# Rainbow triangles in 3-edge-colored graphs

József Balogh, Ping Hu, Bernard Lidický, Florian Pfender,
Jan Volec, Michael Young

SIAM Conference on Discrete Mathematics
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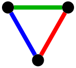
## PROBLEM

*Find a 3-edge-coloring of a complete graph $K_n$ maximizing the number of copies of rainbow colored triangles*  *.*

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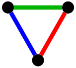
*Find a 3-edge-coloring of a complete graph $K_n$ maximizing the number of copies of rainbow colored triangles*  *.*

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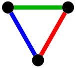
*Find a 3-edge-coloring of a complete graph $K_n$ maximizing the number of copies of rainbow colored triangles*  *.*

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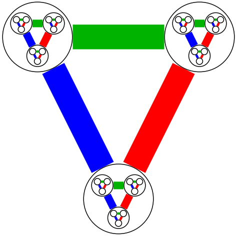
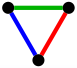
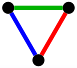
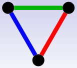
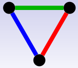
Iterated blow-up of triangle



$\frac{1}{4} =$  .

 denotes graph and/or its density

## $F(n)$ = max # of  over all 3-edge-colorings of $K_n$

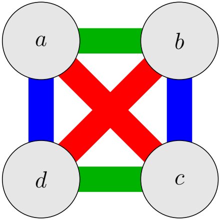
# $F(n) = \max \#$ of  over all 3-edge-colorings of $K_n$

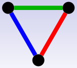
## CONJECTURE (ERDŐS AND SÓS; '72$^-$)

*For all $n > 0$,*

$$F(n) = F(a) + F(b) + F(c) + F(d) + abc + abd + acd + bcd,$$

*where $a + b + c + d = n$; $a, b, c, d$ are as equal as possible, and $F(0) = 0$.*



$F(n) = \max \#$ of  over all 3-edge-colorings of $K_n$

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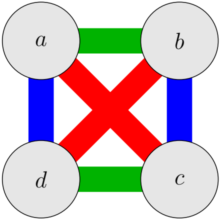
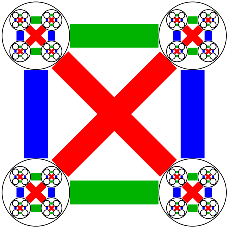
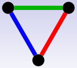




$F(n) = $ max # of  over all 3-edge-colorings of $K_n$

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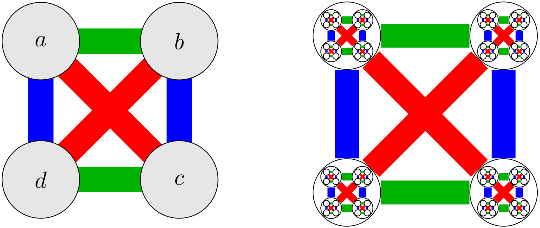
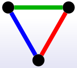
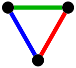
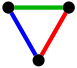


$0.4 =$ 

# Flag algebras application

Construction: $0.4 \leq$ 

- get a matching upper bound  $\approx 0.4$
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- get subgraphs with 0 density
- get extremal construction (stability)

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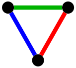
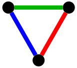
Construction: $0.4 \leq$ 

- get a matching upper bound  $\approx 0.4$
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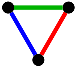
Flag algebras (on 6 vertices) give only

 $\leq 0.4006,$

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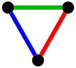
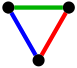
Construction: $0.4 \leq$ 

- get a matching upper bound  $\approx 0.4$
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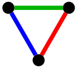
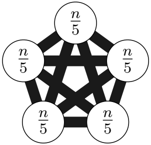
 $\leq 0.4006,$

not enough for rounding.
The iterative extremal construction is causing troubles....

# Not iterated extremal constructions

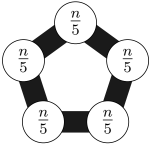
### Theorem (Turán)

*# of edges over $K_l$-free graphs is maximized by*



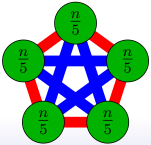
### Theorem (Hatami, Hladký, Kráľ, Norine, Razborov)

*# of $C_5$s over triangle-free graphs is maximized by*



### Theorem (Cummings, Kráľ, Pfender, Sperfeld, Treglown, Young)

*# of monochromatic triangles over 3-edge-colored graphs is minimized by*

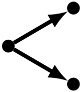
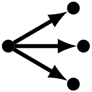


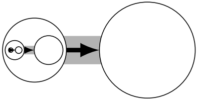
And more... `http://flagmatic.org`

($n$ large enough)

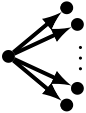
# Iterated extremal constructions

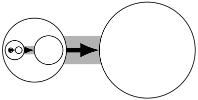
### Theorem (Falgas-Ravry, Vaughan)

*# of*  *and*  *is maximized by*



### Theorem (Huang)

*# of*  *is maximized by*



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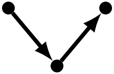
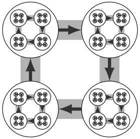
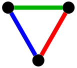
*# of*  *is maximized by*



# Our main result

$F(n) =$ max # of  over all coloring of $K_n$

**Theorem (Balogh, Hu, L., Pfender, Volec, Young)**

*For all $n > n_0$,*

$$F(n) = F(a) + F(b) + F(c) + F(d) + abc + abd + acd + bcd,$$

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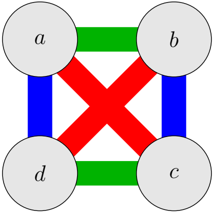
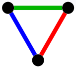
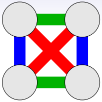
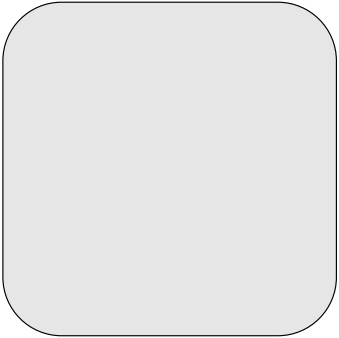


# Sketch of proof

Goal: maximizing  gives edge-coloring like 



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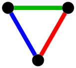
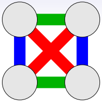
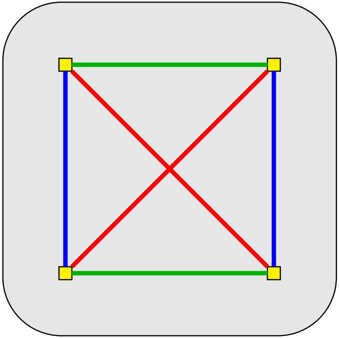
Goal: maximizing  gives edge-coloring like 



- pick a properly 3-edge-colored $K_4$

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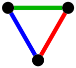
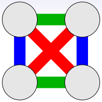
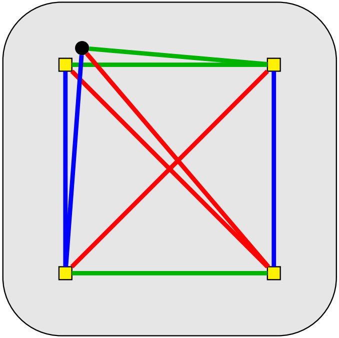
Goal: maximizing  gives edge-coloring like 



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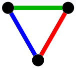
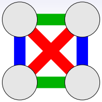
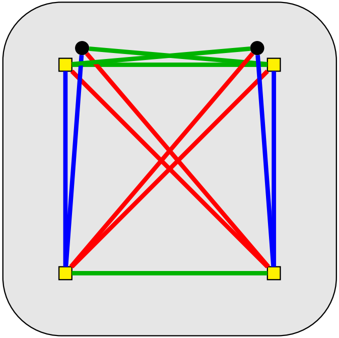
Goal: maximizing  gives edge-coloring like 



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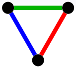
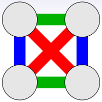
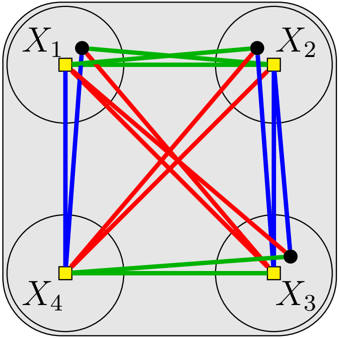
Goal: maximizing  gives edge-coloring like 



- pick a properly 3-edge-colored $K_4$
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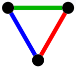
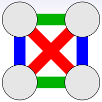
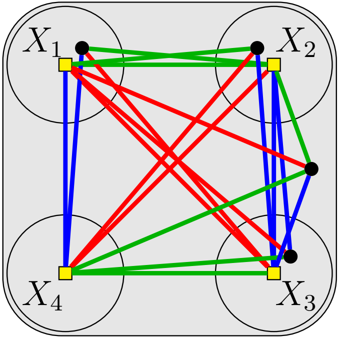
Goal: maximizing  gives edge-coloring like 



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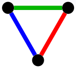
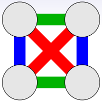
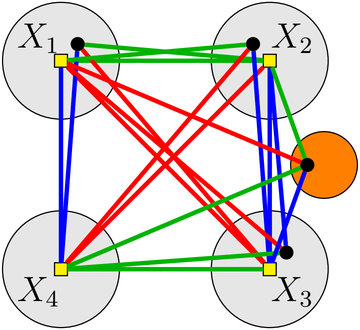
Goal: maximizing  gives edge-coloring like 



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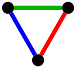
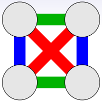
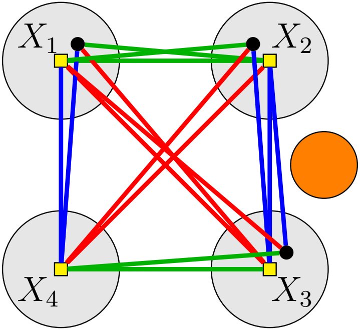
Goal: maximizing  gives edge-coloring like 



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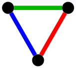
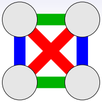
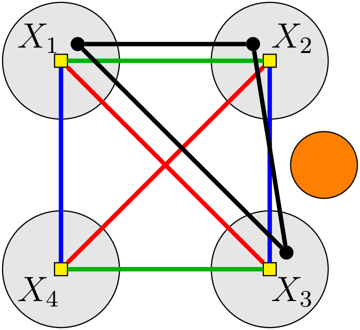
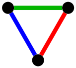
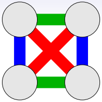
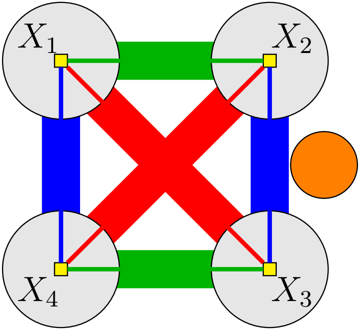
Goal: maximizing  gives edge-coloring like 



- pick a properly 3-edge-colored $K_4$
- partition the rest
- correct edges between $X_i$s

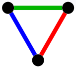
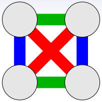
# Sketch of proof

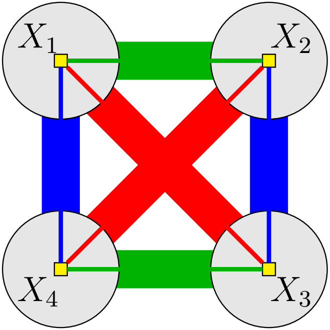
Goal: maximizing  gives edge-coloring like 



- pick a properly 3-edge-colored $K_4$
- partition the rest
- correct edges between $X_i$s
- no orange trash

# SKETCH OF PROOF

Goal: maximizing  gives edge-coloring like 



- pick a properly 3-edge-colored $K_4$
- partition the rest
- correct edges between $X_i$s
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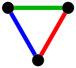
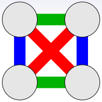
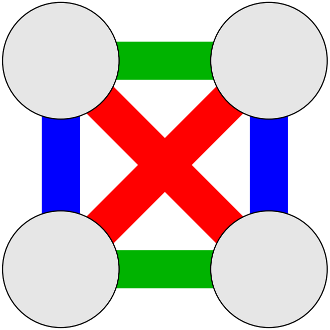
- balance sizes of $X_i$s

# SKETCH OF PROOF

Goal: maximizing  gives edge-coloring like 



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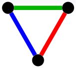
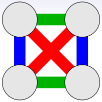
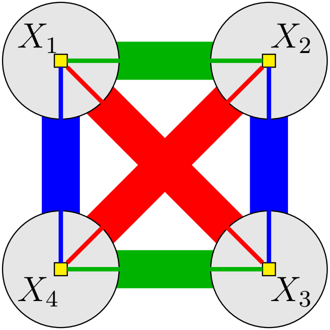
# SKETCH OF PROOF

Goal: maximizing  gives edge-coloring like 



- pick a properly 3-edge-colored $K_4$
- partition the rest
- correct edges between $X_i$s
- no orange trash
- balance sizes of $X_i$s

How to pick the properly 3-edge-colored $K_4$?

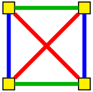
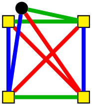
($|X_i|$s close to $0.25n$, few wrongly colored edges, small trash)

# HOW TO PICK $K_4$?

Use Flag Algebras!

# How to pick $K_4$?

Use Flag Algebras!

Try 1: Pick  maximizing 

# How to pick $K_4$?

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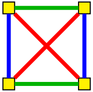
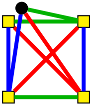
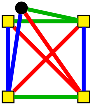
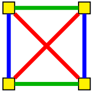
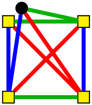
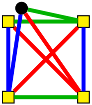
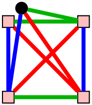
Try 1: Pick  maximizing 

 $(n-5) \geq$

# How to pick $K_4$?

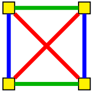
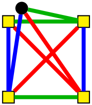
Use Flag Algebras!

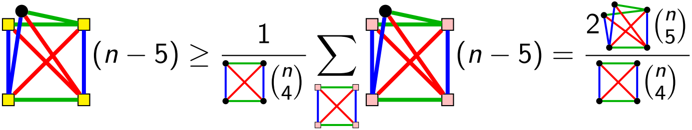
Try 1: Pick  maximizing 

 $(n - 5) \geq \frac{1}{}$  $\binom{n}{4}$ $\sum$   $(n - 5)$

# HOW TO PICK $K_4$?

Use Flag Algebras!

Try 1: Pick  maximizing 



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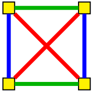
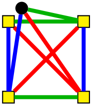
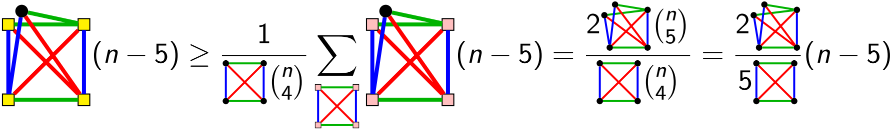
Try 1: Pick  maximizing 



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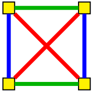
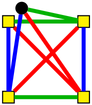
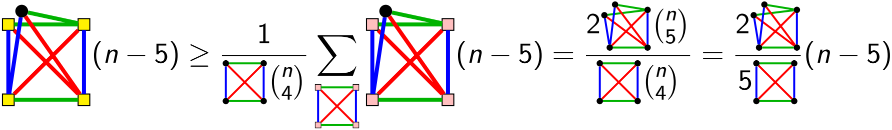
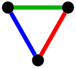
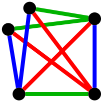
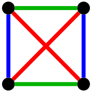
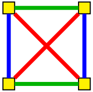
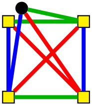
Try 1: Pick  maximizing 

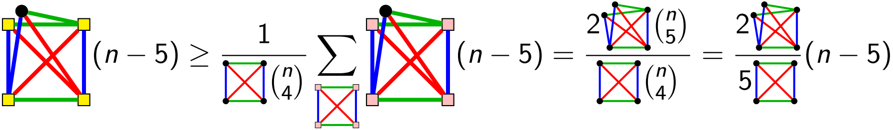


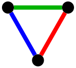
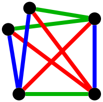
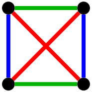
FA:  $\geq 0.4$ then  $> 0.23516$,  $< 0.0952$

# HOW TO PICK $K_4$?

Use Flag Algebras!

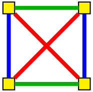
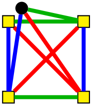
Try 1: Pick  maximizing  $> 0.988$

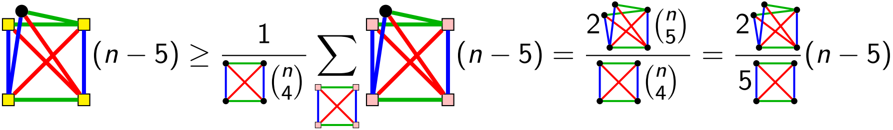


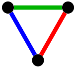
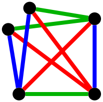
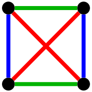
FA:  $\geq 0.4$ then  $> 0.23516$,  $< 0.0952$

# How to pick $K_4$?

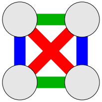
Use Flag Algebras!

Try 1: Pick  maximizing  $> 0.988$



FA:  $\geq 0.4$ then  $> 0.23516$,  $< 0.0952$

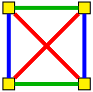
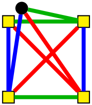
Result for $K_n$:

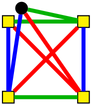
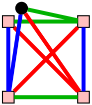
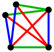
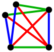


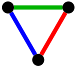
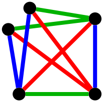
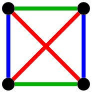
$$|X_1| + |X_2| + |X_3| + |X_4| > 0.988(n-5)$$

# How to pick $K_4$?

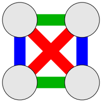
Use Flag Algebras!

Try 1: Pick  maximizing  $> 0.988$

 $(n-5) \geq \frac{1}{\phantom{x}}$  $\binom{n}{4}$ $\sum$   $(n-5) =$ $2$  $\binom{n}{5}$ /  $\binom{n}{4}$ $=$ $2$  / $5$  $(n-5)$

FA:  $\geq 0.4$ then  $> 0.23516$,  $< 0.0952$

Result for $K_n$:

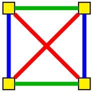


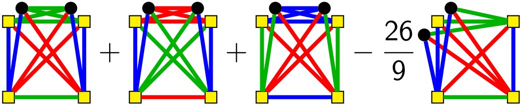
$$|X_1| + |X_2| + |X_3| + |X_4| > 0.988(n-5)$$

Balancing needed...

# HOW TO PICK $K_4$?

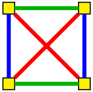
Use Flag Algebras!

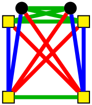
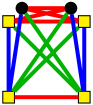
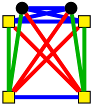
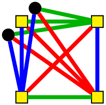
Try 2: Pick  maximizing

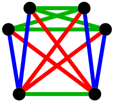
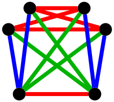
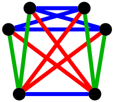
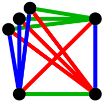


# How to pick $K_4$?

Use Flag Algebras!

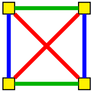
Try 2: Pick  maximizing

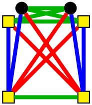
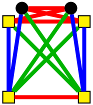
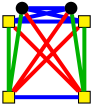
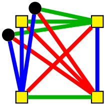
 $+$  $+$  $- \frac{26}{9}$ 

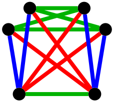
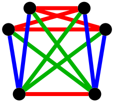
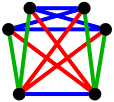
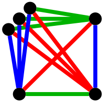
FA: $\frac{4}{15}\Bigg($  $+$  $+$  $\Bigg) - \frac{26}{45}$  $> 0.002629$

# How to pick $K_4$?

Use Flag Algebras!

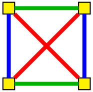
Try 2: Pick  maximizing

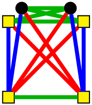
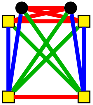
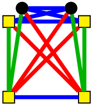
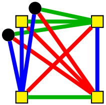
 $+$  $+$  $- \frac{26}{9}$  $> 0.0276$

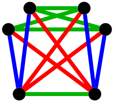
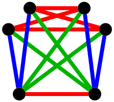
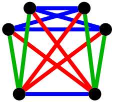
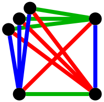
FA: $\frac{4}{15}\Bigg($  $+$  $+$  $\Bigg) - \frac{26}{45}$  $> 0.002629$

# HOW TO PICK $K_4$?

Use Flag Algebras!

Try 2: Pick  maximizing

 $+$  $+$  $- \frac{26}{9}$  $> 0.0276$

FA: $\frac{4}{15}\Bigg($  $+$  $+$  $\Bigg) - \frac{26}{45}$  $> 0.002629$

Final equation:

$$2 \sum_{1 \le i < j \le 4} |X_i||X_j| - |F| - \tfrac{26}{9} \sum_{1 \le i \le 4} |X_i|^2 > 0.0276n^2$$

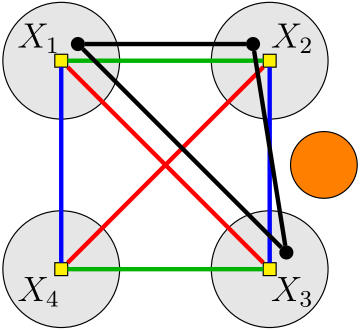
$F$ = wrongly colored edges.

# How the first step worked

$$2 \sum_{1 \leq i < j \leq 4} |X_i||X_j| - |F| - \tfrac{26}{9} \sum_{1 \leq i \leq 4} |X_i|^2 > 0.0276n^2$$

Implies:

$$0.244n < |X_i| < 0.256n$$
$$|\mathit{Trash}| < 0.006n$$
$$|F| < 0.00008\binom{n}{2}$$

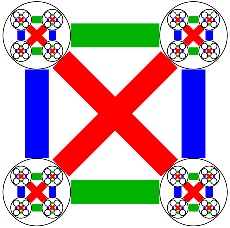


$F$ = wrongly colored edges.

# More results

### Theorem

*# of rainbow $K_3$s is maximized by*



*if on $4^k$ vertices.*

# More results

### Theorem

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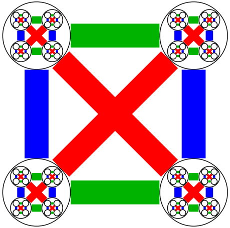
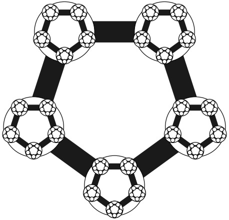


*if on $4^k$ vertices.*

### Theorem

*# of induced $C_5$s is maximized by*

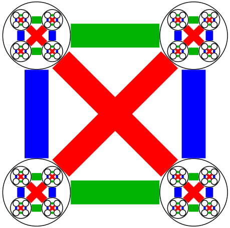


*if on $5^k$ vertices.*

# More results

### Theorem

*# of rainbow $K_3$s is maximized by*



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### Theorem

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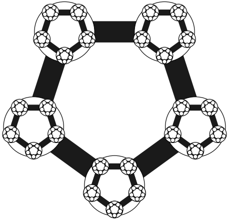
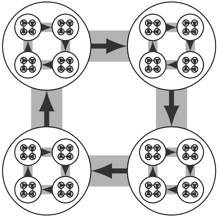


*if on $5^k$ vertices.*

### Theorem

*# of induced oriented $C_4$s is maximized by*

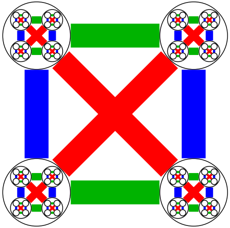


*if on $4^k$ vertices.*

# MORE RESULTS

### THEOREM

*# of rainbow $K_3$s is maximized by*



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### THEOREM

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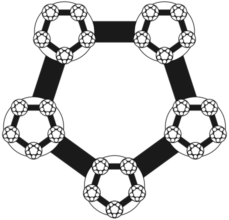

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### THEOREM

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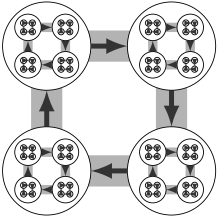
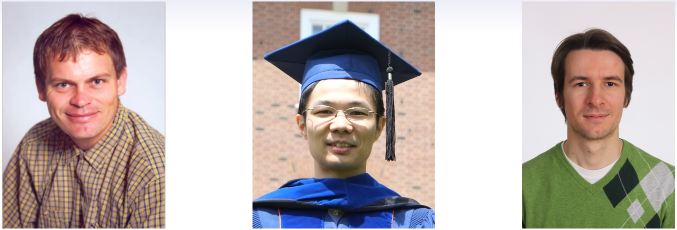


*if on $4^k$ vertices.*

(for all $k$)



Thank you for listening!