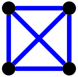
minimize  + 

Theorem (Balogh, Hu, **L.**, Pikhurko, Udvari, Volec '14+)

 +  $\geq \frac{1}{27}$

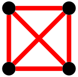
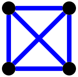
for every permutation graph.

Theorem (Balogh, Hu, **L.**, Pikhurko, Udvari, Volec '14+)

min ( + ) $= \frac{1}{27}$

over permutation graphs (and extremal permutations described using Myers' results).

THEOREM (BALOGH, HU, **L.**, PIKHURKO, UDVARI, VOLEC '14+)

min ( + ) $= \frac{1}{27}$

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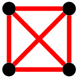
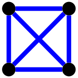
THEOREM (SPERFELD '12; THOMASON '89)

$\frac{1}{35} <$ min ( + ) $< \frac{1}{33}$

over all 2-edge-colored complete graphs.

Flag algebras

Seminal paper:

A. Razborov, Flag Algebras, *Journal of Symbolic Logic* **72** (2007), 1239–1282.

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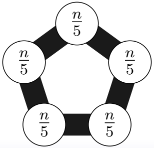
Applications to oriented graphs, hypergraphs, crossing number of complete bipartite graphs, geometry, hypercubes,. . .

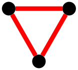
Theorem (Hatami, Hladký, Kráľ, Norine, Razborov 2011; Grzesik 2011)

The number of C_5's in a triangle-free graph on n vertices is at most $(n/5)^5$.



Flag algebras - what are flags

Let G be a 2-edge-colored complete graph on n vertices.



The probability that three random vertices in G span a red triangle.

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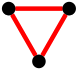

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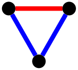

The probability that three random vertices in G span a triangle with one red and two blue edges.

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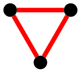

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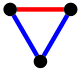

The probability that three random vertices in G span a triangle with one red and two blue edges.

The probability that a random vertex other than v is connected to $v \in V(G)$ by a red edge, i.e., the red degree of v divided by $n - 1$.

Flag algebras - example

Theorem (Mantel 1907)

If a graph G on n vertices has more than $\frac{1}{4}n^2$ edges, then G contain a triangle.

Assume edges are red and non-edges are blue.

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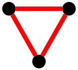 $= 0 \Rightarrow$  $\leq \frac{1}{2}$

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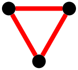
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

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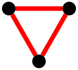
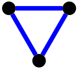
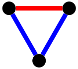
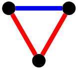
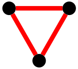
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

$1 =$  $+$  $+$  $+$ 

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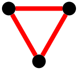
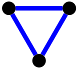
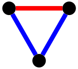
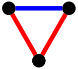
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

$1 =$  $+$  $+$ 

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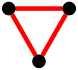
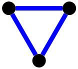
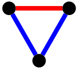
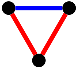
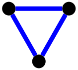
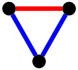
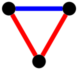
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

$1 =$  $+$  $+$ 

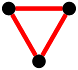
 $= 0$  $+ \frac{1}{3}$  $+ \frac{2}{3}$ 

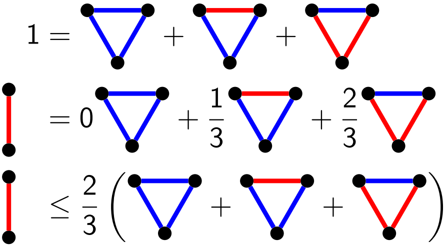
Flag algebras - example

Theorem (Mantel 1907)

If a graph G on n vertices has more than $\frac{1}{4}n^2$ edges, then G contain a triangle.

Assume edges are red and non-edges are blue.

Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)



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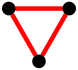
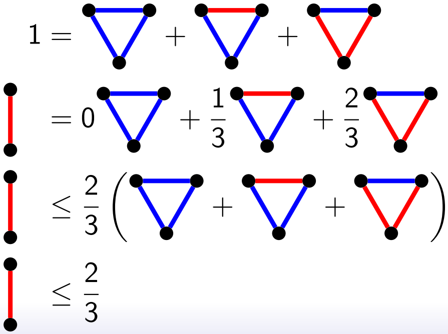
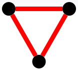
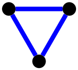
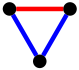
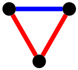
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

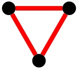


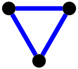
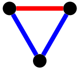
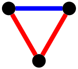
Flag algebras - improvement

Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

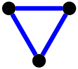
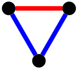
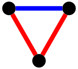
 $= 0$  $+ \frac{1}{3}$  $+ \frac{2}{3}$ 

Flag algebras - improvement

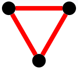
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

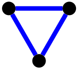
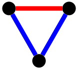
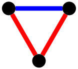
 $= 0$  $+ \frac{1}{3}$  $+ \frac{2}{3}$ 

Idea: find $c_1, c_2, c_3 \in \mathbb{R}$ such that

$0 \leq c_1$  $+ c_2$  $+ c_3$  .

Flag algebras - improvement

Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

 $= 0$  $+ \frac{1}{3}$  $+ \frac{2}{3}$ 

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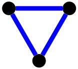
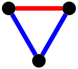
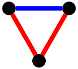
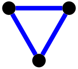
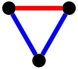
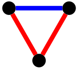
$0 \leq c_1$  $+ c_2$  $+ c_3$  .

Hence

 $\leq c_1$  $+ \left(\frac{1}{3} + c_2\right)$  $+ \left(\frac{2}{3} + c_3\right)$ 

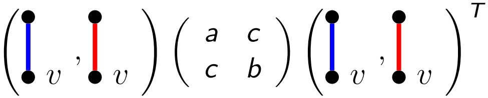
and

 $\leq \max\left\{c_1, \frac{1}{3} + c_2, \frac{2}{3} + c_3\right\}.$

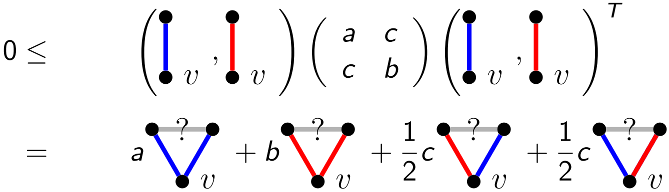
Flag algebras - candidates for c_1, c_2, c_3

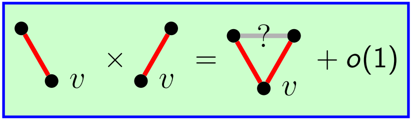
$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0$$

Flag algebras - candidates for c_1, c_2, c_3

$0 \leq$ 

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0$$





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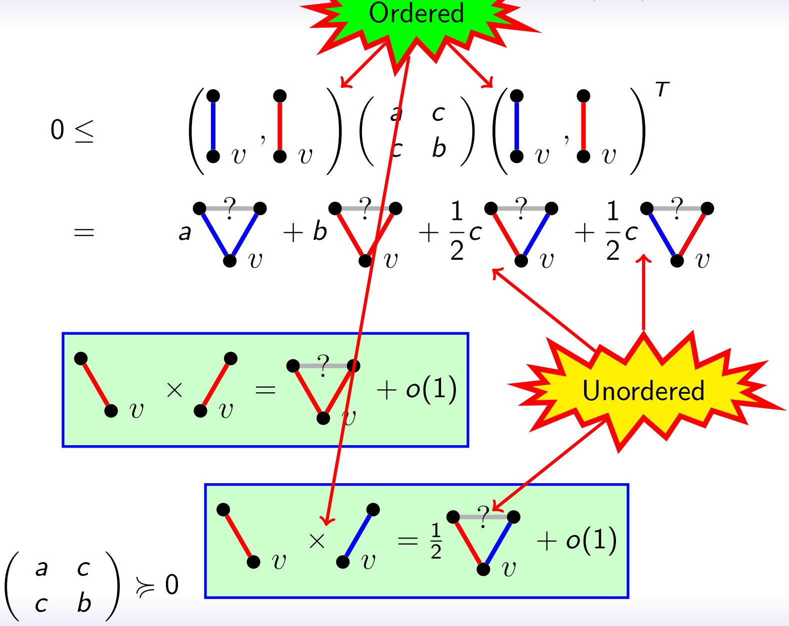
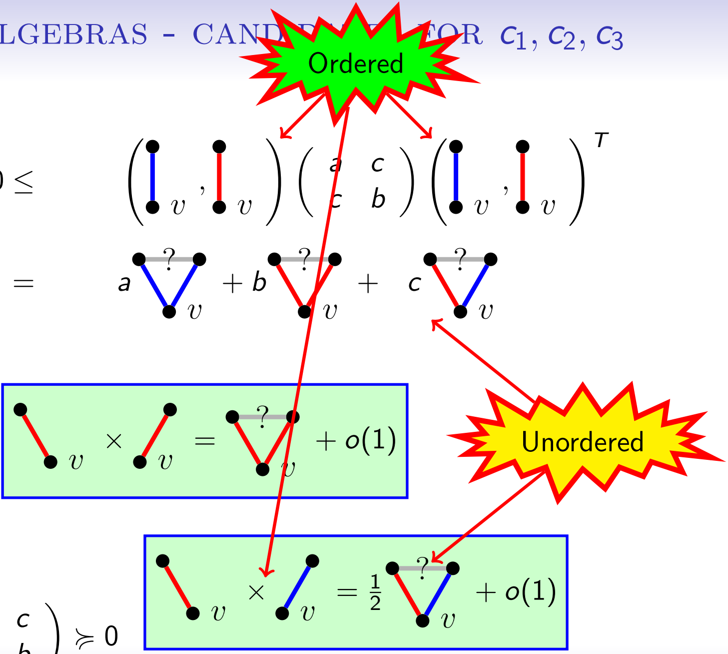
Flag algebras - cand[…] for c_1, c_2, c_3



Flag algebras - cand… for c_1, c_2, c_3



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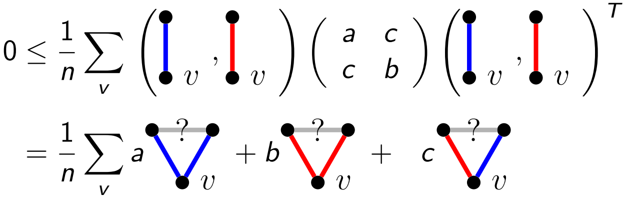

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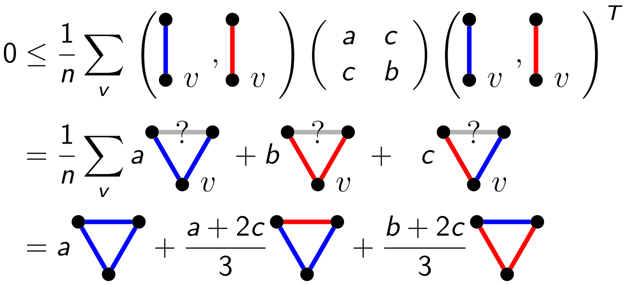

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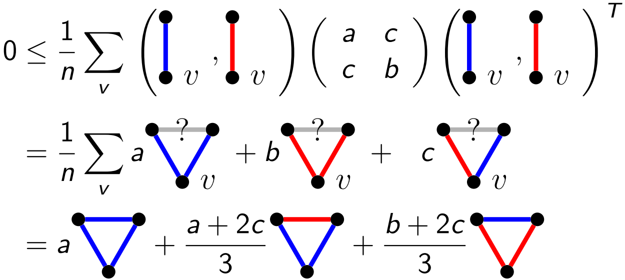
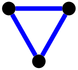
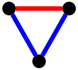
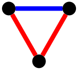
Flag algebras - candidates for c_1, c_2, c_3

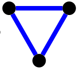
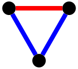
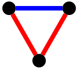


$$c_1 = a, c_2 = \frac{a+2c}{3}, c_3 = \frac{b+2c}{3}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0$$

Flag algebras - using c_1, c_2, c_3

 $=$  $+\ \frac{1}{3}$  $+\ \frac{2}{3}$ 

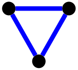
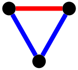
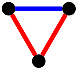
$0 \le a$  $+\ \frac{a+2c}{3}$  $+\ \frac{b+2c}{3}$ 

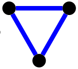
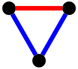
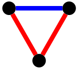
and

 $\le \max\left\{a, \frac{1+a+2c}{3}, \frac{2+b+2c}{3}\right\}.$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0$$

Flag algebras - using c_1, c_2, c_3

 $=$  $+ \frac{1}{3}$  $+ \frac{2}{3}$ 

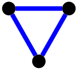
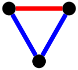
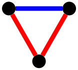
$0 \leq a$  $+ \frac{a+2c}{3}$  $+ \frac{b+2c}{3}$ 

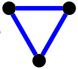
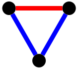
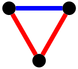
and

 $\leq \max\left\{a, \frac{1+a+2c}{3}, \frac{2+b+2c}{3}\right\}.$

Try

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

 $=$  $+ \frac{1}{3}$  $+ \frac{2}{3}$ 

$0 \leq a$  $+ \frac{a+2c}{3}$  $+ \frac{b+2c}{3}$ 

and

 $\leq \max\left\{a, \frac{1+a+2c}{3}, \frac{2+b+2c}{3}\right\}.$

Try

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

It gives

 $\leq \max\left\{\frac{1}{2}, \frac{1}{6}, \frac{1}{2}\right\} = \frac{1}{2}.$

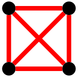
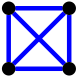
Flag algebras - optimizing a, b, c

 $\leq \max \left\{ a, \frac{1+a+2c}{3}, \frac{2+b+2c}{3} \right\}$

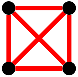
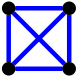
$$(SDP) \begin{cases} \text{Minimize} & d \\ \text{subject to} & a \leq d \\ & \frac{1+a+2c}{3} \leq d \\ & \frac{2+b+2c}{3} \leq d \\ & \begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \end{cases}$$

(SDP) can be solved on computers using CSDP or SDPA.

Back to permutations

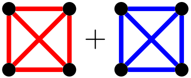
 $+$  $\geq \frac{1}{27}$

Back to permutations

 $+$  $\geq \frac{1}{27}$

- Write a semidefinite program (*SDP*) (with graphs on 7 vertices, 388 constraints, 2 types, $10 + 71$ flags).

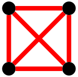
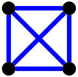
Back to permutations



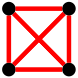
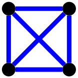
$$\geq \frac{1}{27}$$

- Write a semidefinite program (SDP) (with graphs on 7 vertices, 388 constraints, 2 types, $10 + 71$ flags).
- Solve (SDP) using a computer, obtain $M' \in \mathbb{R}^{f \times f}$.

Back to permutations

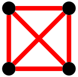
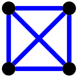
 $+$  $\geq \frac{1}{27} = 0.\overline{037}$

- Write a semidefinite program (SDP) (with graphs on 7 vertices, 388 constraints, 2 types, $10 + 71$ flags).
- Solve (SDP) using a computer, obtain $M' \in \mathbb{R}^{f \times f}$.
- M' gives

 $+$  ≥ 0.0370370369999

and M' has *negative* eigenvalues (-0.1×10^{-12}).

Back to permutations

 $+$  $\geq \frac{1}{27} = 0.\overline{037}$

- Write a semidefinite program (*SDP*) (with graphs on 7 vertices, 388 constraints, 2 types, 10 + 71 flags).
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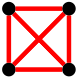
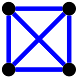
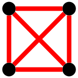
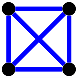
   $+$  ≥ 0.0370370369999

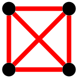
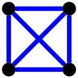
  and M' has *negative* eigenvalues (-0.1×10^{-12}).
- *Round* M' to $M \in \mathbb{Q}^{f \times f}$, such that

   $+$  $\geq \frac{1}{27}$

  and $M \succcurlyeq 0$.

Structure of extremal permutations

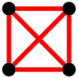
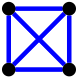
Assuming

 $+$  $= \frac{1}{27}$

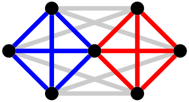
Flag algebra implies:

Structure of extremal permutations

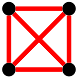
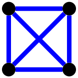
Assuming

 $+$  $= \frac{1}{27}$

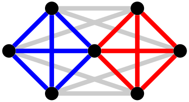
Flag algebra implies:

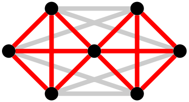
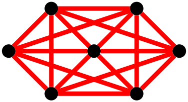
(A)  $= 0$

Assuming

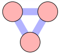
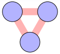
 $+$  $= \frac{1}{27}$

Flag algebra implies:

(A)  $= 0$

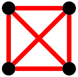
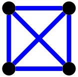
(B) $H =$  $> 0 \Rightarrow H =$ 

After flag algebra (stability)

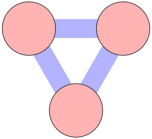
" $+$  is close to $\frac{1}{27} \Rightarrow G$ is close to  or "

Lemma (Stability)

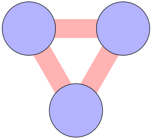
For every $\varepsilon > 0$ there exist n_0 and $\varepsilon' > 0$ such that every admissible graph G of order $n > n_0$ with

 $+$  $\leq \frac{1}{27} + \varepsilon'$

is isomorphic to either



or



after recoloring at most $20\varepsilon n^2$ edges.

After flag algebra (stability sketch)

- Using removal lemma, properties (A) and (B) can be satisfied. (lost εn^2 edges)

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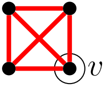
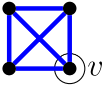
$\frac{1}{27} - \varepsilon \leq$  $+$  $\leq \frac{1}{27} + \varepsilon''$ (1)

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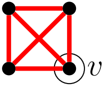
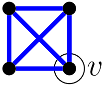
$\frac{1}{27} - \varepsilon \le$ 

$+$ 

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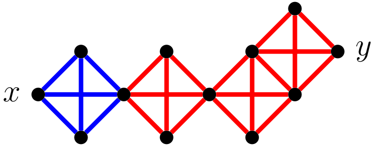
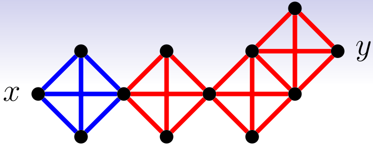
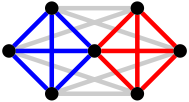
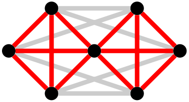
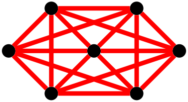
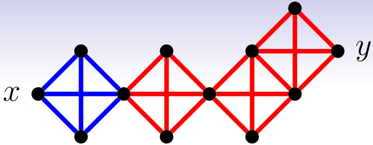
- $x \sim y$ if

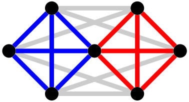


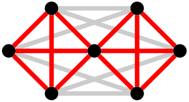
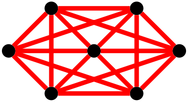
- $x \sim y$ if 

(A)  $= 0$

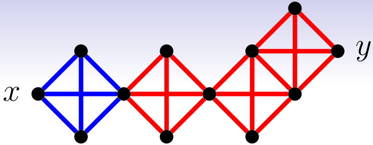
(B) $H =$  $> 0 \Rightarrow H =$ 

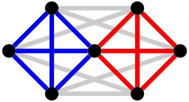
- $x \sim y$ if 

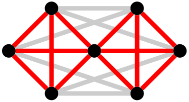
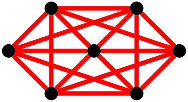
(A)  $= 0$

(B) $H =$  $> 0 \Rightarrow H =$ 

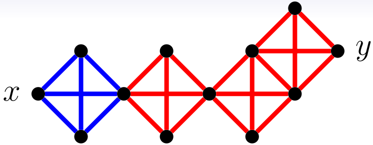
- Every equivalence class is a monochromatic clique.

- $x \sim y$ if 

(A)  $= 0$

(B) $H =$  $> 0 \Rightarrow H =$ 

- Every equivalence class is a monochromatic clique.
- There are three equivalence classes of size $\frac{1}{3}n \pm 16\varepsilon n$ by (1).

- $x \sim y$ if 

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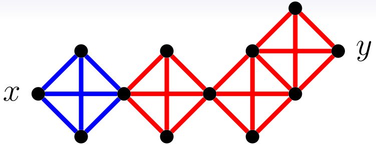
- The classes have the same color

- $x \sim y$ if 

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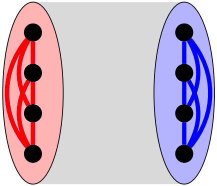
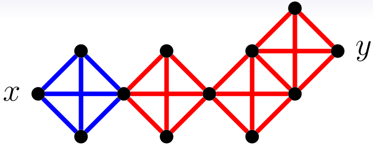


- $x \sim y$ if 

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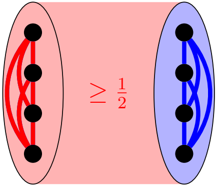
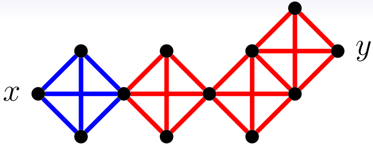


- $x \sim y$ if 

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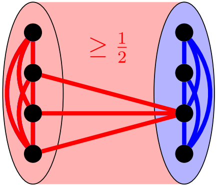
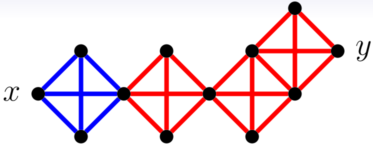


- $x \sim y$ if 

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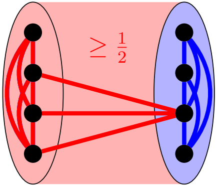

Exact result: By recoloring edges.

Thank you for your attention!

Other permutations - maximizing 1342 and 2413

$$0.19657 \leq \sigma(1342) \leq 2/9 = 0.22222\ldots \qquad \text{AAHHS}$$
$$\sigma(1342) \leq 0.1988373 \qquad \text{BHLPUV}$$

$$51/511 = 0.0998\ldots \leq \sigma(2413) \leq 2/9 = 0.22222 \qquad \text{AAHHS}$$
$$0.1024732 \leq \sigma(2413) \qquad \text{P}$$
$$0.10472\ldots \leq \sigma(2413) \qquad \text{PS}$$
$$\sigma(2413) \leq 0.1047805 \qquad \text{BHLPUV}$$

AAHHS . . . Albert, Atkinson, Handley, Holton, Stromquist 2002
P. . . Presutti 2008
PS. . . Presutti, Stromquist 2010
BHLPUV. . . us