Flag Algebras and Applications to Permutations

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Iowa State University

Nov 7, 2014

Outline

- Extremal problems
- Introduction to the use of Flag Algebras
- Example of automated Flag Algebras approach
- Applications of Flag Algebras in permutations

Problem

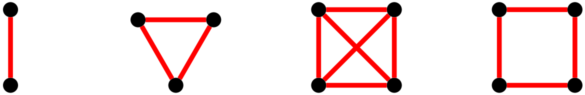
Problem

What is the minimum number of monotone subsequences of size k in a permutation of $[n]$?

Graphs

Definition

A graph G consists of vertices V and edges $E \subseteq \binom{V}{2}$.

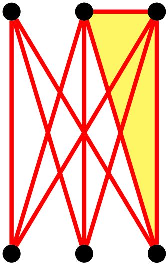


K_2 K_3 K_4 C_4

Extremal problems

Theorem (Mantel 1907)

If a graph G on n vertices has more than $\frac{1}{4}n^2$ edges, then G contains a triangle.



$|E| = 10$

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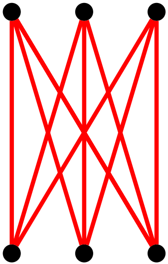


$|E| = 9$

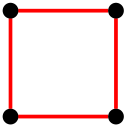
Extremal problems

Theorem (Mantel 1907)

If a graph G on n vertices has more than $\frac{1}{4}n^2$ edges, then G contains a triangle.

Theorem (Dirac 1952)

If all vertices in a graph G on n vertices have degree at least $\frac{n}{2}$, then G is Hamiltonian. ($n \geq 3$)



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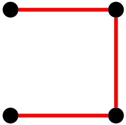

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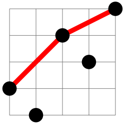
Theorem (Erdős, Szekeres 1935)

Every sequence of $n^2 + 1$ distinct numbers contains a monotone subsequence of size $n + 1$.



(2,1,4,3,5)

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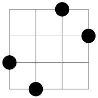


(2,1,4,3)

Extremal problems

Theorem (Mantel 1907)

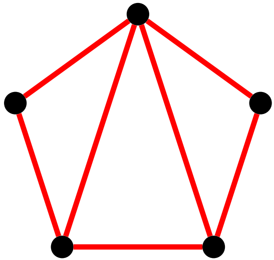
A triangle-free graph contains at most $\frac{1}{4}n^2$ edges.

Problem

Maximize a graph parameter (# of edges) over a class of graphs (triangle-free).

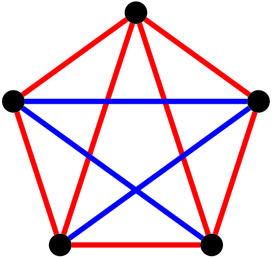
- local condition and global parameter
- threshold
- bound and extremal example

Edge-colored graphs



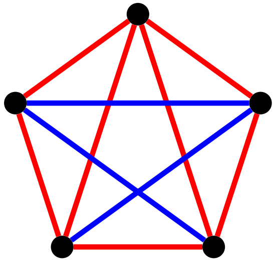
A graph on 5 vertices.

Edge-colored graphs



A 2-edge-colored complete graph K_5 on 5 vertices.

Densities (ratios) in edge-colored graphs



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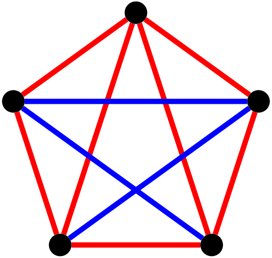
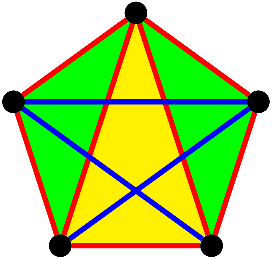


 $= \frac{7}{10}$

Densities (ratios) in edge-colored graphs



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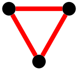 $= \frac{3}{10}$

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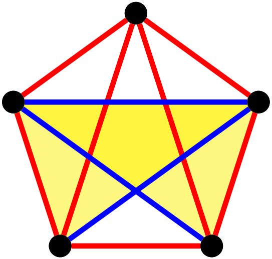

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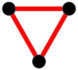
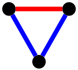
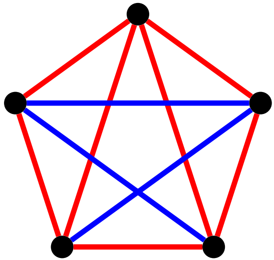
 $= \frac{3}{10}$

 $= \frac{2}{10}$

Densities (ratios) in edge-colored graphs



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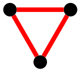

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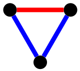

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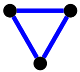
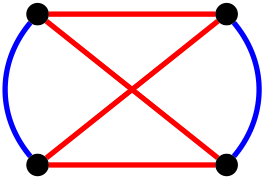


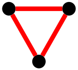
$= 0$

Classical theorems — Mantel's theorem

Theorem (Mantel 1907)

A triangle-free graph contains at most $\frac{1}{4}n^2$ edges.



 $= 0$

 $= \frac{4}{6} = \frac{2}{3} = 0.\overline{6}$

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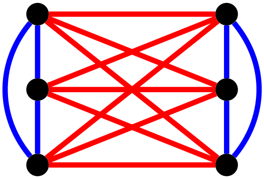

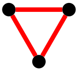 $= 0$

 $= \frac{9}{15} = \frac{3}{5} = 0.6$

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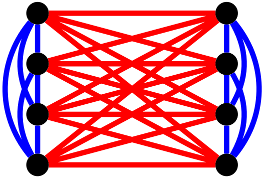

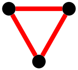 $= 0$

 $= \frac{16}{28} = \frac{4}{7} = 0.571\ldots$

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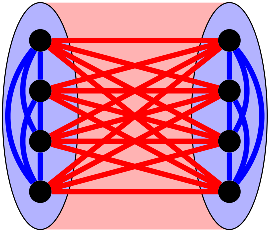

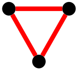 $= 0$

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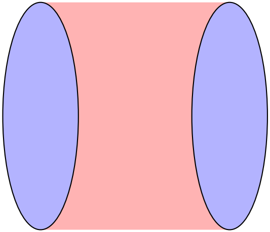

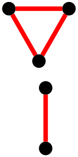

$= 0$

$= \frac{1}{2} + o(1)$

Flag algebras

Seminal paper:

A. Razborov, Flag Algebras, *Journal of Symbolic Logic* **72** (2007), 1239–1282.

David P. Robbins Prize by AMS for Razborov in 2013

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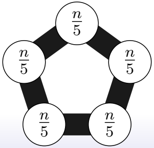
Applications to oriented graphs, hypergraphs, crossing number of graphs, geometry,. . .

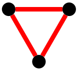
Theorem (Hatami, Hladký, Kráľ, Norine, Razborov 2011; Grzesik 2011)

The number of C_5's in a triangle-free graph on n vertices is at most $(n/5)^5$.



Flag algebras definitions

Let G be a 2-edge-colored complete graph on n vertices.



The probability that three random vertices in G span a red triangle.

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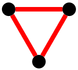

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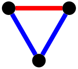

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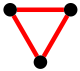

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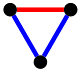

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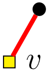

The probability that a random vertex other than v is connected to $v \in V(G)$ by a red edge, i.e., the red degree of v divided by $n-1$.

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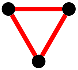

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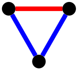

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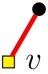

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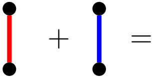

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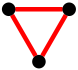

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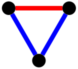

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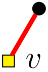

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 $+$  $= 1$

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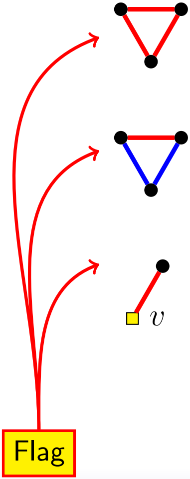

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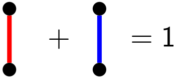
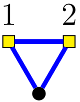
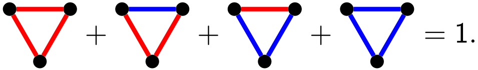




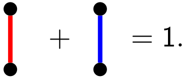
Type is a flag induced by labeled vertices

Flag algebras identities

Let G be a 2-edge-colored complete graph on n vertices. Then



Same kind as



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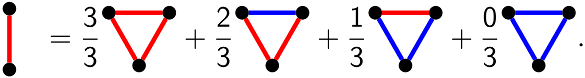
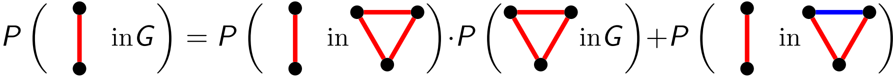


Expanded version where pictures mean graphs:



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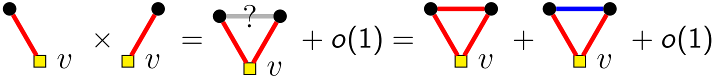

$o(1)$ as $|V(G)| \to \infty$ (will be omitted on next slides)

Flag algebras identities

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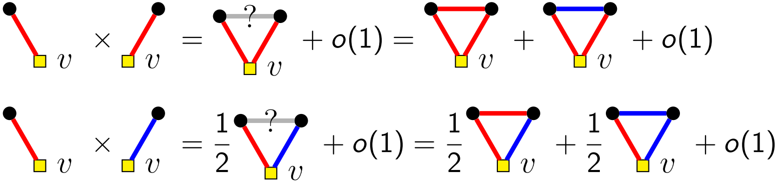

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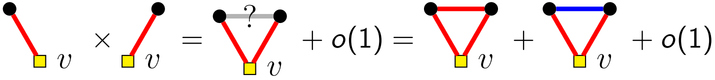
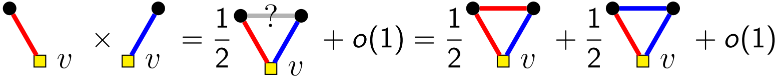
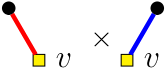
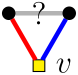






: The probability that choosing two vertices u_1, u_2 other than v gives red vu_1 and blue vu_2.

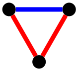
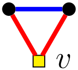


: The probability that choosing two different vertices u_1, u_2 other than v gives one of vu_1 and vu_2 is red and the other is blue.

$o(1)$ as $|V(G)| \to \infty$ (will be omitted on next slides)

Flag algebras identities

Let G be a 2-edge-colored complete graph on n vertices. Then

$\frac{1}{3}$  $= \frac{1}{|V(G)|} \sum_{v \in V(G)}$  v

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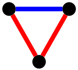
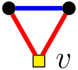
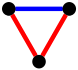
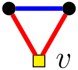
$\frac{1}{3}$  $= \frac{1}{|V(G)|} \sum_{v \in V(G)}$  v

 $\binom{n}{3} = \sum_{v \in V(G)}$  $v \binom{n-1}{2}$

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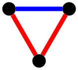
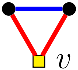
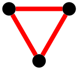
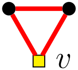
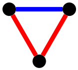
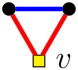
$\frac{1}{3}$  $= \frac{1}{|V(G)|} \sum_{v \in V(G)}$  v

 $= \frac{1}{|V(G)|} \sum_{v \in V(G)}$  v

 $\binom{n}{3} = \sum_{v \in V(G)}$  $v \binom{n-1}{2}$

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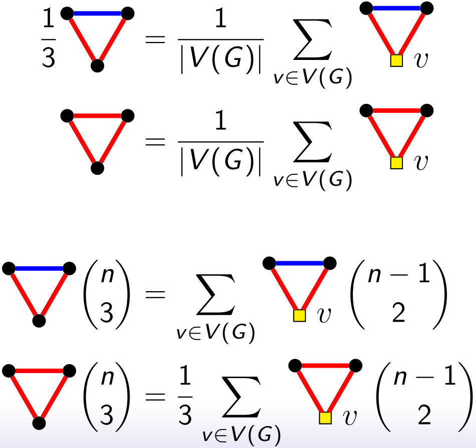
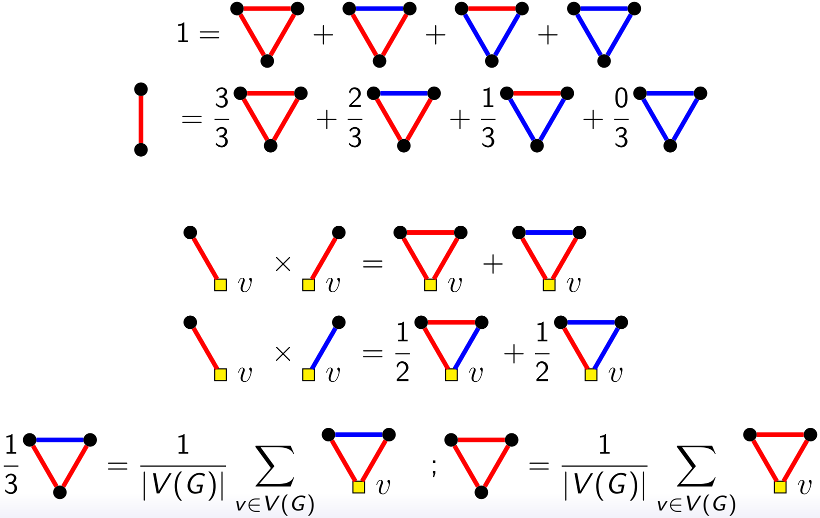


Identities Summary

Let G be a 2-edge-colored complete graph on n vertices. Then



First try for Mantel's theorem

- How to use the equations to prove something
- Gives bounds as well as helps with extremal examples

Example - Mantel's theorem, 1st try

Theorem (Mantel 1907)

A triangle-free graph contains at most $\frac{1}{4}n^2$ edges.

Assume edges are red and non-edges are blue

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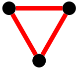
. Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

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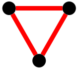
. Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

$0 \leq$ $\Bigg(1 - 2$  $\Bigg)^2$

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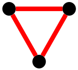
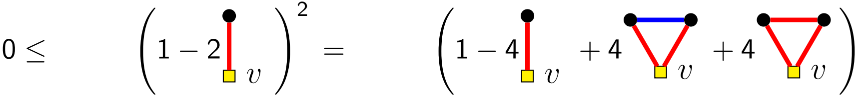
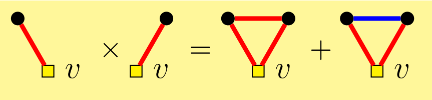
. Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)





EXAMPLE - MANTEL'S THEOREM, 1ST TRY

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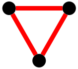
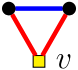
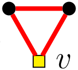
. Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

$0 \leq \frac{1}{n}\sum_v \Big(1 - 2$  $v\Big)^2 = \frac{1}{n}\sum_v \Big(1 - 4$  $v + 4$  $v + 4$  $v\Big)$

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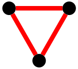
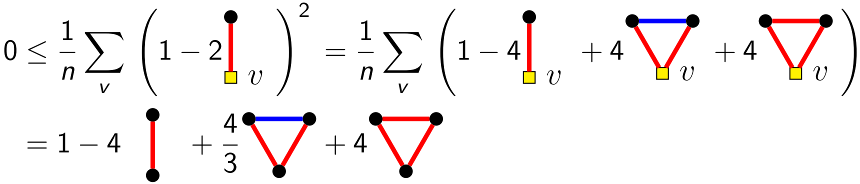
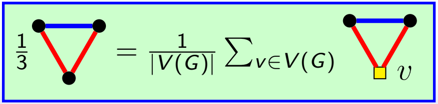
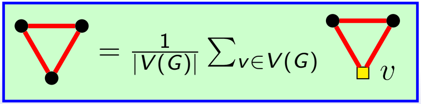
. Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)







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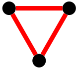
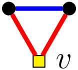
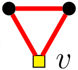
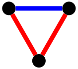
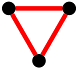
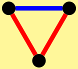
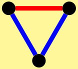
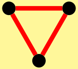
. Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

$0 \leq \frac{1}{n}\sum_v \Big(1 - 2$  $v \Big)^2 = \frac{1}{n}\sum_v \Big(1 - 4$  $v + 4$  $v + 4$  $v\Big)$

$= 1 - 4$  $+ \frac{4}{3}$  $+ 4$ 

 $= \frac{2}{3}$  $+ \frac{1}{3}$  $+$ 

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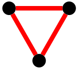
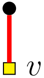
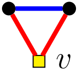
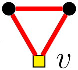
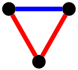
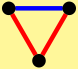
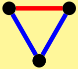
. Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

$0 \leq \frac{1}{n}\sum_v \Big(1 - 2$  $\Big)^2 = \frac{1}{n}\sum_v \Big(1 - 4$  $+ 4$  $+ 4$  $\Big)$

$= 1 - 4$  $+ \frac{4}{3}$ 

 $= \frac{2}{3}$  $+ \frac{1}{3}$ 

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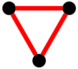
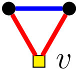
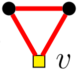
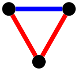
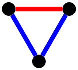
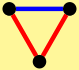
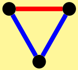
. Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

$0 \leq \frac{1}{n}\sum_v \Big(1 - 2$  $v\Big)^2 = \frac{1}{n}\sum_v \Big(1 - 4$  $v + 4$  $v + 4$  $v\Big)$

$= 1 - 4$  $+ \frac{4}{3}$ 

$= 1 - 2$  $- \frac{2}{3}$ 

2  $= \frac{4}{3}$  $+ \frac{2}{3}$ 

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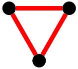
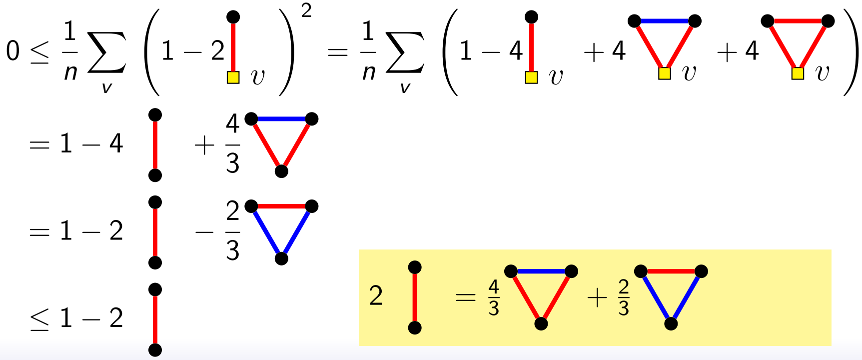
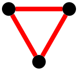
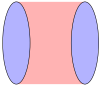
. Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)



Example - stability for Mantel

Assume  $= 0$ and  $= \frac{1}{2}$. Goal is $G =$ .

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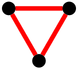
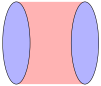
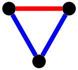
Assume  $= 0$ and  $= \frac{1}{2}$. Goal is $G =$  .

$0 \leq 1 - 2$  $- \dfrac{2}{3}$ 

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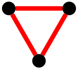
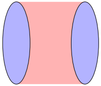
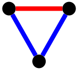
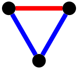
Assume  $= 0$ and  $= \frac{1}{2}$. Goal is $G =$  .

$0 \leq 1 - 2$  $- \frac{2}{3}$ 

$0 \leq -\frac{2}{3}$ 

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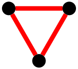
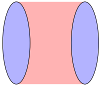
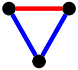
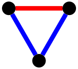
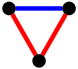
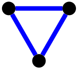
Assume  $= 0$ and  $= \frac{1}{2}$. Goal is $G =$ .

$0 \leq 1 - 2$  $- \frac{2}{3}$ 

$0 \leq -\frac{2}{3}$ 

Only  and  appear.

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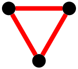
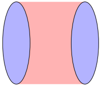
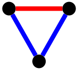
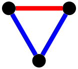
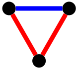
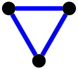
Assume  $= 0$ and  $= \frac{1}{2}$. Goal is $G =$ .

$0 \leq 1 - 2$  $- \frac{2}{3}$ 

$0 \leq -\frac{2}{3}$ 

Only  and  appear.

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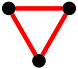
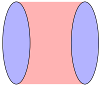
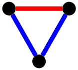
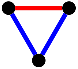
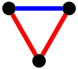
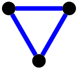
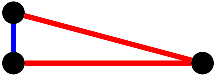
Assume  $= 0$ and  $= \frac{1}{2}$. Goal is $G =$  .

$0 \leq 1 - 2$  $- \frac{2}{3}$ 

$0 \leq -\frac{2}{3}$ 

Only  and  appear.



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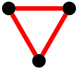
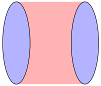
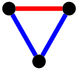
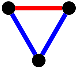
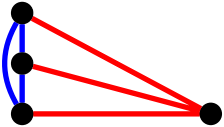
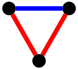
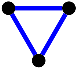
Assume  $= 0$ and  $= \frac{1}{2}$. Goal is $G =$ .

$0 \leq 1 - 2$  $- \frac{2}{3}$ 

$0 \leq -\frac{2}{3}$ 



Only  and  appear.

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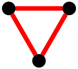
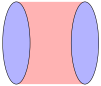
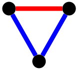
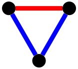
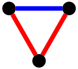
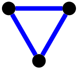
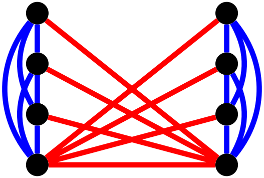
Assume  $= 0$ and  $= \frac{1}{2}$. Goal is $G =$ .

$0 \leq 1 - 2$  $- \frac{2}{3}$ 

$0 \leq -\frac{2}{3}$ 

Only  and  appear.



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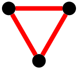
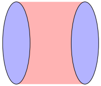
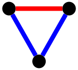
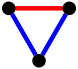
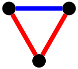
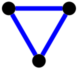
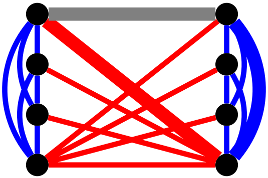
Assume  $= 0$ and  $= \frac{1}{2}$. Goal is $G =$ .

$0 \leq 1 - 2$  $- \frac{2}{3}$ 

$0 \leq -\frac{2}{3}$ 

Only  and  appear.



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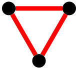
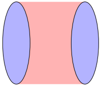
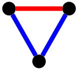
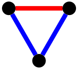
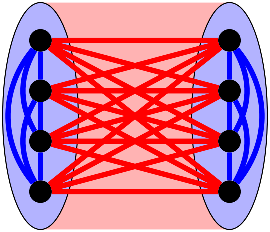
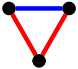
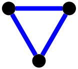
Assume  $= 0$ and  $= \frac{1}{2}$. Goal is $G =$ .

$0 \leq 1 - 2$  $- \frac{2}{3}$ 

$0 \leq -\frac{2}{3}$ 



Only  and  appear.

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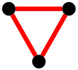
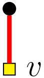
- we optimize on $\mathrm{LIM}^{\mathrm{T}} = \Big\{ q \in \mathrm{LIM} : q\Big($  $\Big) = 0 \Big\}$

$\frac{1}{2} \geq \max_{q \in \mathrm{LIM}^{\mathrm{T}}} q\Big($  $\Big)$

More automatic approach

- How to use computer to guess the right equation for you.

$0 \leq \Big(1 - 2$ 

$\Big)^2$

Example - Mantel's theorem, 2nd try

Theorem (Mantel 1907)

A triangle-free graph contains at most $\frac{1}{4}n^2$ edges.

Assume edges are red and non-edges are blue.

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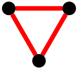
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

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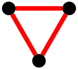
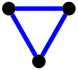
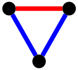
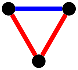
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

 $= 0$  $+ \frac{1}{3}$  $+ \frac{2}{3}$ 

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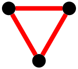
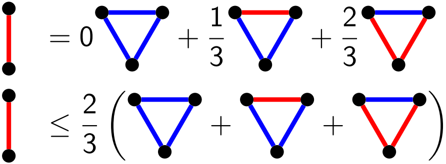
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)



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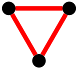
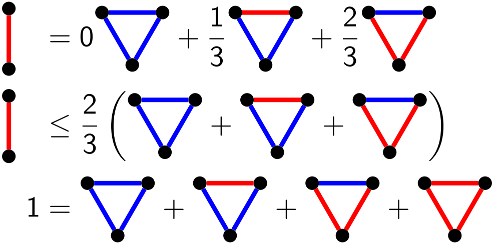
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)



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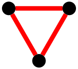
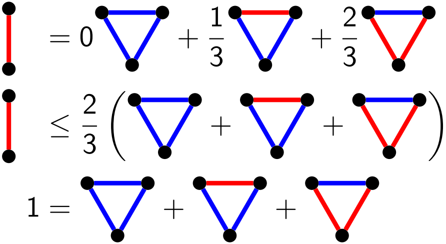
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)



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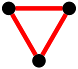
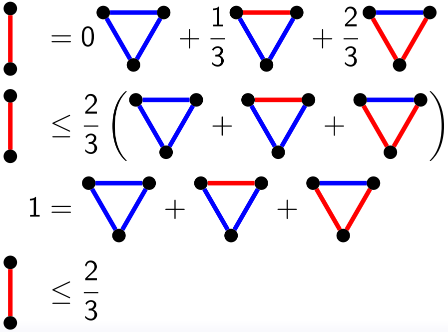
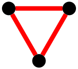
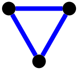
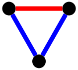
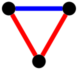
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)



Example - Mantel's theorem, 2nd try

Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

 $= 0$  $+ \frac{1}{3}$  $+ \frac{2}{3}$ 

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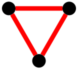
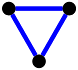
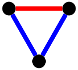
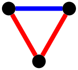
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

 $= 0$  $+ \frac{1}{3}$  $+ \frac{2}{3}$ 

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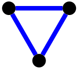
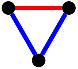
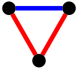
$0 \leq c_1$  $+ c_2$  $+ c_3$  .

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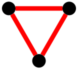
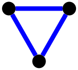
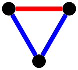
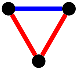
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

 $= 0$  $+ \frac{1}{3}$  $+ \frac{2}{3}$ 

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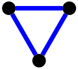
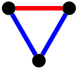
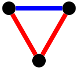
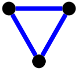
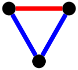
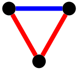
$0 \leq c_1$  $+ c_2$  $+ c_3$  .

After summing together

 $\leq c_1$  $+ \left(\frac{1}{3} + c_2\right)$  $+ \left(\frac{2}{3} + c_3\right)$ 

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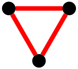
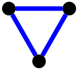
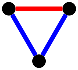
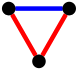
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

 $= 0$  $+ \frac{1}{3}$  $+ \frac{2}{3}$ 

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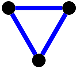
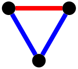
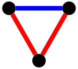
$0 \leq c_1$  $+ c_2$  $+ c_3$  .

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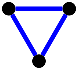
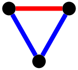
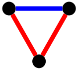
 $\leq c_1$  $+ \left(\frac{1}{3} + c_2\right)$  $+ \left(\frac{2}{3} + c_3\right)$ 

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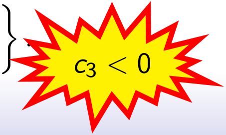

Candidates for c_1, c_2, c_3

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0$$ (matrix is positive semidefinite)

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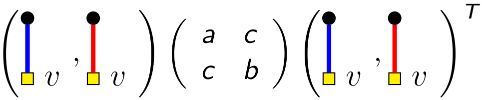
$0 \leq$



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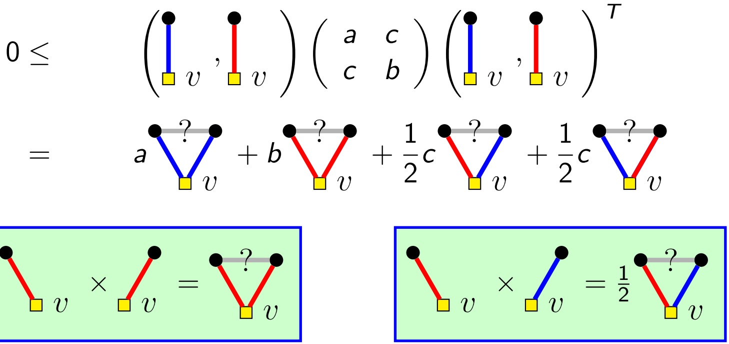

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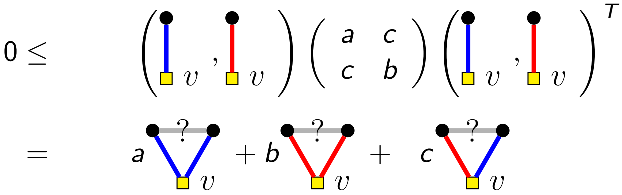

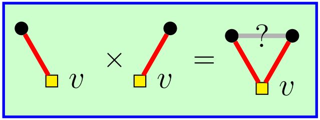

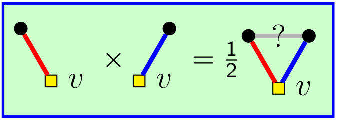

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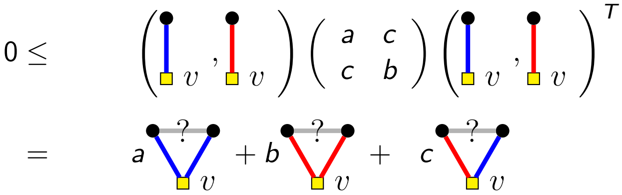

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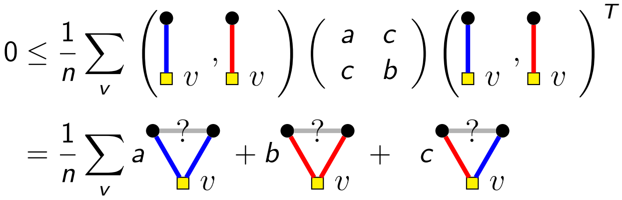

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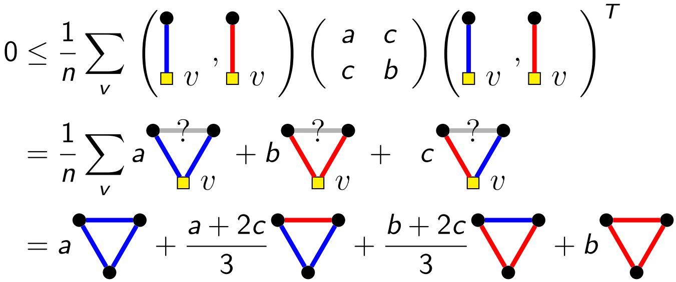

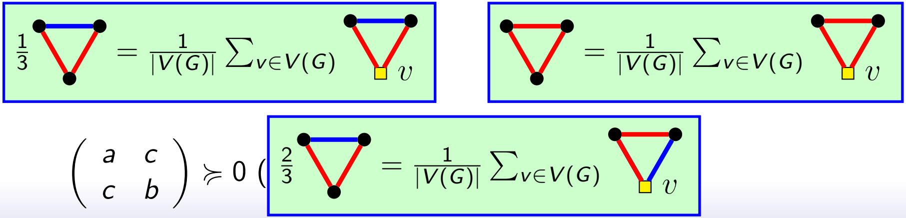

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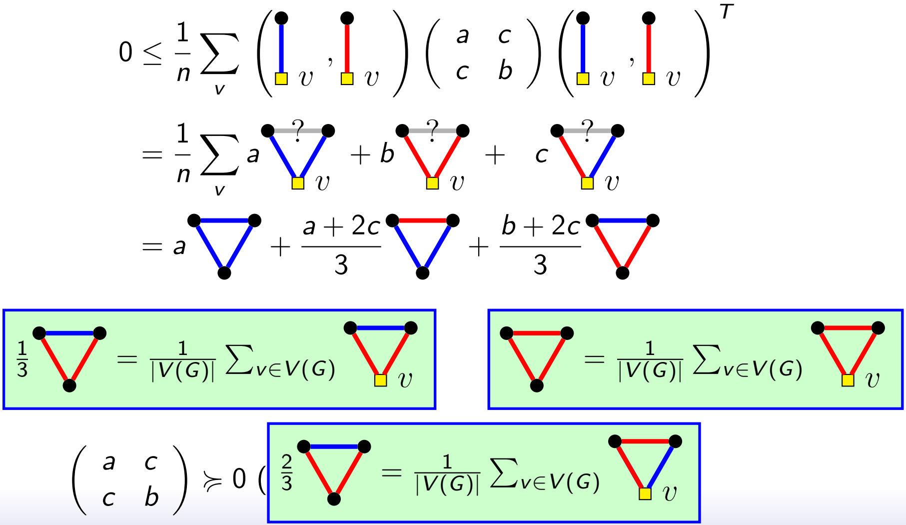

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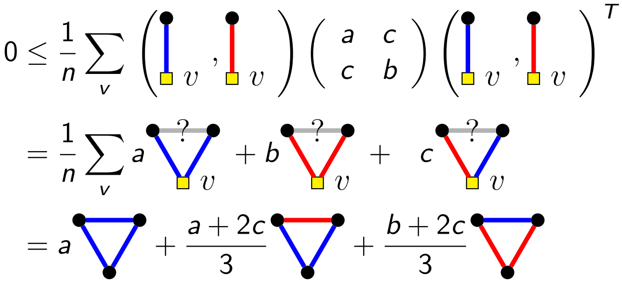
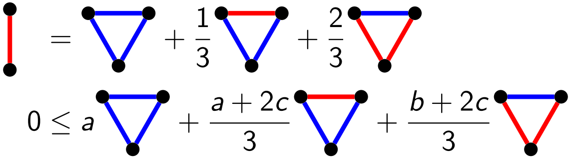


$$c_1 = a,\ c_2 = \frac{a+2c}{3},\ c_3 = \frac{b+2c}{3}$$

$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0$ (matrix is positive semidefinite)

Using c_1, c_2, c_3



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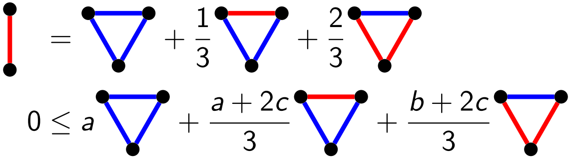

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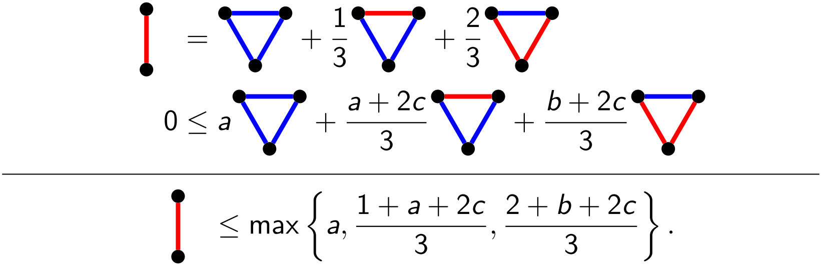

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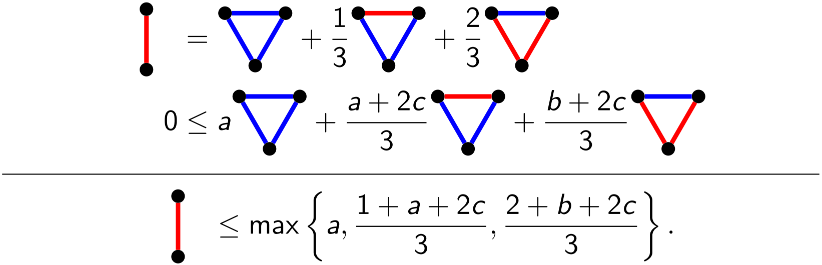

Try

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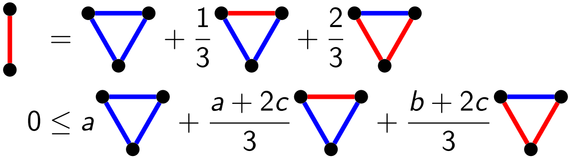

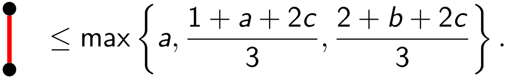

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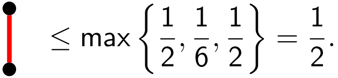


Optimizing a, b, c

 $$\leq \max\left\{a, \frac{1+a+2c}{3}, \frac{2+b+2c}{3}\right\}$$

$$(SDP)\begin{cases} \text{Minimize} & d \\ \text{subject to} & a \leq d \\ & \frac{1+a+2c}{3} \leq d \\ & \frac{2+b+2c}{3} \leq d \\ & \begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \end{cases}$$

(SDP) can be solved on computers using CSDP or SDPA.
Rounding may be needed for exact results.

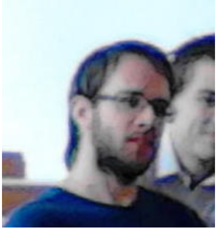
J. Balogh

P. Hu

L.

Permutations

O. Pikhurko



B. Udvari

J. Volec

Permutations and extremal problems

Problem

What is the minimum number of monotone subsequences of size k in a permutation of $[n]$?

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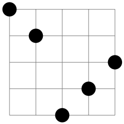

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(5,4,1),(5,4,2),(5,4,3)
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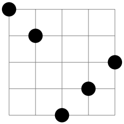

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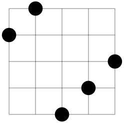


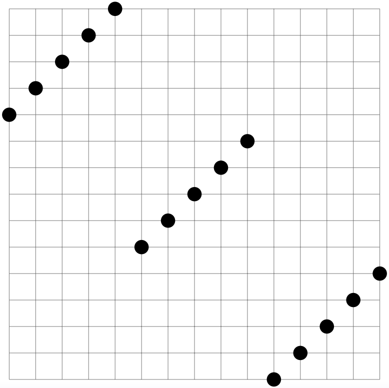
(4,5,1,2,3)

(1,2,3)

Conjecture

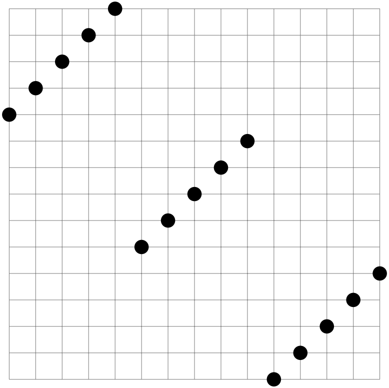
Conjecture (Myers 2002)

The number of monotone subsequences of length k is minimized by a permutation on $[n]$ with $k-1$ increasing runs of as equal lengths as possible.



$k=4, n=15$

Extremal case is not unique



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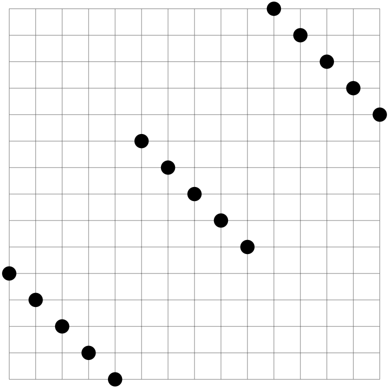

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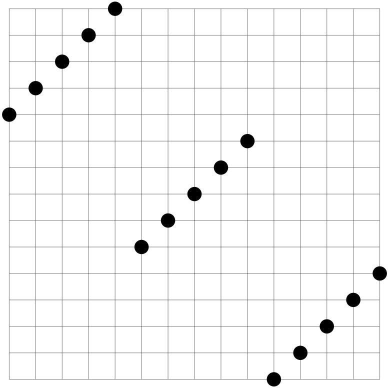

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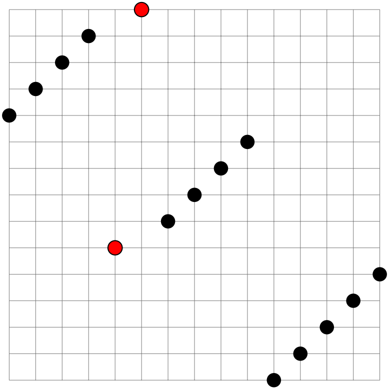
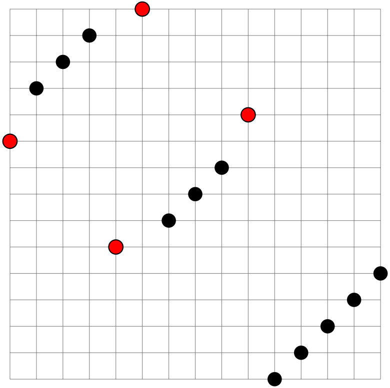
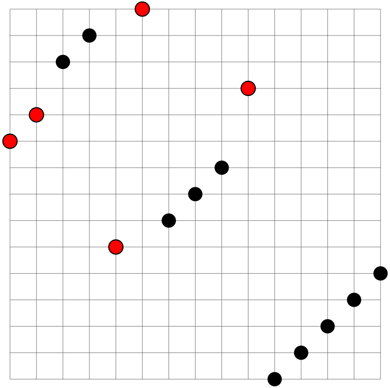


Extremal case is not unique



Extremal case is not unique



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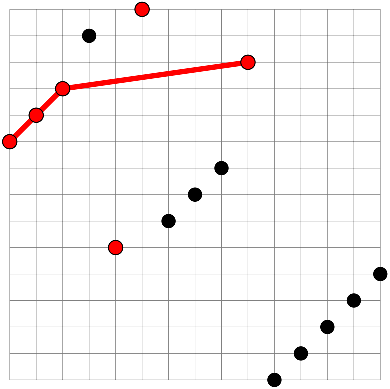
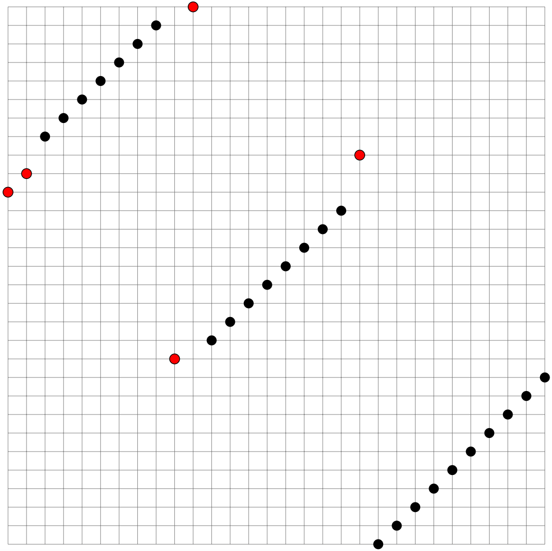


Extremal case is not unique



Conjecture (Myers 2002)

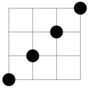
The number of monotone subsequences of length k is minimized by a permutation on $[n]$ with $k-1$ increasing runs of as equal lengths as possible.

Theorem (Samotij, Sudakov '14+)

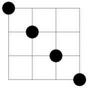
Myers' conjecture is true for sufficiently large k and $n \le k^2 + ck^{3/2}\log k$, where c is an absolute positive constant.

Theorem (Balogh, Hu, **L.**, Pikhurko, Udvari, Volec '14+)

Myers' conjecture is true for $k = 4$ and n sufficiently large.



(1,2,3,4)



(4,3,2,1)

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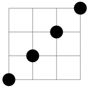

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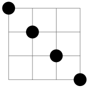


(4,3,2,1)

Use of flag algebras, $k = 5, 6$ also doable, 7 not.

From permutations to permutation graphs

(1,2)

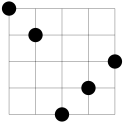
(2,1)

From permutations to permutation graphs

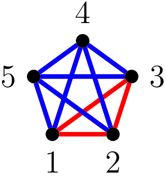
(1,2)

(2,1)

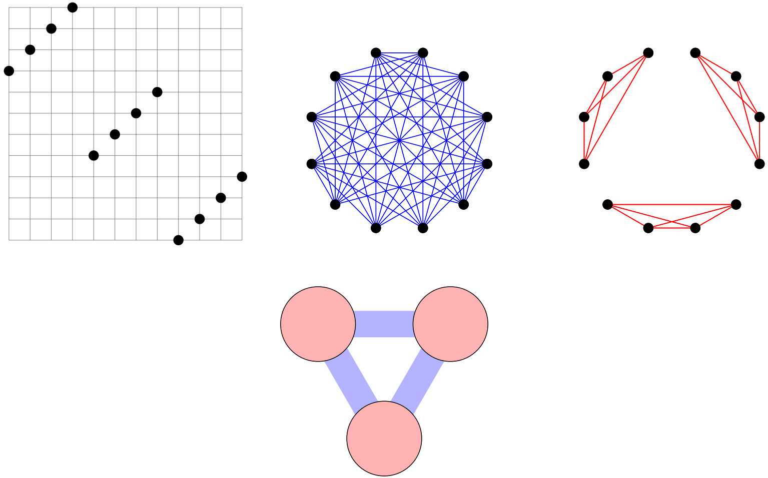
k = 3
n = 5



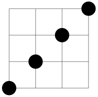
(5,4,1,2,3)



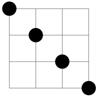
Extremal example ($k = 4$)



As flag algebra question ($k = 4$)

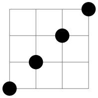


(1,2,3,4)

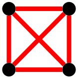


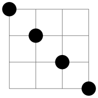
(4,3,2,1)

As flag algebra question $(k = 4)$

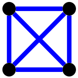


(1,2,3,4)

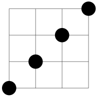




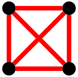
(4,3,2,1)

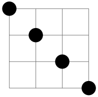


As flag algebra question ($k = 4$)

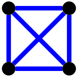


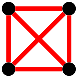
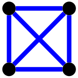
(1,2,3,4)





(4,3,2,1)



minimize  + 

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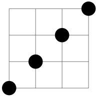
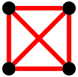
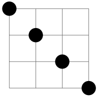


(1,2,3,4)





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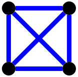
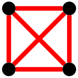
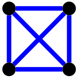
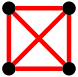
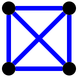


minimize  + 

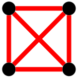
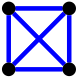
Theorem (Balogh, Hu, **L.**, Pikhurko, Udvari, Volec '14+)

 +  $\geq \frac{1}{27}$

for every permutation graph.

Only for permutation graphs

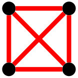
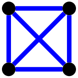
Theorem (Balogh, Hu, L., Pikhurko, Udvari, Volec '14+)

min ( + ) $= \frac{1}{27}$

over permutation graphs (and extremal permutations described using Myers' results).

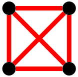
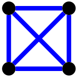
Only for permutation graphs

Theorem (Balogh, Hu, L., Pikhurko, Udvari, Volec '14+)

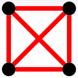
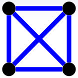
min $\Bigg($  $+$  $\Bigg) = \frac{1}{27}$

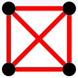
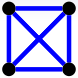
over permutation graphs (and extremal permutations described using Myers' results).

Theorem (Sperfeld '12; Thomason '89)

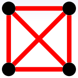
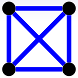
$\frac{1}{35} <$ min $\Bigg($  $+$  $\Bigg) < \frac{1}{33}$

over **all** *sufficiently large 2-edge-colored complete graphs.*

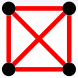
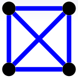
We want to prove  $+$  $\geq \frac{1}{27}$

We want to prove  $+$  $\geq \frac{1}{27}$

- Write a semidefinite program (*SDP*) (with graphs on 7 vertices, 388 constraints).

We want to prove  $+$  $\geq \frac{1}{27}$

- Write a semidefinite program (SDP) (with graphs on 7 vertices, 388 constraints).
- Solve (SDP) using a computer, obtain $M' \in \mathbb{R}^{f \times f}$.

We want to prove  $+$  $\geq \frac{1}{27} = 0.\overline{037}$

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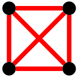
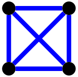
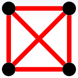
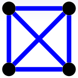
 $+$  ≥ 0.0370370369999

We want to prove  $+$  $\geq \frac{1}{27} = 0.\overline{037}$

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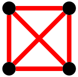
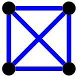
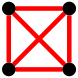
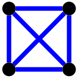
 $+$  ≥ 0.0370370369999

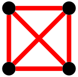
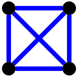
- *Round* M' to $M \in \mathbb{Q}^{f \times f}$, such that

 $+$  $\geq \frac{1}{27}$

and $M \succcurlyeq 0$.

Structure of extremal permutations

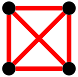
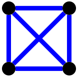
Assuming

 $+$  $= \frac{1}{27}$

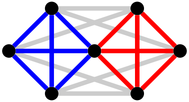
Flag algebra implies:

Structure of extremal permutations

Assuming

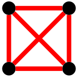
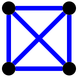
 $+$  $= \frac{1}{27}$

Flag algebra implies:

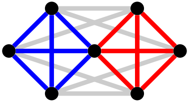
(A)  $= o(1)$

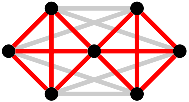
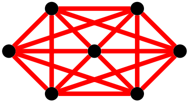
Structure of extremal permutations

Assuming

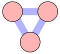
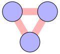
 $+$  $= \frac{1}{27}$

Flag algebra implies:

(A)  $= o(1)$

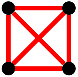
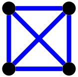
(B) Almost all  are  .

After flag algebra (stability)

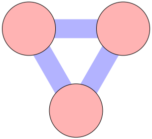
" +  is close to $\frac{1}{27}$ $\Rightarrow$ G is close to  or "

Lemma (Stability)

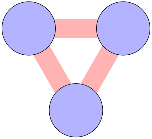
For every $\varepsilon > 0$ there exist n_0 and $\varepsilon' > 0$ such that every admissible graph G of order $n > n_0$ with

 $+$  $\leq \frac{1}{27} + \varepsilon'$

is isomorphic to either

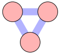
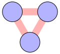


or



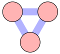
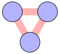
after recoloring at most $20\varepsilon n^2$ edges.

After flag algebra (stability sketch)

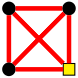
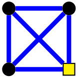
" +  is close to $\frac{1}{27} \Rightarrow$ G is close to  or "

- Using removal lemma, properties (A) and (B) can be satisfied entirely. (lost εn^2 edges)

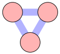
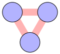
After flag algebra (stability sketch)

“ +  is close to $\frac{1}{27}$ $\Rightarrow$ G is close to  or ”

- Using removal lemma, properties (A) and (B) can be satisfied entirely. (lost εn^2 edges)
- For all $v \in V(G) \setminus X$, where $|X| \leq 2\varepsilon n$ vertices

$\frac{1}{27} - \varepsilon \leq$  $_v$ +  $_v$ $\leq \frac{1}{27} + \varepsilon''$ (1)

After flag algebra (stability sketch)

" +  is close to $\frac{1}{27}$ $\Rightarrow$ G is close to  or "

- Using removal lemma, properties (A) and (B) can be satisfied entirely. (lost εn^2 edges)
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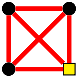
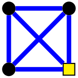
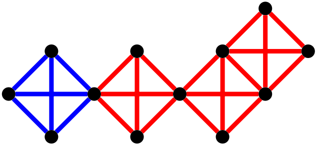
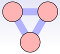
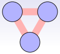
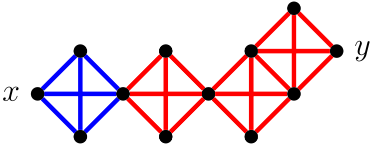
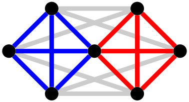
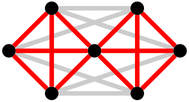
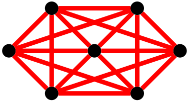
$\frac{1}{27} - \varepsilon \leq$  v $+$  v $\leq \frac{1}{27} + \varepsilon''$ (1)

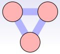
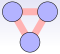
- $x \sim y$ if x  y

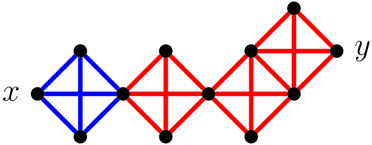
“ +  is close to $\frac{1}{27}$ $\Rightarrow$ G is close to  or ”

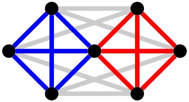
- $x \sim y$ if 

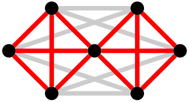
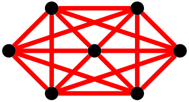
(A)  $= 0$

(B) All  are  .

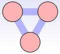
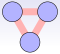
" +  is close to $\frac{1}{27}$ $\Rightarrow$ G is close to  or "

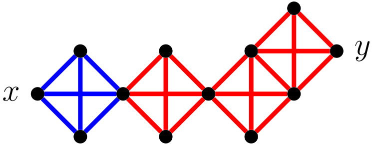
- $x \sim y$ if 

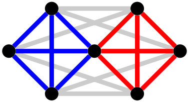
(A)  $= 0$

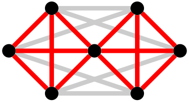
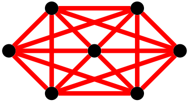
(B) All  are  .

- Every equivalence class is a monochromatic clique.

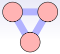
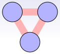
" +  is close to $\frac{1}{27} \Rightarrow G$ is close to  or "

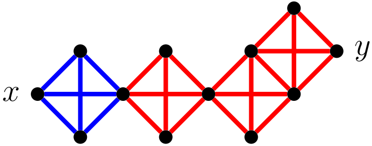
- $x \sim y$ if 

(A)  $= 0$

(B) All  are  .

- Every equivalence class is a monochromatic clique.
- There are three equivalence classes of size $\frac{1}{3}n \pm 16\varepsilon n$ by (1).

" +  is close to $\frac{1}{27} \Rightarrow G$ is close to  or "

- $x \sim y$ if 

- Every equivalence class is a monochromatic clique.
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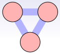
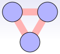
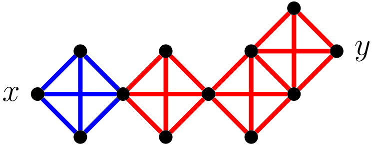
- The classes have the same color

" +  is close to $\frac{1}{27}$ $\Rightarrow$ G is close to  or "

- $x \sim y$ if 

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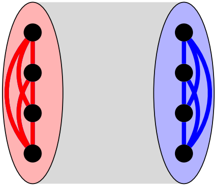
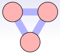
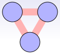
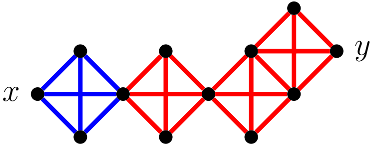


" +  is close to $\frac{1}{27}$ $\Rightarrow$ G is close to  or "

- $x \sim y$ if 

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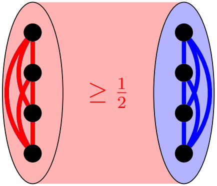
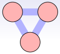
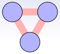
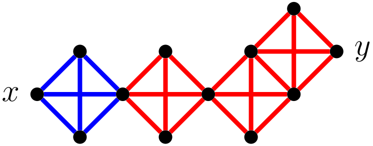


“ +  is close to $\frac{1}{27}$ $\Rightarrow$ G is close to  or ”

- $x \sim y$ if



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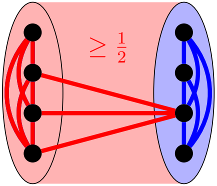
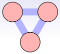
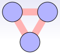
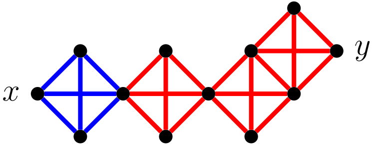


" +  is close to $\frac{1}{27}$ $\Rightarrow$ G is close to  or "

- $x \sim y$ if 

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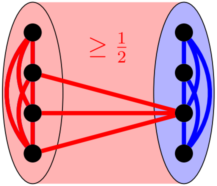

Exact result: By recoloring edges.

Other permutations - maximizing 1342 and 2413

$$0.19657 \leq \sigma(1342) \leq 2/9 = 0.22222\ldots \qquad \text{AAHHS}$$
$$\sigma(1342) \leq 0.1988373 \qquad \text{BHLPUV}$$

$$51/511 = 0.0998\ldots \leq \sigma(2413) \leq 2/9 = 0.22222 \qquad \text{AAHHS}$$
$$0.1024732 \leq \sigma(2413) \qquad \text{P}$$
$$0.10472\ldots \leq \sigma(2413) \qquad \text{PS}$$
$$\sigma(2413) \leq 0.1047805 \qquad \text{BHLPUV}$$

AAHHS . . . Albert, Atkinson, Handley, Holton, Stromquist 2002
P. . . Presutti 2008
PS. . . Presutti, Stromquist 2010
BHLPUV. . . us

Thank you for your attention!