

NUMBER OF MONOTONE SUBSEQUENCES OF LENGTH FOUR IN PERMUTATIONS

József Balogh, Ping Hu, Bernard Lidický, Oleg Pikhurko,
Balázs Udvari, Jan Volec

University of Illinois at Urbana-Champaign, University of Warwick,
University of Szeged

Atlanta Lecture Series XI
January 25, 2014

PERMUTATIONS AND EXTREMAL PROBLEMS

PROBLEM

What is the minimum number of monotone subsequences of size k in a permutation of $[n]$?

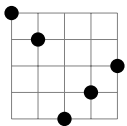
PERMUTATIONS AND EXTREMAL PROBLEMS

PROBLEM

What is the minimum number of monotone subsequences of size k in a permutation of $[n]$?

$$k = 3$$

$$n = 5$$



$$(5,4,1,2,3)$$

$$(5,4,1), (5,4,2), (5,4,3)$$

$$(1,2,3)$$

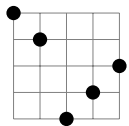
PERMUTATIONS AND EXTREMAL PROBLEMS

PROBLEM

What is the minimum number of monotone subsequences of size k in a permutation of $[n]$?

$$k = 3$$

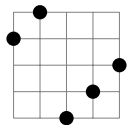
$$n = 5$$



$(5,4,1), (5,4,2), (5,4,3)$

$(1,2,3)$

$(5,4,1,2,3)$



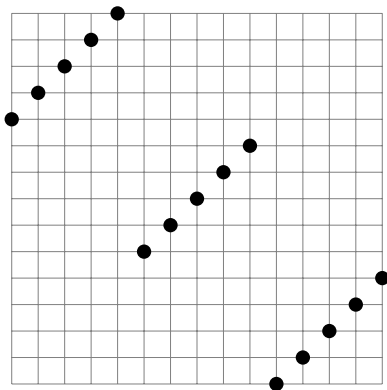
$(1,2,3)$

$(4,5,1,2,3)$

CONJECTURE

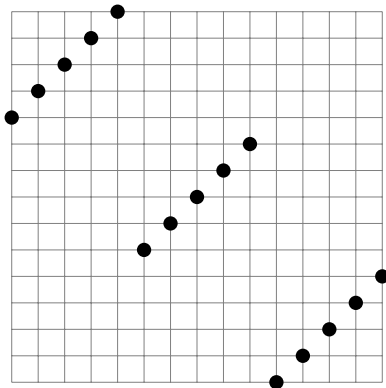
CONJECTURE (MYERS 2002)

The number of monotone subsequences of length k is minimized by a permutation on $[n]$ with $k - 1$ increasing runs of as equal lengths as possible.

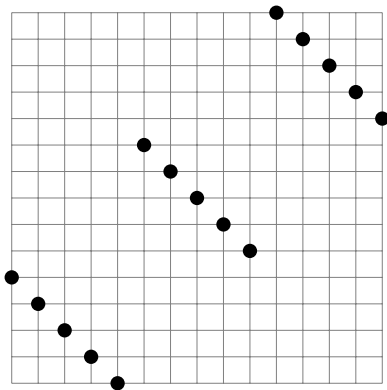


$$k = 4, n = 15$$

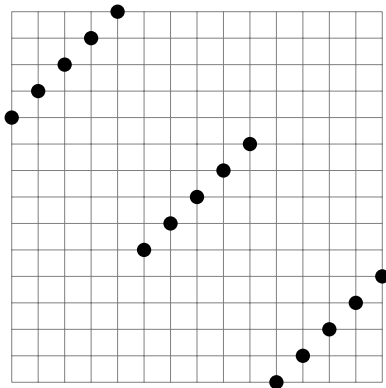
EXTREMAL CASE IS NOT UNIQUE



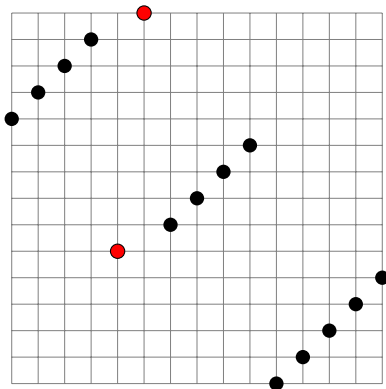
EXTREMAL CASE IS NOT UNIQUE



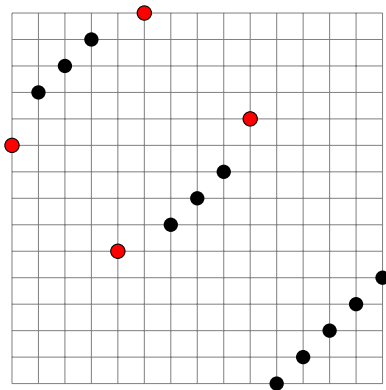
EXTREMAL CASE IS NOT UNIQUE



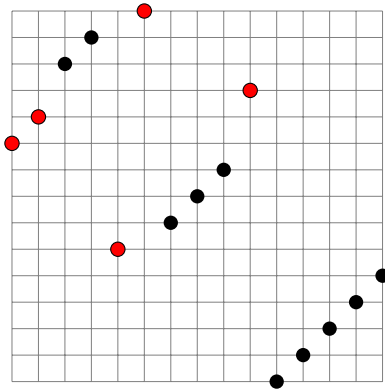
EXTREMAL CASE IS NOT UNIQUE



EXTREMAL CASE IS NOT UNIQUE



EXTREMAL CASE IS NOT UNIQUE

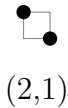
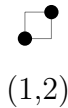


THEOREM (BALOGH, HU, L., PIKHURKO, UDVARI, VOLEC '14+)
Myers' conjecture is true for $k = 4$ and n sufficiently large.

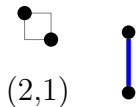
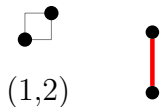
THEOREM (BALOGH, HU, L., PIKHURKO, UDVARI, VOLEC '14+)
Myers' conjecture is true for $k = 4$ and n sufficiently large.

We translate the problem to graphs and use flag algebras.

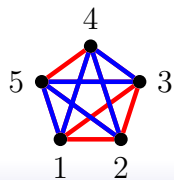
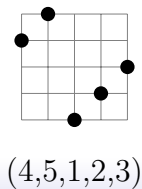
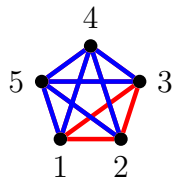
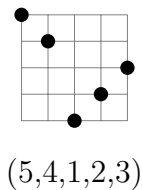
FROM PERMUTATIONS TO PERMUTATION GRAPHS



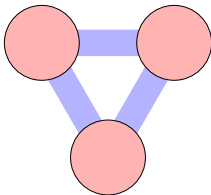
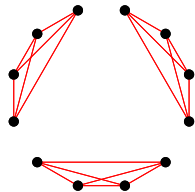
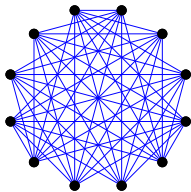
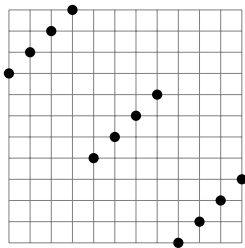
FROM PERMUTATIONS TO PERMUTATION GRAPHS



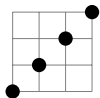
$k = 3$
 $n = 5$



EXTREMAL EXAMPLE ($k = 4$)



AS FLAG ALGEBRA QUESTION ($k = 4$)

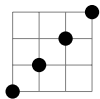


(1,2,3,4)



(4,3,2,1)

AS FLAG ALGEBRA QUESTION ($k = 4$)



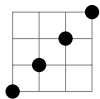
(1,2,3,4)



(4,3,2,1)



AS FLAG ALGEBRA QUESTION ($k = 4$)





(1,2,3,4)

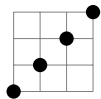


(4,3,2,1)



minimize  + 

AS FLAG ALGEBRA QUESTION ($k = 4$)



(1,2,3,4)



(4,3,2,1)



minimize  + 

THEOREM (BALOGH, HU, L., PIKHURKO, UDVARI, VOLEC '14+)

$$\text{Red square graph} + \text{Blue square graph} \geq \frac{1}{27}$$

for every permutation graph.

THEOREM (BALOGH, HU, L., PIKHURKO, UDVARI, VOLEC '14+)

$$\min \left(\begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \right) + \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \right) = \frac{1}{27}$$

over permutation graphs (and extremal permutations described using Myers' results).

THEOREM (BALOGH, HU, L., PIKHURKO, UDVARI, VOLEC '14+)

$$\min \left(\begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \right) + \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} = \frac{1}{27}$$

over permutation graphs (and extremal permutations described using Myers' results).

THEOREM (SPERFELD '12; THOMASON '89)

$$\frac{1}{35} < \min \left(\begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \right) + \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} < \frac{1}{33}$$

over all 2-edge-colored complete graphs.

FLAG ALGEBRAS

Seminal paper:

A. Razborov, Flag Algebras, *Journal of Symbolic Logic* **72** (2007), 1239–1282.

David P. Robbins Prize by AMS for Razborov in 2013

FLAG ALGEBRAS

Seminal paper:

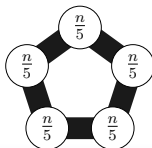
A. Razborov, Flag Algebras, *Journal of Symbolic Logic* **72** (2007), 1239–1282.

David P. Robbins Prize by AMS for Razborov in 2013

Applications to oriented graphs, hypergraphs, crossing number of complete bipartite graphs, geometry, hypercubes,...

THEOREM (HATAMI, HLADKÝ, KRÁL, NORINE, RAZBOROV 2011; GRZESIK 2011)

The number of C_5 's in a triangle-free graph on n vertices is at most $(n/5)^5$.



FLAG ALGEBRAS - WHAT ARE FLAGS

Let G be a 2-edge-colored complete graph on n vertices.



The probability that three random vertices in G span a red triangle.

FLAG ALGEBRAS - WHAT ARE FLAGS

Let G be a 2-edge-colored complete graph on n vertices.



The probability that three random vertices in G span a red triangle.



The probability that three random vertices in G span a triangle with one red and two blue edges.

FLAG ALGEBRAS - WHAT ARE FLAGS

Let G be a 2-edge-colored complete graph on n vertices.



The probability that three random vertices in G span a red triangle.



The probability that three random vertices in G span a triangle with one red and two blue edges.



The probability that a random vertex other than v is connected to $v \in V(G)$ by a red edge, i.e., the red degree of v divided by $n - 1$.

FLAG ALGEBRAS - EXAMPLE

THEOREM (MANTEL 1907)

If a graph G on n vertices has more than $\frac{1}{4}n^2$ edges, then G contain a triangle.

Assume **edges are red** and **non-edges are blue**.

FLAG ALGEBRAS - EXAMPLE

THEOREM (MANTEL 1907)

If a graph G on n vertices has more than $\frac{1}{4}n^2$ edges, then G contain a triangle.

Assume edges are red and non-edges are blue.



$$\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} = 0 \Rightarrow \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \leq \frac{1}{2}$$

FLAG ALGEBRAS - EXAMPLE

THEOREM (MANTEL 1907)

If a graph G on n vertices has more than $\frac{1}{4}n^2$ edges, then G contain a triangle.

Assume edges are red and non-edges are blue.



Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)

FLAG ALGEBRAS - EXAMPLE

THEOREM (MANTEL 1907)

If a graph G on n vertices has more than $\frac{1}{4}n^2$ edges, then G contain a triangle.

Assume edges are red and non-edges are blue.

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)



$$1 = \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array}$$

FLAG ALGEBRAS - EXAMPLE

THEOREM (MANTEL 1907)

If a graph G on n vertices has more than $\frac{1}{4}n^2$ edges, then G contain a triangle.

Assume edges are red and non-edges are blue.

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)



$$1 = \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}$$

FLAG ALGEBRAS - EXAMPLE

THEOREM (MANTEL 1907)

If a graph G on n vertices has more than $\frac{1}{4}n^2$ edges, then G contain a triangle.

Assume edges are red and non-edges are blue.

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)

$$1 = \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array}$$



$$\begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} = 0 + \frac{1}{3} \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array}$$

FLAG ALGEBRAS - EXAMPLE

THEOREM (MANTEL 1907)

If a graph G on n vertices has more than $\frac{1}{4}n^2$ edges, then G contain a triangle.

Assume edges are red and non-edges are blue.

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)

$$1 = \begin{array}{c} \bullet \\ \diagdown \text{ (blue)} \\ \bullet \\ \diagup \text{ (blue)} \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \text{ (red)} \\ \bullet \\ \diagup \text{ (blue)} \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \text{ (blue)} \\ \bullet \\ \diagup \text{ (red)} \\ \bullet \end{array}$$

$$\begin{array}{c} \bullet \\ \text{---} \text{ (red)} \\ \bullet \\ \text{---} \text{ (red)} \\ \bullet \end{array} = 0 = \begin{array}{c} \bullet \\ \diagdown \text{ (blue)} \\ \bullet \\ \diagup \text{ (blue)} \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet \\ \diagdown \text{ (red)} \\ \bullet \\ \diagup \text{ (blue)} \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \\ \diagdown \text{ (blue)} \\ \bullet \\ \diagup \text{ (red)} \\ \bullet \end{array}$$



$$\begin{array}{c} \bullet \\ \text{---} \text{ (red)} \\ \bullet \\ \text{---} \text{ (red)} \\ \bullet \end{array} \leq \frac{2}{3} \left(\begin{array}{c} \bullet \\ \diagdown \text{ (blue)} \\ \bullet \\ \diagup \text{ (blue)} \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \text{ (red)} \\ \bullet \\ \diagup \text{ (blue)} \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \text{ (blue)} \\ \bullet \\ \diagup \text{ (red)} \\ \bullet \end{array} \right)$$

FLAG ALGEBRAS - EXAMPLE

THEOREM (MANTEL 1907)



If a graph G on n vertices has more than $\frac{1}{4}n^2$ edges, then G contain a triangle.

Assume edges are red and non-edges are blue.

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)



$$\begin{aligned}
 1 &= \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \\
 \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} &= 0 \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \\
 \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} &\leq \frac{2}{3} \left(\begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right) \\
 \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} &\leq \frac{2}{3}
 \end{aligned}$$

FLAG ALGEBRAS - IMPROVEMENT

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)

$$\text{red vertical line} = 0 \cdot \text{blue triangle} + \frac{1}{3} \cdot \text{red top triangle} + \frac{2}{3} \cdot \text{red bottom triangle}$$

FLAG ALGEBRAS - IMPROVEMENT



Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)

$$\text{img alt="vertical red line with two dots" data-bbox="285 255 305 355"} = 0 \cdot \text{img alt="triangle with all blue edges" data-bbox="385 260 465 350"} + \frac{1}{3} \cdot \text{img alt="triangle with top edge red, other two blue" data-bbox="520 260 600 350"} + \frac{2}{3} \cdot \text{img alt="triangle with bottom edge red, other two blue" data-bbox="655 260 735 350" data-bbox="280 250 735 355"/>$$

Idea: find $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$0 \leq c_1 \cdot \text{img alt="triangle with all blue edges" data-bbox="355 475 435 565"} + c_2 \cdot \text{img alt="triangle with top edge red, other two blue" data-bbox="495 475 575 565"} + c_3 \cdot \text{img alt="triangle with bottom edge red, other two blue" data-bbox="635 475 715 565"} .$$

FLAG ALGEBRAS - IMPROVEMENT

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)

$$\text{red edge} = 0 \cdot \text{blue triangle} + \frac{1}{3} \cdot \text{red triangle} + \frac{2}{3} \cdot \text{red triangle}$$

Idea: find $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$0 \leq c_1 \cdot \text{blue triangle} + c_2 \cdot \text{red triangle} + c_3 \cdot \text{red triangle}.$$

Hence

$$\text{red edge} \leq c_1 \cdot \text{blue triangle} + \left(\frac{1}{3} + c_2\right) \cdot \text{red triangle} + \left(\frac{2}{3} + c_3\right) \cdot \text{red triangle}$$

and

$$\text{red edge} \leq \max \left\{ c_1, \frac{1}{3} + c_2, \frac{2}{3} + c_3 \right\}.$$

FLAG ALGEBRAS - CANDIDATES FOR c_1, c_2, c_3

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0$$

FLAG ALGEBRAS - CANDIDATES FOR c_1, c_2, c_3

$$0 \leq \left(\begin{array}{c} \bullet \\ \color{blue}{|} \\ \bullet \\ v \end{array} , \begin{array}{c} \bullet \\ \color{red}{|} \\ \bullet \\ v \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ \color{blue}{|} \\ \bullet \\ v \end{array} , \begin{array}{c} \bullet \\ \color{red}{|} \\ \bullet \\ v \end{array} \right)^T$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0$$

FLAG ALGEBRAS - CANDIDATES FOR $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$

$$\begin{aligned}
 0 &\leq \left(\begin{array}{c} \bullet \\ | \\ \bullet \\ v \end{array}, \begin{array}{c} \bullet \\ | \\ \bullet \\ v \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ | \\ \bullet \\ v \end{array}, \begin{array}{c} \bullet \\ | \\ \bullet \\ v \end{array} \right)^T \\
 &= a \begin{array}{c} \bullet \quad ? \\ \diagdown \quad \diagup \\ \bullet \\ v \end{array} + b \begin{array}{c} \bullet \quad ? \\ \diagup \quad \diagdown \\ \bullet \\ v \end{array} + \frac{1}{2}c \begin{array}{c} \bullet \quad ? \\ \diagdown \quad \diagdown \\ \bullet \\ v \end{array} + \frac{1}{2}c \begin{array}{c} \bullet \quad ? \\ \diagup \quad \diagup \\ \bullet \\ v \end{array}
 \end{aligned}$$

$$\begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ v \end{array} \times \begin{array}{c} \bullet \\ \diagup \\ \bullet \\ v \end{array} = \begin{array}{c} \bullet \quad ? \\ \diagdown \quad \diagup \\ \bullet \\ v \end{array} + o(1)$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0$$

FLAG ALGEBRAS - CANDIDATES FOR C_1, C_2, C_3

Ordered

$$0 \leq \left(\begin{array}{c} \bullet \\ | \\ \bullet \\ v \end{array}, \begin{array}{c} \bullet \\ | \\ \bullet \\ v \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ | \\ \bullet \\ v \end{array}, \begin{array}{c} \bullet \\ | \\ \bullet \\ v \end{array} \right)^T$$

$$= a \begin{array}{c} \bullet \quad ? \\ \diagdown \quad \diagup \\ \bullet \\ v \end{array} + b \begin{array}{c} \bullet \quad ? \\ \diagup \quad \diagdown \\ \bullet \\ v \end{array} + \frac{1}{2}c \begin{array}{c} \bullet \quad ? \\ \diagdown \quad \diagup \\ \bullet \\ v \end{array} + \frac{1}{2}c \begin{array}{c} \bullet \quad ? \\ \diagup \quad \diagdown \\ \bullet \\ v \end{array}$$

$$\begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ v \end{array} \times \begin{array}{c} \bullet \\ \diagup \\ \bullet \\ v \end{array} = \begin{array}{c} \bullet \quad ? \\ \diagdown \quad \diagup \\ \bullet \\ v \end{array} + o(1)$$

Unordered

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \not\approx 0 \quad \begin{array}{c} \bullet \\ \diagdown \\ \bullet \\ v \end{array} \times \begin{array}{c} \bullet \\ \diagup \\ \bullet \\ v \end{array} = \frac{1}{2} \begin{array}{c} \bullet \quad ? \\ \diagdown \quad \diagup \\ \bullet \\ v \end{array} + o(1)$$

FLAG ALGEBRAS - CANDIDATES FOR C_1, C_2, C_3

Ordered

$$0 \leq \left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array} v, \begin{array}{c} \bullet \\ | \\ \bullet \end{array} v \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array} v, \begin{array}{c} \bullet \\ | \\ \bullet \end{array} v \right)^T$$

$$= a \begin{array}{c} \bullet \\ \text{?} \\ \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} v + b \begin{array}{c} \bullet \\ \text{?} \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} v + c \begin{array}{c} \bullet \\ \text{?} \\ \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} v$$

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} v \times \begin{array}{c} \bullet \\ | \\ \bullet \end{array} v = \begin{array}{c} \bullet \\ \text{?} \\ \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} v + o(1)$$

Unordered

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \not\approx 0 \quad \begin{array}{c} \bullet \\ | \\ \bullet \end{array} v \times \begin{array}{c} \bullet \\ | \\ \bullet \end{array} v = \frac{1}{2} \begin{array}{c} \bullet \\ \text{?} \\ \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} v + o(1)$$

FLAG ALGEBRAS - CANDIDATES FOR c_1, c_2, c_3

$$\begin{aligned}
 0 &\leq \frac{1}{n} \sum_v \left(\begin{array}{c} \bullet \\ | \\ \bullet_v \end{array}, \begin{array}{c} \bullet \\ | \\ \bullet_v \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ | \\ \bullet_v \end{array}, \begin{array}{c} \bullet \\ | \\ \bullet_v \end{array} \right)^T \\
 &= \frac{1}{n} \sum_v a \begin{array}{c} \bullet \quad ? \\ \diagdown \quad \diagup \\ \bullet_v \end{array} + b \begin{array}{c} \bullet \quad ? \\ \diagup \quad \diagdown \\ \bullet_v \end{array} + c \begin{array}{c} \bullet \quad ? \\ \diagdown \quad \diagup \\ \bullet_v \end{array}
 \end{aligned}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0$$

FLAG ALGEBRAS - CANDIDATES FOR c_1, c_2, c_3

$$\begin{aligned}
 0 &\leq \frac{1}{n} \sum_v \left(\begin{array}{c} \bullet \\ | \\ \bullet_v \end{array}, \begin{array}{c} \bullet \\ | \\ \bullet_v \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ | \\ \bullet_v \end{array}, \begin{array}{c} \bullet \\ | \\ \bullet_v \end{array} \right)^T \\
 &= \frac{1}{n} \sum_v a \begin{array}{c} \bullet \quad ? \\ \diagdown \quad / \\ \bullet_v \end{array} + b \begin{array}{c} \bullet \quad ? \\ / \quad \diagdown \\ \bullet_v \end{array} + c \begin{array}{c} \bullet \quad ? \\ / \quad \diagup \\ \bullet_v \end{array} \\
 &= a \begin{array}{c} \bullet \quad \bullet \\ / \quad \diagdown \\ \bullet_v \end{array} + \frac{a+2c}{3} \begin{array}{c} \bullet \quad \bullet \\ / \quad \diagdown \\ \bullet_v \end{array} + \frac{b+2c}{3} \begin{array}{c} \bullet \quad \bullet \\ / \quad \diagup \\ \bullet_v \end{array}
 \end{aligned}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0$$

FLAG ALGEBRAS - CANDIDATES FOR c_1, c_2, c_3

$$0 \leq \frac{1}{n} \sum_v \left(\begin{array}{c} \bullet \\ \text{blue} \\ \bullet \\ v \end{array}, \begin{array}{c} \bullet \\ \text{red} \\ \bullet \\ v \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ \text{blue} \\ \bullet \\ v \end{array}, \begin{array}{c} \bullet \\ \text{red} \\ \bullet \\ v \end{array} \right)^T$$

$$= \frac{1}{n} \sum_v a \begin{array}{c} \bullet \\ \text{?} \\ \bullet \\ v \end{array} + b \begin{array}{c} \bullet \\ \text{?} \\ \bullet \\ v \end{array} + c \begin{array}{c} \bullet \\ \text{?} \\ \bullet \\ v \end{array}$$

$$= a \begin{array}{c} \bullet \\ \text{blue} \\ \bullet \\ v \end{array} + \frac{a+2c}{3} \begin{array}{c} \bullet \\ \text{blue/red} \\ \bullet \\ v \end{array} + \frac{b+2c}{3} \begin{array}{c} \bullet \\ \text{red/blue} \\ \bullet \\ v \end{array}$$

$$c_1 = a, c_2 = \frac{a+2c}{3}, c_3 = \frac{b+2c}{3}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0$$

FLAG ALGEBRAS - USING c_1, c_2, c_3

$$\begin{aligned} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} &= \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \\ 0 \leq a & \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{a+2c}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{b+2c}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \end{aligned}$$

and

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \leq \max \left\{ a, \frac{1+a+2c}{3}, \frac{2+b+2c}{3} \right\}.$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0$$

FLAG ALGEBRAS - USING c_1, c_2, c_3

$$\begin{aligned} \text{Diagram} &= \text{Diagram}_1 + \frac{1}{3} \text{Diagram}_2 + \frac{2}{3} \text{Diagram}_3 \\ 0 \leq a &+ \frac{a+2c}{3} + \frac{b+2c}{3} \end{aligned}$$

The diagram on the left is a vertical red line segment with two black dots at its ends. The three diagrams on the right are triangles with three black dots at their vertices. In the first triangle, all three edges are blue. In the second triangle, the top edge is red and the two bottom edges are blue. In the third triangle, the two bottom edges are red and the top edge is blue.

and

$$\text{Diagram} \leq \max \left\{ a, \frac{1+a+2c}{3}, \frac{2+b+2c}{3} \right\}.$$

Try

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

FLAG ALGEBRAS - USING c_1, c_2, c_3

$$\begin{aligned}
 \text{Diagram 1} &= \text{Diagram 2} + \frac{1}{3} \text{Diagram 3} + \frac{2}{3} \text{Diagram 4} \\
 0 \leq a \text{Diagram 5} &+ \frac{a+2c}{3} \text{Diagram 6} + \frac{b+2c}{3} \text{Diagram 7}
 \end{aligned}$$

The diagrams are triangles with three vertices. Diagram 1 is a vertical red line. Diagram 2 has a blue top edge and blue sides. Diagram 3 has a red top edge and blue sides. Diagram 4 has a blue top edge and red sides. Diagram 5 has a blue top edge and blue sides. Diagram 6 has a red top edge and blue sides. Diagram 7 has a blue top edge and red sides.

and

$$\text{Diagram 1} \leq \max \left\{ a, \frac{1+a+2c}{3}, \frac{2+b+2c}{3} \right\}.$$

Try

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

It gives

$$\text{Diagram 1} \leq \max \left\{ \frac{1}{2}, \frac{1}{6}, \frac{1}{2} \right\} = \frac{1}{2}.$$

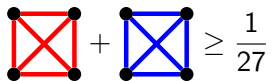
FLAG ALGEBRAS - OPTIMIZING a, b, c

$$\bullet \text{---} \bullet \leq \max \left\{ a, \frac{1+a+2c}{3}, \frac{2+b+2c}{3} \right\}$$

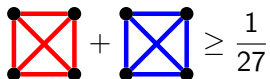
$$(SDP) \left\{ \begin{array}{l} \text{Minimize } d \\ \text{subject to } a \leq d \\ \frac{1+a+2c}{3} \leq d \\ \frac{2+b+2c}{3} \leq d \\ \begin{pmatrix} a & c \\ c & b \end{pmatrix} \succeq 0 \end{array} \right.$$

(*SDP*) can be solved on computers using CSDP or SDPA.

BACK TO PERMUTATIONS

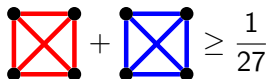

$$\text{Red Square} + \text{Blue Square} \geq \frac{1}{27}$$

BACK TO PERMUTATIONS


$$\text{Red Square} + \text{Blue Square} \geq \frac{1}{27}$$

- Write a semidefinite program (*SDP*) (with graphs on 7 vertices, 388 constraints, 2 types, 10 + 71 flags).

BACK TO PERMUTATIONS

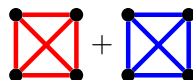


The image shows two square graphs, each with four vertices and four edges. The left graph has red edges, and the right graph has blue edges. Both graphs include the four outer edges of the square and two diagonal edges. The two graphs are separated by a plus sign, followed by a greater-than-or-equal-to sign and the fraction 1/27.

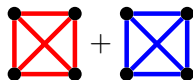
$$\text{Red Graph} + \text{Blue Graph} \geq \frac{1}{27}$$

- Write a semidefinite program (*SDP*) (with graphs on 7 vertices, 388 constraints, 2 types, 10 + 71 flags).
- Solve (*SDP*) using a computer, obtain $M' \in \mathbb{R}^{f \times f}$.

BACK TO PERMUTATIONS

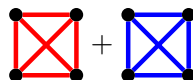

$$+ \geq \frac{1}{27} = 0.\overline{037}$$

- Write a semidefinite program (*SDP*) (with graphs on 7 vertices, 388 constraints, 2 types, 10 + 71 flags).
- Solve (*SDP*) using a computer, obtain $M' \in \mathbb{R}^{f \times f}$.
- M' gives

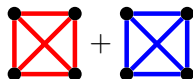

$$\geq 0.0370370369999$$

and M' has *negative* eigenvalues (-0.1×10^{-12}).

BACK TO PERMUTATIONS

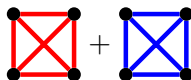

$$+ \geq \frac{1}{27} = 0.\overline{037}$$

- Write a semidefinite program (*SDP*) (with graphs on 7 vertices, 388 constraints, 2 types, 10 + 71 flags).
- Solve (*SDP*) using a computer, obtain $M' \in \mathbb{R}^{f \times f}$.
- M' gives


$$\geq 0.0370370369999$$

and M' has *negative* eigenvalues (-0.1×10^{-12}).

- Round M' to $M \in \mathbb{Q}^{f \times f}$, such that


$$\geq \frac{1}{27}$$

and $M \succcurlyeq 0$.

STRUCTURE OF EXTREMAL PERMUTATIONS

Assuming

$$\begin{array}{c} \bullet & & \bullet \\ \diagdown & & / \\ \bullet & & \bullet \\ \diagup & & \diagdown \\ \bullet & & \bullet \end{array} + \begin{array}{c} \bullet & & \bullet \\ \diagdown & & / \\ \bullet & & \bullet \\ \diagup & & \diagdown \\ \bullet & & \bullet \end{array} = \frac{1}{27}$$

Flag algebra implies:

STRUCTURE OF EXTREMAL PERMUTATIONS

Assuming

$$\begin{array}{c} \bullet & & \bullet \\ \diagdown & & / \\ \bullet & & \bullet \\ \diagup & & \diagdown \\ \bullet & & \bullet \end{array} + \begin{array}{c} \bullet & & \bullet \\ \diagup & & \diagdown \\ \bullet & & \bullet \\ \diagdown & & / \\ \bullet & & \bullet \end{array} = \frac{1}{27}$$

Flag algebra implies:

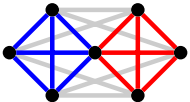
(A) $\begin{array}{c} \bullet & & \bullet \\ \diagdown & & / \\ \bullet & & \bullet \\ \diagup & & \diagdown \\ \bullet & & \bullet \end{array} = 0$

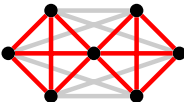
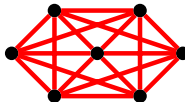
STRUCTURE OF EXTREMAL PERMUTATIONS

Assuming





$$\begin{array}{c} \bullet & & \bullet \\ \diagdown & & / \\ \bullet & & \bullet \\ \diagup & & \diagdown \\ \bullet & & \bullet \end{array} + \begin{array}{c} \bullet & & \bullet \\ \diagup & & \diagdown \\ \bullet & & \bullet \\ \diagdown & & / \\ \bullet & & \bullet \end{array} = \frac{1}{27}$$

Flag algebra implies:

(A)  = 0



(B) $H =$  $> 0 \Rightarrow H =$ 

AFTER FLAG ALGEBRA (STABILITY)

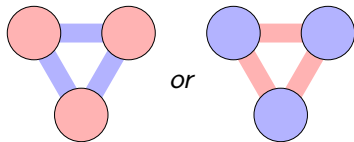
“ +  is close to $\frac{1}{27} \Rightarrow G$ is close to  or ”

LEMMA (STABILITY)

For every $\varepsilon > 0$ there exist n_0 and $\varepsilon' > 0$ such that every admissible graph G of order $n > n_0$ with

$$\text{ + \text{} \leq \frac{1}{27} + \varepsilon'$$

is isomorphic to either



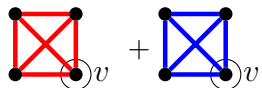
after recoloring at most $20\varepsilon n^2$ edges.

AFTER FLAG ALGEBRA (STABILITY SKETCH)

- Using removal lemma, properties (A) and (B) can be satisfied.
(lost εn^2 edges)

AFTER FLAG ALGEBRA (STABILITY SKETCH)

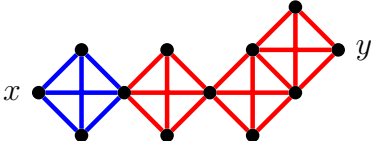
- Using removal lemma, properties (A) and (B) can be satisfied.
(lost εn^2 edges)
- For all $v \in V(G) \setminus X$, where $|X| \leq 2\varepsilon n$ vertices

$$\frac{1}{27} - \varepsilon \leq \text{red square} + \text{blue square} \leq \frac{1}{27} + \varepsilon'' \quad (1)$$


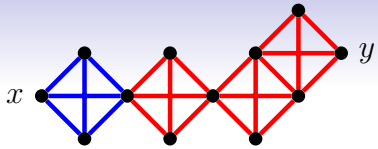
AFTER FLAG ALGEBRA (STABILITY SKETCH)

- Using removal lemma, properties (A) and (B) can be satisfied. (lost εn^2 edges)
- For all $v \in V(G) \setminus X$, where $|X| \leq 2\varepsilon n$ vertices

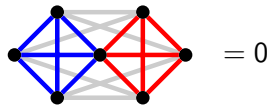
$$\frac{1}{27} - \varepsilon \leq \left(\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \right)_v + \left(\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \right)_v \leq \frac{1}{27} + \varepsilon'' \quad (1)$$

- $x \sim y$ if 

• $x \sim y$ if

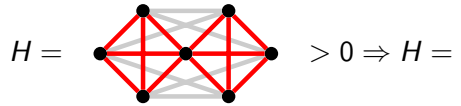


(A)



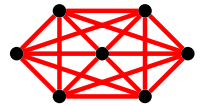
$= 0$

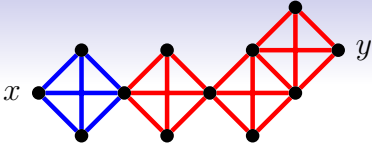
(B)

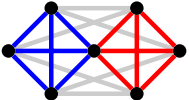


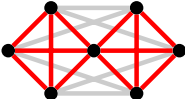
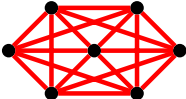
$H =$

$> 0 \Rightarrow H =$

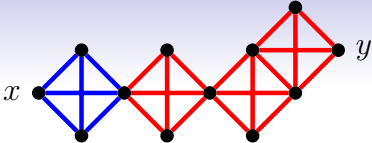


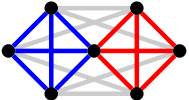
- $x \sim y$ if 

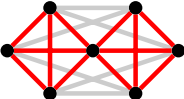
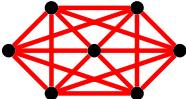
(A)  = 0

(B) $H =$  $> 0 \Rightarrow H =$ 

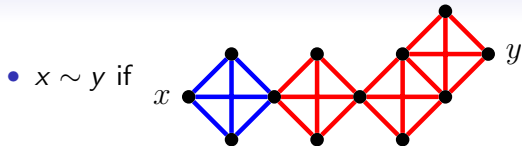
- Every equivalence class is a monochromatic clique.

- $x \sim y$ if 

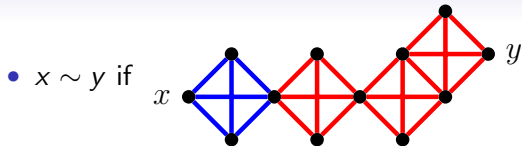
(A)  = 0

(B) $H =$  $> 0 \Rightarrow H =$ 

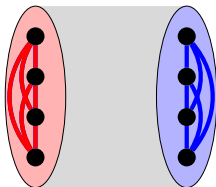
- Every equivalence class is a monochromatic clique.
- There are three equivalence classes of size $\frac{1}{3}n \pm 16\epsilon n$ by (1).

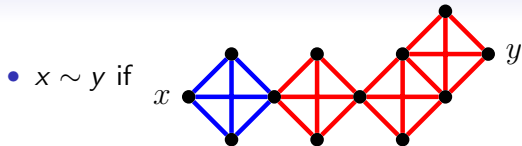


- Every equivalence class is a monochromatic clique.
- There are three equivalence classes of size $\frac{1}{3}n \pm 16\epsilon n$ by (1).
- The classes have the same color

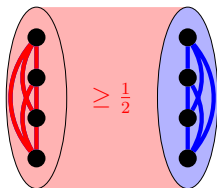


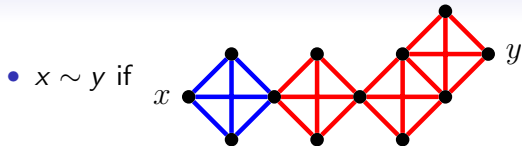
- Every equivalence class is a monochromatic clique.
- There are three equivalence classes of size $\frac{1}{3}n \pm 16\epsilon n$ by (1).
- The classes have the same color



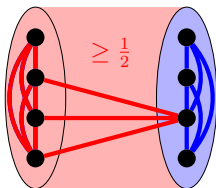


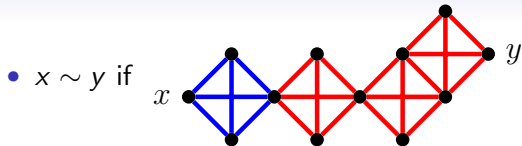
- Every equivalence class is a monochromatic clique.
- There are three equivalence classes of size $\frac{1}{3}n \pm 16\epsilon n$ by (1).
- The classes have the same color



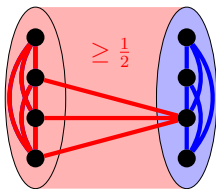


- Every equivalence class is a monochromatic clique.
- There are three equivalence classes of size $\frac{1}{3}n \pm 16\epsilon n$ by (1).
- The classes have the same color





- Every equivalence class is a monochromatic clique.
- There are three equivalence classes of size $\frac{1}{3}n \pm 16\epsilon n$ by (1).
- The classes have the same color



Exact result: By recoloring edges.

Thank you for your attention!

OTHER PERMUTATIONS - MAXIMIZING 1342 AND 2413

$$0.19657 \leq \sigma(1342) \leq 2/9 = 0.22222\dots \quad \text{AAHHS}$$
$$\sigma(1342) \leq 0.1988373 \quad \text{BHLPUV}$$

$$51/511 = 0.0998\dots \leq \sigma(2413) \leq 2/9 = 0.22222 \quad \text{AAHHS}$$
$$0.1024732 \leq \sigma(2413) \quad \text{P}$$
$$0.10472\dots \leq \sigma(2413) \quad \text{PS}$$
$$\sigma(2413) \leq 0.1047805 \quad \text{BHLPUV}$$

AAHHS ... Albert, Atkinson, Handley, Holton, Stromquist 2002

P... Presutti 2008

PS... Presutti, Stromquist 2010

BHLPUV... us