

FLAG ALGEBRAS AND APPLICATIONS TO PERMUTATIONS

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Iowa State University

Nov 7, 2014

OUTLINE

- Extremal problems
- Introduction to the use of Flag Algebras
- Example of automated Flag Algebras approach
- Applications of Flag Algebras in permutations

PROBLEM

PROBLEM

What is the minimum number of monotone subsequences of size k in a permutation of $[n]$?

GRAPHS

DEFINITION

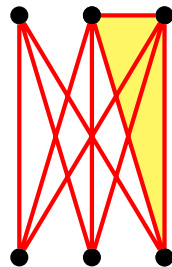
A graph G consists of vertices V and edges $E \subseteq \binom{V}{2}$.

 K_2  K_3  K_4  C_4

EXTREMAL PROBLEMS

THEOREM (MANTEL 1907)

If a graph G on n vertices has more than $\frac{1}{4}n^2$ edges, then G contains a triangle.

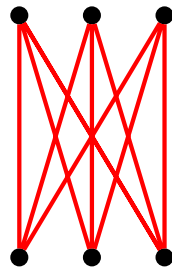


$$|E| = 10$$

EXTREMAL PROBLEMS

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If a graph G on n vertices has more than $\frac{1}{4}n^2$ edges, then G contains a triangle.



$$|E| = 9$$

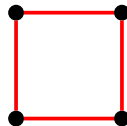
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THEOREM (DIRAC 1952)

If all vertices in a graph G on n vertices have degree at least $\frac{n}{2}$, then G is Hamiltonian. ($n \geq 3$)



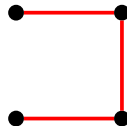
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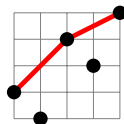
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THEOREM (ERDŐS, SZEKERES 1935)

Every sequence of $n^2 + 1$ distinct numbers contains a monotone subsequence of size $n + 1$.



$(2,1,4,3,5)$

EXTREMAL PROBLEMS

THEOREM (MANTEL 1907)

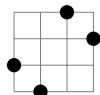
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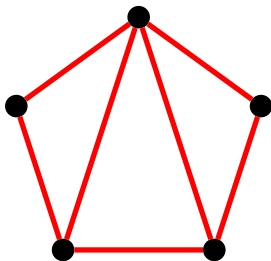
A triangle-free graph contains at most $\frac{1}{4}n^2$ edges.

PROBLEM

Maximize a graph parameter (# of edges) over a class of graphs (triangle-free).

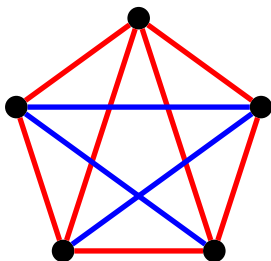
- local condition and global parameter
- threshold
- bound and extremal example

EDGE-COLORED GRAPHS



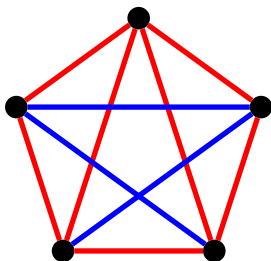
A graph on 5 vertices.

EDGE-COLORED GRAPHS

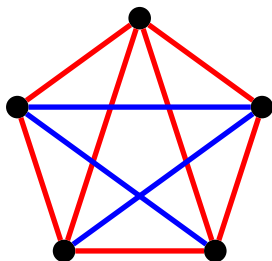


A 2-edge-colored complete graph K_5 on 5 vertices.

DENSITIES (RATIOS) IN EDGE-COLORED GRAPHS

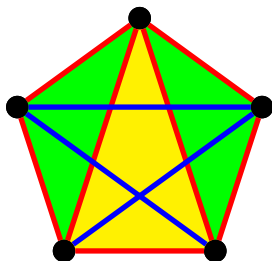


DENSITIES (RATIOS) IN EDGE-COLORED GRAPHS



$$\text{red edge} = \frac{7}{10}$$

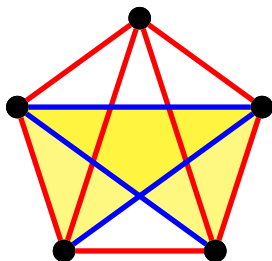
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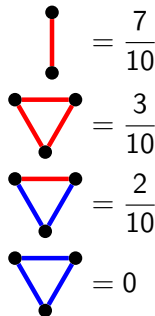
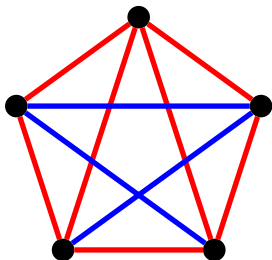
$$= \frac{3}{10}$$

DENSITIES (RATIOS) IN EDGE-COLORED GRAPHS



$$\begin{array}{l}
 \begin{array}{c} \bullet \\ | \\ \bullet \end{array} = \frac{7}{10} \\
 \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} = \frac{3}{10} \\
 \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} = \frac{2}{10}
 \end{array}$$

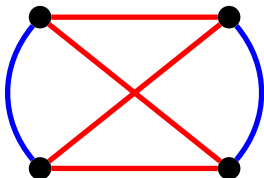
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CLASSICAL THEOREMS — MANTEL'S THEOREM

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A triangle-free graph contains at most $\frac{1}{4}n^2$ edges.

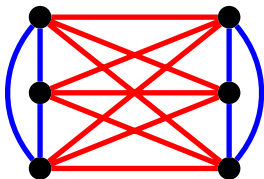


$$\begin{aligned}
 & \text{Triangle} = 0 \\
 & \text{Path of length 2} = \frac{4}{6} = \frac{2}{3} = 0.\bar{6}
 \end{aligned}$$

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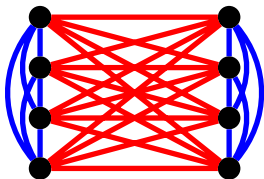


$$\begin{aligned}
 & \text{Triangle} = 0 \\
 & \text{Path of length 2} = \frac{9}{15} = \frac{3}{5} = 0.6
 \end{aligned}$$

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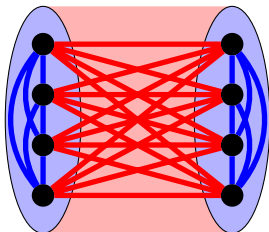


$$\begin{aligned}
 & \text{Triangle} = 0 \\
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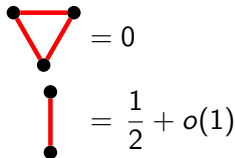
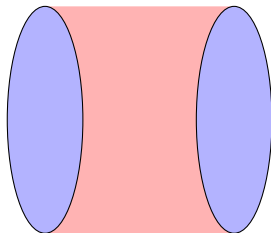


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FLAG ALGEBRAS

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A. Razborov, Flag Algebras, *Journal of Symbolic Logic* **72** (2007), 1239–1282.

David P. Robbins Prize by AMS for Razborov in 2013

FLAG ALGEBRAS

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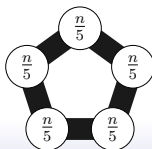
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Applications to oriented graphs, hypergraphs, crossing number of graphs, geometry, . . .

THEOREM (HATAMI, HLADKÝ, KRÁL, NORINE, RAZBOROV 2011; GRZESIK 2011)

The number of C_5 's in a triangle-free graph on n vertices is at most $(n/5)^5$.



FLAG ALGEBRAS DEFINITIONS

Let G be a 2-edge-colored complete graph on n vertices.



The probability that three random vertices in G span a red triangle.

FLAG ALGEBRAS DEFINITIONS

Let G be a 2-edge-colored complete graph on n vertices.



The probability that three random vertices in G span a red triangle.



The probability that three random vertices in G span a triangle with one red and two blue edges.

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The probability that a random vertex other than v is connected to $v \in V(G)$ by a red edge, i.e., the red degree of v divided by $n - 1$.

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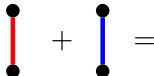
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$$\text{red edge} + \text{blue edge} = 1$$

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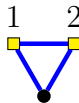


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$$+ = 1$$



Flag

Type is a flag induced by labeled vertices

FLAG ALGEBRAS IDENTITIES

Let G be a 2-edge-colored complete graph on n vertices. Then

$$\text{Red Triangle} + \text{Red Triangle with Blue Top Edge} + \text{Red Triangle with Blue Bottom Edge} + \text{Blue Triangle} = 1.$$

Same kind as

$$\text{Red Edge} + \text{Blue Edge} = 1.$$

FLAG ALGEBRAS IDENTITIES

Let G be a 2-edge-colored complete graph on n vertices. Then

$$\text{edge} = \frac{3}{3} \text{triangle}_{\text{red}} + \frac{2}{3} \text{triangle}_{\text{red-blue}} + \frac{1}{3} \text{triangle}_{\text{blue}} + \frac{0}{3} \text{triangle}_{\text{blue-red}}.$$

Expanded version where pictures mean graphs:

$$P\left(\begin{array}{c} \bullet \\ \text{red edge} \\ \bullet \end{array} \text{ in } G\right) = P\left(\begin{array}{c} \bullet \\ \text{red edge} \\ \bullet \end{array} \text{ in } \begin{array}{c} \bullet & \bullet \\ \text{red triangle} \\ \bullet \end{array}\right) \cdot P\left(\begin{array}{c} \bullet & \bullet \\ \text{red triangle} \\ \bullet \end{array} \text{ in } G\right) + P\left(\begin{array}{c} \bullet \\ \text{red edge} \\ \bullet \end{array} \text{ in } \begin{array}{c} \bullet & \bullet \\ \text{blue triangle} \\ \bullet \end{array}\right)$$

FLAG ALGEBRAS IDENTITIES

Let G be a 2-edge-colored complete graph on n vertices. Then

$$\begin{array}{c} \bullet \\ \text{red} \\ \square v \end{array} \times \begin{array}{c} \bullet \\ \text{red} \\ \square v \end{array} = \begin{array}{c} \bullet \quad ? \quad \bullet \\ \text{red} \quad \text{red} \\ \square v \end{array} + o(1) = \begin{array}{c} \bullet \text{---} \bullet \\ \text{red} \quad \text{red} \\ \square v \end{array} + \begin{array}{c} \bullet \text{---} \bullet \\ \text{blue} \quad \text{red} \\ \square v \end{array} + o(1)$$

$o(1)$ as $|V(G)| \rightarrow \infty$ (will be omitted on next slides)

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$$\begin{array}{c} \bullet \\ \text{red} \\ \square v \end{array} \times \begin{array}{c} \bullet \\ \text{blue} \\ \square v \end{array} = \frac{1}{2} \begin{array}{c} \bullet \quad ? \quad \bullet \\ \text{red} \quad \text{blue} \\ \square v \end{array} + o(1) = \frac{1}{2} \begin{array}{c} \bullet \quad \text{red} \quad \bullet \\ \text{red} \quad \text{blue} \\ \square v \end{array} + \frac{1}{2} \begin{array}{c} \bullet \quad \text{blue} \quad \bullet \\ \text{red} \quad \text{blue} \\ \square v \end{array} + o(1)$$

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$\begin{array}{c} \bullet \\ \text{red} \\ \square v \end{array} \times \begin{array}{c} \bullet \\ \text{blue} \\ \square v \end{array}$: The probability that choosing two vertices u_1, u_2 other than v gives red vu_1 and blue vu_2 .

$\begin{array}{c} \bullet \quad ? \quad \bullet \\ \text{red} \quad \text{blue} \\ \square v \end{array}$: The probability that choosing two **different** vertices u_1, u_2 other than v gives one of vu_1 and vu_2 is red and the other is blue.

$o(1)$ as $|V(G)| \rightarrow \infty$ (will be omitted on next slides)

FLAG ALGEBRAS IDENTITIES

Let G be a 2-edge-colored complete graph on n vertices. Then

$$\frac{1}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \text{ } \square \text{ } v \end{array}$$

FLAG ALGEBRAS IDENTITIES

Let G be a 2-edge-colored complete graph on n vertices. Then

$$\frac{1}{3} \text{ (triangle with 2 red edges) } = \frac{1}{|V(G)|} \sum_{v \in V(G)} \text{ (triangle with 2 red edges, vertex } v \text{ highlighted) }$$

$$\text{ (triangle with 2 red edges) } \binom{n}{3} = \sum_{v \in V(G)} \text{ (triangle with 2 red edges, vertex } v \text{ highlighted) } \binom{n-1}{2}$$

FLAG ALGEBRAS IDENTITIES

Let G be a 2-edge-colored complete graph on n vertices. Then

$$\frac{1}{3} \text{ (triangle with 2 red edges, 1 blue edge) } = \frac{1}{|V(G)|} \sum_{v \in V(G)} \text{ (triangle with 2 red edges, 1 blue edge, vertex } v \text{ highlighted) }$$

$$\text{ (triangle with 3 red edges) } = \frac{1}{|V(G)|} \sum_{v \in V(G)} \text{ (triangle with 3 red edges, vertex } v \text{ highlighted) }$$

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FLAG ALGEBRAS IDENTITIES

Let G be a 2-edge-colored complete graph on n vertices. Then

$$\frac{1}{3} \begin{array}{c} \bullet \text{---} \text{blue} \text{---} \bullet \\ \diagup \text{red} \diagdown \\ \bullet \end{array} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{array}{c} \bullet \text{---} \text{blue} \text{---} \bullet \\ \diagup \text{red} \diagdown \\ \square v \end{array}$$

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$$\begin{array}{c} \bullet \text{---} \text{red} \text{---} \bullet \\ \diagup \text{red} \diagdown \\ \bullet \end{array} \binom{n}{3} = \frac{1}{3} \sum_{v \in V(G)} \begin{array}{c} \bullet \text{---} \text{red} \text{---} \bullet \\ \diagup \text{red} \diagdown \\ \square v \end{array} \binom{n-1}{2}$$

IDENTITIES SUMMARY

Let G be a 2-edge-colored complete graph on n vertices. Then

$$1 = \begin{array}{c} \text{red triangle} \end{array} + \begin{array}{c} \text{red-blue triangle} \end{array} + \begin{array}{c} \text{blue triangle} \end{array} + \begin{array}{c} \text{blue triangle} \end{array}$$

$$\begin{array}{c} \text{red edge} \end{array} = \frac{3}{3} \begin{array}{c} \text{red triangle} \end{array} + \frac{2}{3} \begin{array}{c} \text{red-blue triangle} \end{array} + \frac{1}{3} \begin{array}{c} \text{blue triangle} \end{array} + \frac{0}{3} \begin{array}{c} \text{blue triangle} \end{array}$$

$$\begin{array}{c} \text{red edge} \\ \square v \end{array} \times \begin{array}{c} \text{red edge} \\ \square v \end{array} = \begin{array}{c} \text{red triangle} \\ \square v \end{array} + \begin{array}{c} \text{red-blue triangle} \\ \square v \end{array}$$

$$\begin{array}{c} \text{red edge} \\ \square v \end{array} \times \begin{array}{c} \text{blue edge} \\ \square v \end{array} = \frac{1}{2} \begin{array}{c} \text{red triangle} \\ \square v \end{array} + \frac{1}{2} \begin{array}{c} \text{blue triangle} \\ \square v \end{array}$$

$$\frac{1}{3} \begin{array}{c} \text{red triangle} \end{array} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{array}{c} \text{red triangle} \\ \square v \end{array} ; \begin{array}{c} \text{red-blue triangle} \end{array} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{array}{c} \text{red triangle} \\ \square v \end{array}$$

First try for Mantel's theorem

- How to use the equations to prove something
- Gives bounds as well as helps with extremal examples

EXAMPLE - MANTEL'S THEOREM, 1ST TRY

THEOREM (MANTEL 1907)

A triangle-free graph contains at most $\frac{1}{4}n^2$ edges.

Assume edges are red and non-edges are blue



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$$0 \leq \left(1 - 2 \begin{array}{c} \bullet \\ \text{red edge} \\ \text{yellow square} \end{array} v \right)^2$$





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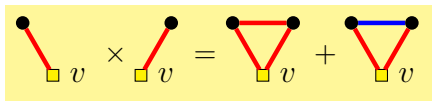
THEOREM (MANTEL 1907)





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

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




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

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

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

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




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


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



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

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

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

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

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
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

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
$$= 1 - 4 \text{  } + \frac{4}{3} \text{  }$$

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
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EXAMPLE - STABILITY FOR MANTEL




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$$0 \leq 1 - 2 \cdot \text{red edge} - \frac{2}{3} \cdot \text{blue triangle}$$




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
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Only  and  appear.

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
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
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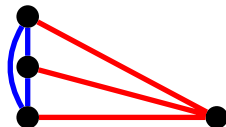
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
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


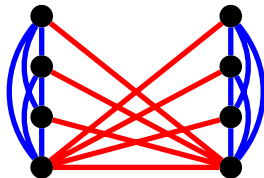
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
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



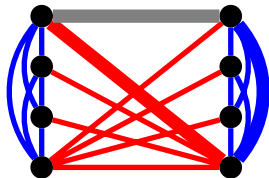
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
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


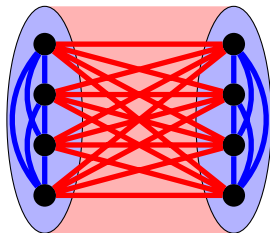
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- we optimize on $\text{LIM}^T = \left\{ q \in \text{LIM} : q \left(\begin{array}{c} \bullet \text{---} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right) = 0 \right\}$

$$\frac{1}{2} \geq \max_{q \in \text{LIM}^T} q \left(\begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \right)$$

More automatic approach

- How to use computer to guess the right equation for you.

$$0 \leq \left(1 - 2 \overset{\bullet}{\underset{\square}{\text{v}}} \right)^2$$

EXAMPLE - MANTEL'S THEOREM, 2ND TRY

THEOREM (MANTEL 1907)

A triangle-free graph contains at most $\frac{1}{4}n^2$ edges.



Assume **edges are red** and **non-edges are blue**.

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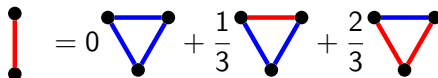
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$$\text{edge} = 0 \cdot \text{triangle} + \frac{1}{3} \cdot \text{triangle} + \frac{2}{3} \cdot \text{triangle}$$


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$$\begin{aligned}
 \text{Two red edges} &= 0 \cdot \text{Blue triangle} + \frac{1}{3} \cdot \text{Triangle with 1 red edge} + \frac{2}{3} \cdot \text{Triangle with 2 red edges} \\
 &\leq \frac{2}{3} \left(\text{Blue triangle} + \text{Triangle with 1 red edge} + \text{Triangle with 2 red edges} \right)
 \end{aligned}$$

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$$\begin{aligned}
 \text{Red path of length 2} &= 0 \cdot \text{Blue triangle} + \frac{1}{3} \cdot \text{Triangle with 2 red edges} + \frac{2}{3} \cdot \text{Triangle with 3 red edges} \\
 &\leq \frac{2}{3} \left(\text{Blue triangle} + \text{Triangle with 2 red edges} + \text{Triangle with 3 red edges} \right) \\
 1 &= \text{Blue triangle} + \text{Triangle with 2 red edges} + \text{Triangle with 3 red edges}
 \end{aligned}$$

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 &\leq \frac{2}{3} \left(\text{Blue triangle} + \text{Triangle with 2 red edges} + \text{Triangle with 3 red edges} \right) \\
 1 &= \text{Blue triangle} + \text{Triangle with 2 red edges} + \text{Triangle with 3 red edges}
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Assume edges are red and non-edges are blue.

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$$\begin{aligned}
 \text{Red edge} &= 0 \cdot \text{Blue triangle} + \frac{1}{3} \cdot \text{Triangle with 2 red edges} + \frac{2}{3} \cdot \text{Triangle with 3 red edges} \\
 &\leq \frac{2}{3} \left(\text{Blue triangle} + \text{Triangle with 2 red edges} + \text{Triangle with 3 red edges} \right) \\
 &= 1 \cdot \text{Blue triangle} + \text{Triangle with 2 red edges} + \text{Triangle with 3 red edges} \\
 &\leq \frac{2}{3}
 \end{aligned}$$

EXAMPLE - MANTEL'S THEOREM, 2ND TRY

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)

$$\text{Red edge} = 0 \cdot \text{Blue triangle} + \frac{1}{3} \cdot \text{Triangle with 2 blue edges, 1 red edge} + \frac{2}{3} \cdot \text{Triangle with 1 blue edge, 2 red edges}$$

EXAMPLE - MANTEL'S THEOREM, 2ND TRY



Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)

$$\text{red edge} = 0 \cdot \text{blue triangle} + \frac{1}{3} \cdot \text{triangle with 2 red edges} + \frac{2}{3} \cdot \text{triangle with 3 red edges}$$

Idea: find $c_1, c_2, c_3 \in \mathbb{R}$ such that for every graph G

$$0 \leq c_1 \cdot \text{blue triangle} + c_2 \cdot \text{triangle with 2 red edges} + c_3 \cdot \text{triangle with 3 red edges}.$$

EXAMPLE - MANTEL'S THEOREM, 2ND TRY

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)

$$\text{red edge} = 0 \cdot \text{blue triangle} + \frac{1}{3} \cdot \text{triangle with 2 red edges} + \frac{2}{3} \cdot \text{triangle with 3 red edges}$$

Idea: find $c_1, c_2, c_3 \in \mathbb{R}$ such that for every graph G

$$0 \leq c_1 \cdot \text{blue triangle} + c_2 \cdot \text{triangle with 2 red edges} + c_3 \cdot \text{triangle with 3 red edges}.$$



After summing together

$$\text{red edge} \leq c_1 \cdot \text{blue triangle} + \left(\frac{1}{3} + c_2\right) \cdot \text{triangle with 2 red edges} + \left(\frac{2}{3} + c_3\right) \cdot \text{triangle with 3 red edges}$$

and

$$\text{red edge} \leq \max \left\{ (0 + c_1), \frac{1}{3} + c_2, \frac{2}{3} + c_3 \right\}.$$

EXAMPLE - MANTEL'S THEOREM, 2ND TRY

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)

$$\text{red edge} = 0 \cdot \text{blue triangle} + \frac{1}{3} \cdot \text{triangle with 2 red edges} + \frac{2}{3} \cdot \text{triangle with 3 red edges}$$

Idea: find $c_1, c_2, c_3 \in \mathbb{R}$ such that for every graph G

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After summing together

$$\text{red edge} \leq c_1 \cdot \text{blue triangle} + \left(\frac{1}{3} + c_2\right) \cdot \text{triangle with 2 red edges} + \left(\frac{2}{3} + c_3\right) \cdot \text{triangle with 3 red edges}$$

and

$$\text{red edge} \leq \max \left\{ (0 + c_1), \frac{1}{3} + c_2, \frac{2}{3} + c_3 \right\}$$

★ $c_3 < 0$ ★

CANDIDATES FOR c_1, c_2, c_3

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

CANDIDATES FOR c_1, c_2, c_3

$$0 \leq \left(\begin{array}{c} \bullet \\ \text{blue line} \\ \text{yellow square} \\ v \end{array}, \begin{array}{c} \bullet \\ \text{red line} \\ \text{yellow square} \\ v \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ \text{blue line} \\ \text{yellow square} \\ v \end{array}, \begin{array}{c} \bullet \\ \text{red line} \\ \text{yellow square} \\ v \end{array} \right)^T$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

CANDIDATES FOR c_1, c_2, c_3

$$\begin{aligned}
 0 &\leq \left(\begin{array}{c} \bullet \\ \text{blue} \\ \square v \end{array}, \begin{array}{c} \bullet \\ \text{red} \\ \square v \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ \text{blue} \\ \square v \end{array}, \begin{array}{c} \bullet \\ \text{red} \\ \square v \end{array} \right)^T \\
 &= a \begin{array}{c} \bullet \quad ? \quad \bullet \\ \text{blue} \quad \text{blue} \\ \square v \end{array} + b \begin{array}{c} \bullet \quad ? \quad \bullet \\ \text{red} \quad \text{red} \\ \square v \end{array} + \frac{1}{2}c \begin{array}{c} \bullet \quad ? \quad \bullet \\ \text{red} \quad \text{blue} \\ \square v \end{array} + \frac{1}{2}c \begin{array}{c} \bullet \quad ? \quad \bullet \\ \text{blue} \quad \text{red} \\ \square v \end{array}
 \end{aligned}$$

$$\begin{array}{c} \bullet \\ \text{red} \\ \square v \end{array} \times \begin{array}{c} \bullet \\ \text{red} \\ \square v \end{array} = \begin{array}{c} \bullet \quad ? \quad \bullet \\ \text{red} \quad \text{red} \\ \square v \end{array}$$

$$\begin{array}{c} \bullet \\ \text{red} \\ \square v \end{array} \times \begin{array}{c} \bullet \\ \text{blue} \\ \square v \end{array} = \frac{1}{2} \begin{array}{c} \bullet \quad ? \quad \bullet \\ \text{red} \quad \text{blue} \\ \square v \end{array}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

CANDIDATES FOR c_1, c_2, c_3

$$\begin{aligned}
 0 &\leq \left(\begin{array}{c} \bullet \\ \text{blue} \\ \square v \end{array}, \begin{array}{c} \bullet \\ \text{red} \\ \square v \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ \text{blue} \\ \square v \end{array}, \begin{array}{c} \bullet \\ \text{red} \\ \square v \end{array} \right)^T \\
 &= a \begin{array}{c} \bullet \text{---} ? \text{---} \bullet \\ \text{blue} \quad \text{blue} \\ \square v \end{array} + b \begin{array}{c} \bullet \text{---} ? \text{---} \bullet \\ \text{red} \quad \text{red} \\ \square v \end{array} + c \begin{array}{c} \bullet \text{---} ? \text{---} \bullet \\ \text{red} \quad \text{blue} \\ \square v \end{array}
 \end{aligned}$$

$$\begin{array}{c} \bullet \\ \text{red} \\ \square v \end{array} \times \begin{array}{c} \bullet \\ \text{red} \\ \square v \end{array} = \begin{array}{c} \bullet \text{---} ? \text{---} \bullet \\ \text{red} \quad \text{red} \\ \square v \end{array}$$

$$\begin{array}{c} \bullet \\ \text{red} \\ \square v \end{array} \times \begin{array}{c} \bullet \\ \text{blue} \\ \square v \end{array} = \frac{1}{2} \begin{array}{c} \bullet \text{---} ? \text{---} \bullet \\ \text{red} \quad \text{blue} \\ \square v \end{array}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

CANDIDATES FOR c_1, c_2, c_3

$$\begin{aligned}
 0 &\leq \left(\begin{pmatrix} \bullet & \bullet \\ \text{yellow square} & v \end{pmatrix}, \begin{pmatrix} \bullet & \bullet \\ \text{yellow square} & v \end{pmatrix} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{pmatrix} \bullet & \bullet \\ \text{yellow square} & v \end{pmatrix}, \begin{pmatrix} \bullet & \bullet \\ \text{yellow square} & v \end{pmatrix} \right)^T \\
 &= a \begin{pmatrix} \bullet & ? & \bullet \\ & \text{yellow square} & v \end{pmatrix} + b \begin{pmatrix} \bullet & ? & \bullet \\ & \text{yellow square} & v \end{pmatrix} + c \begin{pmatrix} \bullet & ? & \bullet \\ & \text{yellow square} & v \end{pmatrix}
 \end{aligned}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

CANDIDATES FOR c_1, c_2, c_3

$$\begin{aligned}
 0 &\leq \frac{1}{n} \sum_v \left(\begin{array}{c} \bullet \\ \text{blue} \\ \text{yellow } v \end{array}, \begin{array}{c} \bullet \\ \text{red} \\ \text{yellow } v \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ \text{blue} \\ \text{yellow } v \end{array}, \begin{array}{c} \bullet \\ \text{red} \\ \text{yellow } v \end{array} \right)^T \\
 &= \frac{1}{n} \sum_v a \begin{array}{c} \bullet \text{---} ? \text{---} \bullet \\ \text{blue} \quad \text{blue} \\ \text{yellow } v \end{array} + b \begin{array}{c} \bullet \text{---} ? \text{---} \bullet \\ \text{red} \quad \text{red} \\ \text{yellow } v \end{array} + c \begin{array}{c} \bullet \text{---} ? \text{---} \bullet \\ \text{red} \quad \text{blue} \\ \text{yellow } v \end{array}
 \end{aligned}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

CANDIDATES FOR c_1, c_2, c_3

$$\begin{aligned}
 0 &\leq \frac{1}{n} \sum_v \left(\begin{array}{c} \bullet \\ \text{blue} \\ \text{yellow } v \end{array}, \begin{array}{c} \bullet \\ \text{red} \\ \text{yellow } v \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ \text{blue} \\ \text{yellow } v \end{array}, \begin{array}{c} \bullet \\ \text{red} \\ \text{yellow } v \end{array} \right)^T \\
 &= \frac{1}{n} \sum_v a \begin{array}{c} \bullet \text{---} ? \text{---} \bullet \\ \text{blue} \diagdown \quad \diagup \\ \text{yellow } v \end{array} + b \begin{array}{c} \bullet \text{---} ? \text{---} \bullet \\ \text{red} \diagdown \quad \diagup \\ \text{yellow } v \end{array} + c \begin{array}{c} \bullet \text{---} ? \text{---} \bullet \\ \text{red} \diagdown \quad \text{blue} \diagup \\ \text{yellow } v \end{array} \\
 &= a \begin{array}{c} \bullet \text{---} \bullet \\ \text{blue} \diagdown \quad \diagup \\ \bullet \end{array} + \frac{a+2c}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \text{blue} \diagdown \quad \text{red} \diagup \\ \bullet \end{array} + \frac{b+2c}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \text{red} \diagdown \quad \text{blue} \diagup \\ \bullet \end{array} + b \begin{array}{c} \bullet \text{---} \bullet \\ \text{red} \diagdown \quad \diagup \\ \bullet \end{array}
 \end{aligned}$$

$$\frac{1}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \text{blue} \diagdown \quad \diagup \\ \bullet \end{array} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{array}{c} \bullet \text{---} \bullet \\ \text{blue} \diagdown \quad \text{red} \diagup \\ \text{yellow } v \end{array}$$

$$\begin{array}{c} \bullet \text{---} \bullet \\ \text{red} \diagdown \quad \diagup \\ \bullet \end{array} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{array}{c} \bullet \text{---} \bullet \\ \text{red} \diagdown \quad \text{red} \diagup \\ \text{yellow } v \end{array}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \left(\frac{2}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \text{blue} \diagdown \quad \text{red} \diagup \\ \bullet \end{array} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{array}{c} \bullet \text{---} \bullet \\ \text{blue} \diagdown \quad \text{red} \diagup \\ \text{yellow } v \end{array} \right)$$

CANDIDATES FOR c_1, c_2, c_3

$$\begin{aligned}
 0 &\leq \frac{1}{n} \sum_v \left(\begin{array}{c} \bullet \\ \text{blue} \\ \text{yellow } v \end{array}, \begin{array}{c} \bullet \\ \text{red} \\ \text{yellow } v \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ \text{blue} \\ \text{yellow } v \end{array}, \begin{array}{c} \bullet \\ \text{red} \\ \text{yellow } v \end{array} \right)^T \\
 &= \frac{1}{n} \sum_v a \begin{array}{c} \bullet \text{ ? } \bullet \\ \text{blue} \\ \text{yellow } v \end{array} + b \begin{array}{c} \bullet \text{ ? } \bullet \\ \text{red} \\ \text{yellow } v \end{array} + c \begin{array}{c} \bullet \text{ ? } \bullet \\ \text{red} \text{ blue} \\ \text{yellow } v \end{array} \\
 &= a \begin{array}{c} \bullet \text{---} \bullet \\ \text{blue} \\ \bullet \end{array} + \frac{a+2c}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \text{red} \text{ blue} \\ \bullet \end{array} + \frac{b+2c}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \text{red} \\ \bullet \end{array}
 \end{aligned}$$

$$\frac{1}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \text{red} \text{ blue} \\ \bullet \end{array} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{array}{c} \bullet \text{---} \bullet \\ \text{red} \\ \text{yellow } v \end{array}$$

$$\begin{array}{c} \bullet \text{---} \bullet \\ \text{red} \\ \bullet \end{array} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{array}{c} \bullet \text{---} \bullet \\ \text{red} \\ \text{yellow } v \end{array}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \left(\frac{2}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \text{red} \text{ blue} \\ \bullet \end{array} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{array}{c} \bullet \text{---} \bullet \\ \text{red} \text{ blue} \\ \text{yellow } v \end{array} \right)$$

CANDIDATES FOR c_1, c_2, c_3

$$\begin{aligned}
 0 &\leq \frac{1}{n} \sum_v \left(\begin{array}{c} \bullet \\ \text{blue} \\ \square_v \end{array}, \begin{array}{c} \bullet \\ \text{red} \\ \square_v \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ \text{blue} \\ \square_v \end{array}, \begin{array}{c} \bullet \\ \text{red} \\ \square_v \end{array} \right)^T \\
 &= \frac{1}{n} \sum_v a \begin{array}{c} \bullet \text{---} ? \text{---} \bullet \\ \text{blue} \diagdown \quad \diagup \\ \square_v \end{array} + b \begin{array}{c} \bullet \text{---} ? \text{---} \bullet \\ \text{red} \diagdown \quad \diagup \\ \square_v \end{array} + c \begin{array}{c} \bullet \text{---} ? \text{---} \bullet \\ \text{red} \diagdown \quad \text{blue} \diagup \\ \square_v \end{array} \\
 &= a \begin{array}{c} \bullet \text{---} \bullet \\ \text{blue} \diagdown \quad \diagup \\ \bullet \end{array} + \frac{a+2c}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \text{blue} \diagdown \quad \text{red} \diagup \\ \bullet \end{array} + \frac{b+2c}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \text{red} \diagdown \quad \text{blue} \diagup \\ \bullet \end{array} \\
 c_1 &= a, \quad c_2 = \frac{a+2c}{3}, \quad c_3 = \frac{b+2c}{3}
 \end{aligned}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

USING c_1, c_2, c_3

$$\begin{array}{c}
 \text{Diagram of a red edge} \\
 \text{---} \\
 \text{Diagram of a blue triangle}
 \end{array}
 =
 \begin{array}{c}
 \text{Diagram of a blue triangle} \\
 + \frac{1}{3} \text{Diagram of a triangle with red top edge and blue bottom edges} \\
 + \frac{2}{3} \text{Diagram of a triangle with red bottom edges and blue top edge}
 \end{array}$$

$$0 \leq a \begin{array}{c} \text{Diagram of a blue triangle} \end{array} + \frac{a+2c}{3} \begin{array}{c} \text{Diagram of a triangle with red top edge and blue bottom edges} \end{array} + \frac{b+2c}{3} \begin{array}{c} \text{Diagram of a triangle with red bottom edges and blue top edge} \end{array}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

USING c_1, c_2, c_3

$$\begin{array}{c}
 \text{Diagram of a single edge} = \text{Diagram of a blue triangle} + \frac{1}{3} \text{Diagram of a triangle with top edge red} + \frac{2}{3} \text{Diagram of a triangle with bottom edge red} \\
 0 \leq a \text{Diagram of a blue triangle} + \frac{a+2c}{3} \text{Diagram of a triangle with top edge red} + \frac{b+2c}{3} \text{Diagram of a triangle with bottom edge red}
 \end{array}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

USING c_1, c_2, c_3

$$\begin{array}{c}
 \text{Diagram of } K_2 \\
 = \\
 \text{Diagram of } K_3 \text{ (all blue)} + \frac{1}{3} \text{Diagram of } K_3 \text{ (top blue, bottom red)} + \frac{2}{3} \text{Diagram of } K_3 \text{ (top blue, bottom red)} \\
 0 \leq a \text{Diagram of } K_3 \text{ (all blue)} + \frac{a+2c}{3} \text{Diagram of } K_3 \text{ (top blue, bottom red)} + \frac{b+2c}{3} \text{Diagram of } K_3 \text{ (top blue, bottom red)}
 \end{array}$$

$$\text{Diagram of } K_2 \leq \max \left\{ a, \frac{1+a+2c}{3}, \frac{2+b+2c}{3} \right\}.$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

USING c_1, c_2, c_3

$$\begin{aligned}
 \text{edge} &= \text{triangle}_{\text{blue}} + \frac{1}{3} \text{triangle}_{\text{blue/red}} + \frac{2}{3} \text{triangle}_{\text{red/blue}} \\
 0 &\leq a \text{triangle}_{\text{blue}} + \frac{a+2c}{3} \text{triangle}_{\text{blue/red}} + \frac{b+2c}{3} \text{triangle}_{\text{red/blue}}
 \end{aligned}$$

$$\text{edge} \leq \max \left\{ a, \frac{1+a+2c}{3}, \frac{2+b+2c}{3} \right\}.$$

Try

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

USING c_1, c_2, c_3

$$\begin{aligned}
 \text{Diagram 1} &= \text{Diagram 2} + \frac{1}{3} \text{Diagram 3} + \frac{2}{3} \text{Diagram 4} \\
 0 &\leq a \text{Diagram 5} + \frac{a+2c}{3} \text{Diagram 6} + \frac{b+2c}{3} \text{Diagram 7}
 \end{aligned}$$

The diagrams are triangles with vertices represented by black dots. Diagram 1 is a vertical red edge. Diagram 2 has a blue top edge and blue bottom edges. Diagram 3 has a red top edge and blue bottom edges. Diagram 4 has a blue top edge and red bottom edges. Diagram 5 has a blue top edge and blue bottom edges. Diagram 6 has a red top edge and blue bottom edges. Diagram 7 has a blue top edge and red bottom edges.

$$\text{Diagram 1} \leq \max \left\{ a, \frac{1+a+2c}{3}, \frac{2+b+2c}{3} \right\}.$$

Try

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

It gives

$$\text{Diagram 1} \leq \max \left\{ \frac{1}{2}, \frac{1}{6}, \frac{1}{2} \right\} = \frac{1}{2}.$$

OPTIMIZING a, b, c

$$\text{red line segment} \leq \max \left\{ a, \frac{1+a+2c}{3}, \frac{2+b+2c}{3} \right\}$$

$$(SDP) \left\{ \begin{array}{ll} \text{Minimize} & d \\ \text{subject to} & a \leq d \\ & \frac{1+a+2c}{3} \leq d \\ & \frac{2+b+2c}{3} \leq d \\ & \begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \end{array} \right.$$

(SDP) can be solved on computers using CSDP or SDPA.
Rounding may be needed for exact results.



J. Balogh

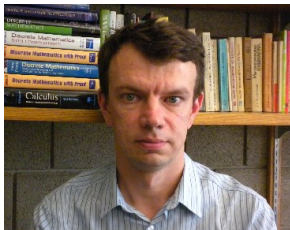


P. Hu



L.

Permutations



O. Pikhurko



B. Udvari



J. Volec

PERMUTATIONS AND EXTREMAL PROBLEMS

PROBLEM

What is the minimum number of monotone subsequences of size k in a permutation of $[n]$?

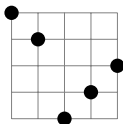
PERMUTATIONS AND EXTREMAL PROBLEMS

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What is the minimum number of monotone subsequences of size k in a permutation of $[n]$?

$$k = 3$$

$$n = 5$$



$$(5,4,1), (5,4,2), (5,4,3)$$

$$(1,2,3)$$

$$(5,4,1,2,3)$$

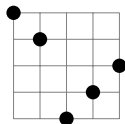
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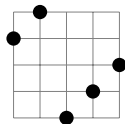
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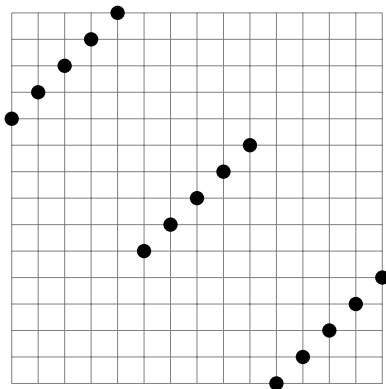
$(1,2,3)$

$(4,5,1,2,3)$

CONJECTURE

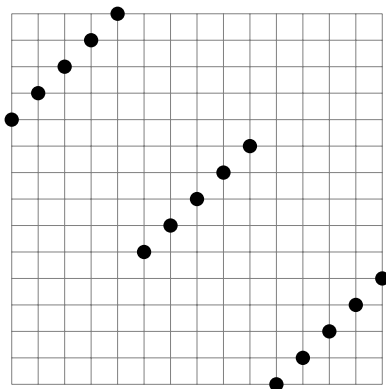
CONJECTURE (MYERS 2002)

The number of monotone subsequences of length k is minimized by a permutation on $[n]$ with $k - 1$ increasing runs of as equal lengths as possible.

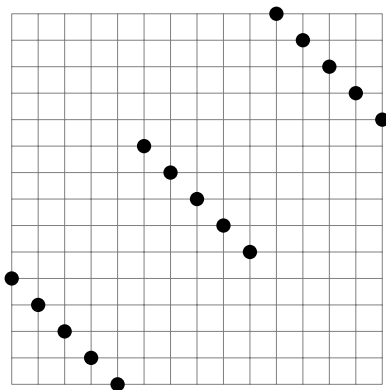


$$k = 4, n = 15$$

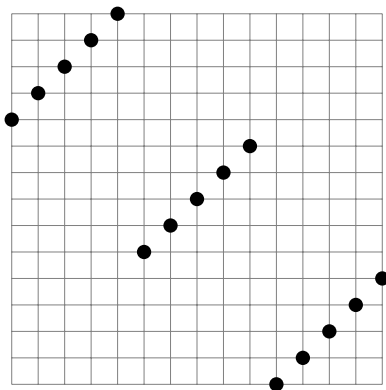
EXTREMAL CASE IS NOT UNIQUE



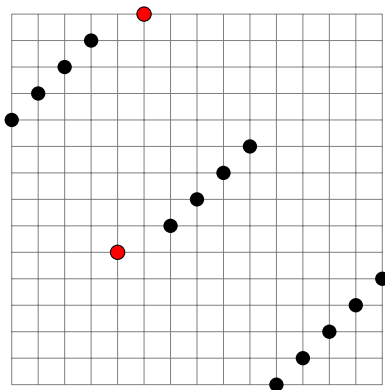
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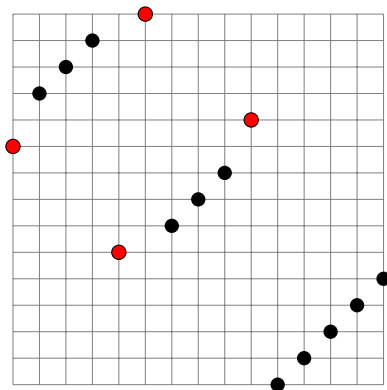
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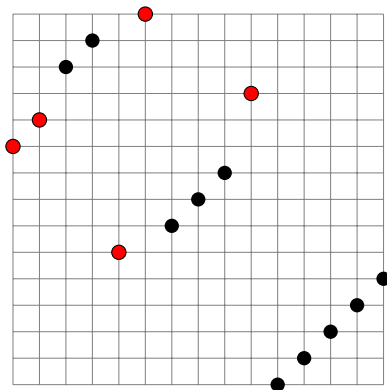
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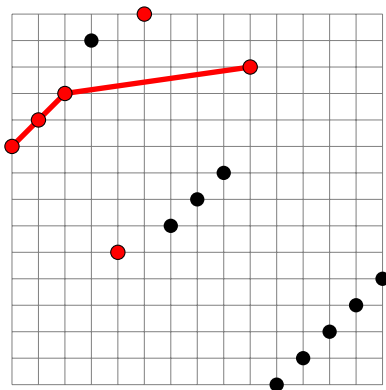
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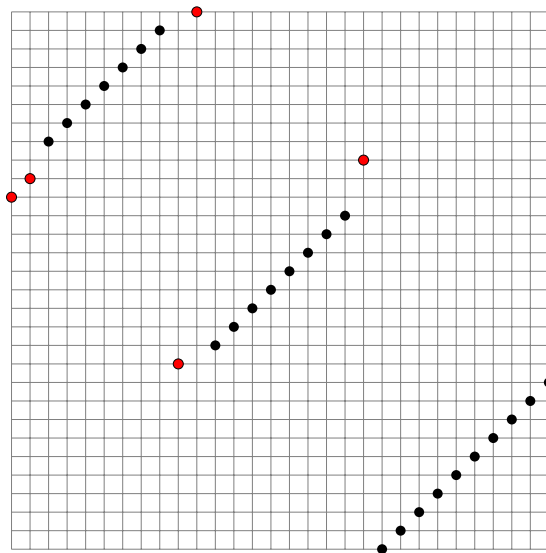
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Myers' conjecture is true for sufficiently large k and $n \leq k^2 + ck^{3/2} \log k$, where c is an absolute positive constant.

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Myers' conjecture is true for $k = 4$ and n sufficiently large.



(1,2,3,4)



(4,3,2,1)

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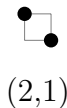
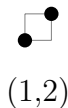
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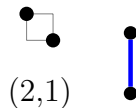
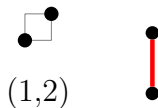
(4,3,2,1)

Use of flag algebras, $k = 5, 6$ also doable, 7 not.

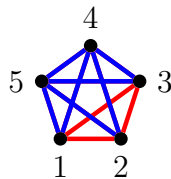
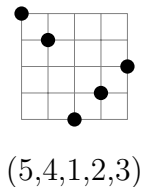
FROM PERMUTATIONS TO PERMUTATION GRAPHS



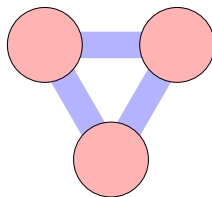
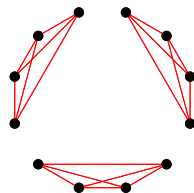
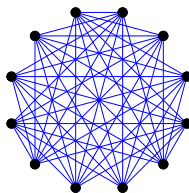
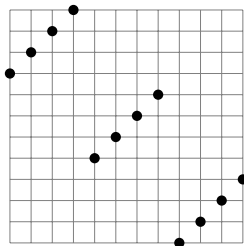
FROM PERMUTATIONS TO PERMUTATION GRAPHS



$k = 3$
 $n = 5$



EXTREMAL EXAMPLE ($k = 4$)



AS FLAG ALGEBRA QUESTION ($k = 4$)



(1,2,3,4)

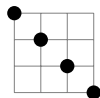


(4,3,2,1)

AS FLAG ALGEBRA QUESTION ($k = 4$)



$(1,2,3,4)$



$(4,3,2,1)$



AS FLAG ALGEBRA QUESTION ($k = 4$)



$(1,2,3,4)$

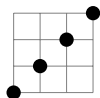


$(4,3,2,1)$



minimize  + 

AS FLAG ALGEBRA QUESTION ($k = 4$)



(1,2,3,4)



(4,3,2,1)



minimize  + 

THEOREM (BALOGH, HU, L., PIKHURKO, UDVARI, VOLEC '14+)

$$\text{Red } K_4 + \text{Blue } K_4 \geq \frac{1}{27}$$

for every permutation graph.

ONLY FOR PERMUTATION GRAPHS

THEOREM (BALOGH, HU, L., PIKHURKO, UDVARI, VOLEC '14+)

$$\min \left(\text{red square with diagonals} + \text{blue square with diagonals} \right) = \frac{1}{27}$$

over permutation graphs (and extremal permutations described using Myers' results).

ONLY FOR PERMUTATION GRAPHS

THEOREM (BALOGH, HU, L., PIKHURKO, UDVARI, VOLEC '14+)

$$\min \left(\begin{array}{c} \bullet \quad \bullet \\ \text{red edges} \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \text{blue edges} \\ \bullet \quad \bullet \end{array} \right) = \frac{1}{27}$$



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THEOREM (SPERFELD '12; THOMASON '89)

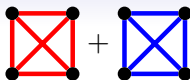
$$\frac{1}{35} < \min \left(\begin{array}{c} \bullet \quad \bullet \\ \text{red edges} \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \text{blue edges} \\ \bullet \quad \bullet \end{array} \right) < \frac{1}{33}$$

*over **all** sufficiently large 2-edge-colored complete graphs.*

We want to prove

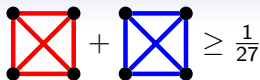
 $+$  $\geq \frac{1}{27}$

We want to prove


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$$\begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array} + \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array} \geq \frac{1}{27} = 0.\overline{037}$$

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We want to prove

$$\begin{array}{c} \bullet & & \bullet \\ \text{red square with} & + & \text{blue square with} \\ \text{diagonals} & & \text{diagonals} \\ \bullet & & \bullet \end{array} \geq \frac{1}{27} = 0.\overline{037}$$

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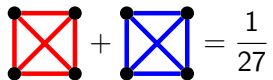
- Round M' to $M \in \mathbb{Q}^{f \times f}$, such that

$$\begin{array}{c} \bullet & & \bullet \\ \text{red square with} & + & \text{blue square with} \\ \text{diagonals} & & \text{diagonals} \\ \bullet & & \bullet \end{array} \geq \frac{1}{27}$$

and $M \succcurlyeq 0$.

STRUCTURE OF EXTREMAL PERMUTATIONS

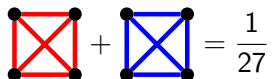
Assuming


$$+ = \frac{1}{27}$$

Flag algebra implies:

STRUCTURE OF EXTREMAL PERMUTATIONS

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Flag algebra implies:

(A)



$$= o(1)$$

STRUCTURE OF EXTREMAL PERMUTATIONS

Assuming





$$\begin{array}{c} \bullet & & \bullet \\ \diagup & & \diagdown \\ \bullet & & \bullet \\ \diagdown & & \diagup \end{array} + \begin{array}{c} \bullet & & \bullet \\ \diagdown & & \diagup \\ \bullet & & \bullet \\ \diagup & & \diagdown \end{array} = \frac{1}{27}$$

Flag algebra implies:

(A)  = o(1)

(B) Almost all  are .

AFTER FLAG ALGEBRA (STABILITY)

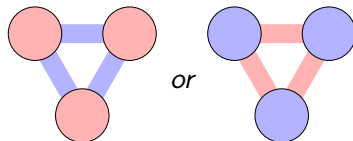
" +  is close to $\frac{1}{27} \Rightarrow G$ is close to  or "

LEMMA (STABILITY)

For every $\varepsilon > 0$ there exist n_0 and $\varepsilon' > 0$ such that every admissible graph G of order $n > n_0$ with





$$\left(\text{square with red edges and diagonal} \right) + \left(\text{square with blue edges and diagonal} \right) \leq \frac{1}{27} + \varepsilon'$$

is isomorphic to either







after recoloring at most $20\varepsilon n^2$ edges.

AFTER FLAG ALGEBRA (STABILITY SKETCH)

“ +  is close to $\frac{1}{27} \Rightarrow G$ is close to  or ”

- Using removal lemma, properties (A) and (B) can be satisfied entirely. (lost εn^2 edges)





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- For all $v \in V(G) \setminus X$, where $|X| \leq 2\varepsilon n$ vertices

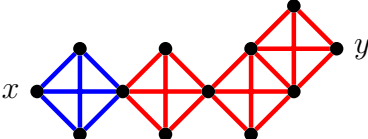
$$\frac{1}{27} - \varepsilon \leq \left(\text{Red square with both diagonals} \right)_v + \left(\text{Blue square with both diagonals} \right)_v \leq \frac{1}{27} + \varepsilon'' \quad (1)$$

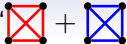


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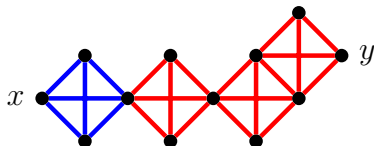
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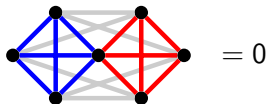
- $x \sim y$ if 

" is close to $\frac{1}{27} \Rightarrow G$ is close to  or 

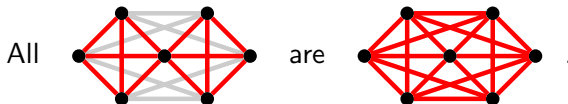
• $x \sim y$ if







(A)

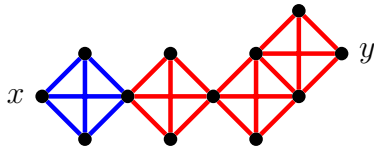


(B)

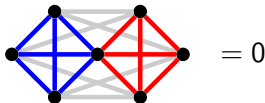


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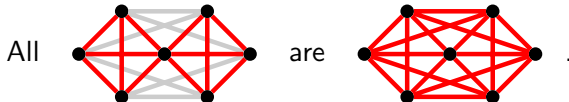
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

(A)

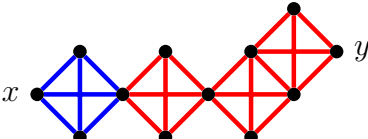


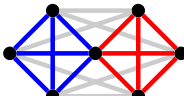
(B)



- Every equivalence class is a monochromatic clique.





" +  is close to $\frac{1}{27} \Rightarrow G$ is close to  or 

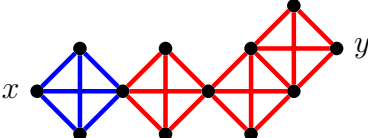
• $x \sim y$ if 

(A)  = 0



(B) All  are .

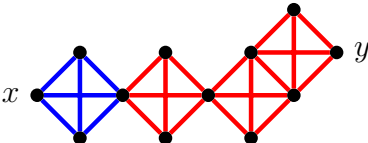
- Every equivalence class is a monochromatic clique.
- There are three equivalence classes of size $\frac{1}{3}n \pm 16\epsilon n$ by (1).

“ +  is close to $\frac{1}{27} \Rightarrow G$ is close to  or ”

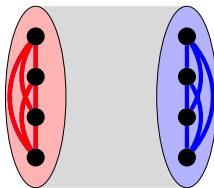
• $x \sim y$ if 



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- The classes have the same color

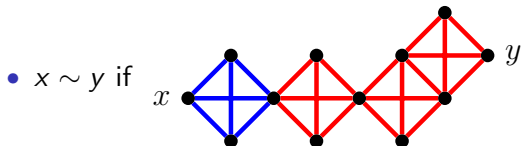
“ $\square + \square$ is close to $\frac{1}{27} \Rightarrow G$ is close to  or ”

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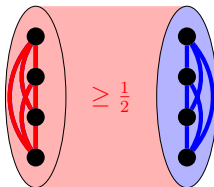
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


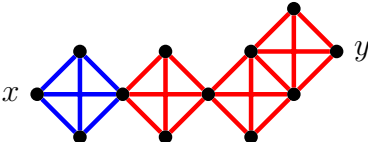
“ $\square_{\text{red}} + \square_{\text{blue}}$ is close to $\frac{1}{27} \Rightarrow G$ is close to  or ”



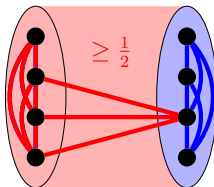
- Every equivalence class is a monochromatic clique.
- There are three equivalence classes of size $\frac{1}{3}n \pm 16\epsilon n$ by (1).
- The classes have the same color







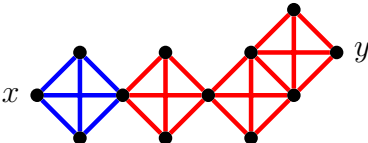
“ $\square + \square$ is close to $\frac{1}{27} \Rightarrow G$ is close to  or ”

- $x \sim y$ if 

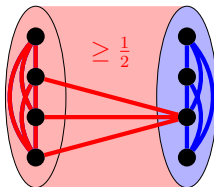
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“ +  is close to $\frac{1}{27} \Rightarrow G$ is close to  or 

• $x \sim y$ if 

- Every equivalence class is a monochromatic clique.
- There are three equivalence classes of size $\frac{1}{3}n \pm 16\epsilon n$ by (1).
- The classes have the same color



Exact result: By recoloring edges.

OTHER PERMUTATIONS - MAXIMIZING 1342 AND 2413

$$0.19657 \leq \sigma(1342) \leq 2/9 = 0.22222 \dots \quad \text{AAHHS}$$

$$\sigma(1342) \leq 0.1988373 \quad \text{BHLPUV}$$

$$51/511 = 0.0998 \dots \leq \sigma(2413) \leq 2/9 = 0.22222 \quad \text{AAHHS}$$

$$0.1024732 \leq \sigma(2413) \quad \text{P}$$

$$0.10472 \dots \leq \sigma(2413) \quad \text{PS}$$

$$\sigma(2413) \leq 0.1047805 \quad \text{BHLPUV}$$

AAHHS ... Albert, Atkinson, Handley, Holton, Stromquist 2002

P... Presutti 2008

PS... Presutti, Stromquist 2010

BHLPUV... us

Thank you for your attention!