Upper bounds on small Ramsey numbers

Bernard Lidický Florian Pfender

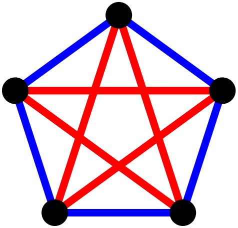
Iowa State University
University of Colorado Denver

Atlanta Lecture Series in Combinatorics and Graph Theory XIV
Feb 15, 2015

Definition

$R(G_1, G_2, \dots, G_k)$ is the smallest integer n such that any k-edge coloring of K_n contains a copy of G_i in color i for some $1 \leq i \leq k$.

$R(K_3, K_3) > 5$

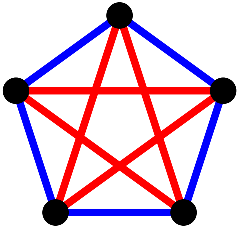


$R(K_3, K_3) \leq 6$

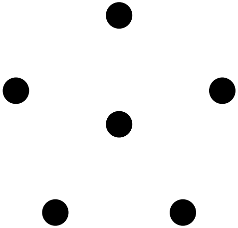
Definition

$R(G_1, G_2, \ldots, G_k)$ is the smallest integer n such that any k-edge coloring of K_n contains a copy of G_i in color i for some $1 \leq i \leq k$.

$R(K_3, K_3) > 5$



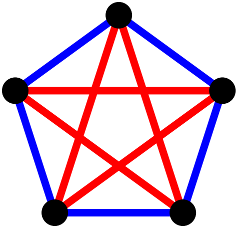
$R(K_3, K_3) \leq 6$



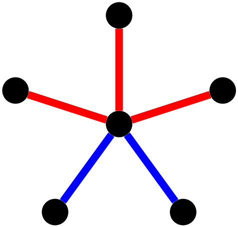
Definition

$R(G_1, G_2, \ldots, G_k)$ is the smallest integer n such that any k-edge coloring of K_n contains a copy of G_i in color i for some $1 \leq i \leq k$.

$R(K_3, K_3) > 5$



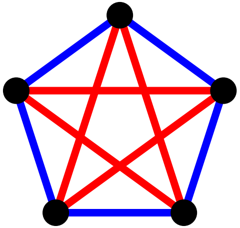
$R(K_3, K_3) \leq 6$



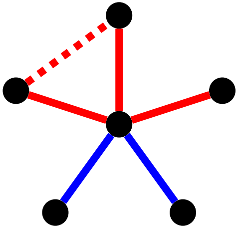
Definition

$R(G_1, G_2, \ldots, G_k)$ is the smallest integer n such that any k-edge coloring of K_n contains a copy of G_i in color i for some $1 \leq i \leq k$.

$R(K_3, K_3) > 5$



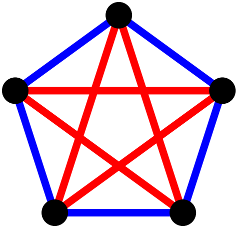
$R(K_3, K_3) \leq 6$



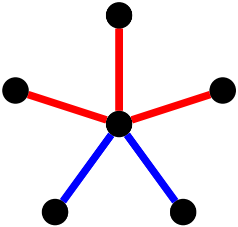
Definition

$R(G_1, G_2, \ldots, G_k)$ is the smallest integer n such that any k-edge coloring of K_n contains a copy of G_i in color i for some $1 \le i \le k$.

$R(K_3, K_3) > 5$



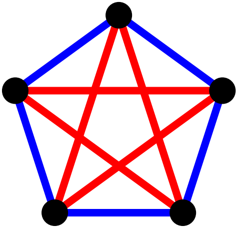
$R(K_3, K_3) \le 6$



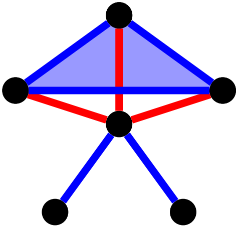
Definition

$R(G_1, G_2, \ldots, G_k)$ is the smallest integer n such that any k-edge coloring of K_n contains a copy of G_i in color i for some $1 \leq i \leq k$.

$R(K_3, K_3) > 5$



$R(K_3, K_3) \leq 6$



Theorem (Ramsey 1930)

$R(K_m, K_n)$ is finite.

THEOREM (RAMSEY 1930)
$R(K_m, K_n)$ is finite.

$R(G_1, \ldots, G_k)$ is finite

Questions:

- study how $R(G_1, \ldots, G_k)$ grows if $G_1, \ldots, G_k$ grow (large)
- study $R(G_1, \ldots, G_k)$ for fixed $G_1, \ldots, G_k$ (small)

Theorem (Ramsey 1930)
$R(K_m, K_n)$ is finite.

$R(G_1, \ldots, G_k)$ is finite

Questions:

- study how $R(G_1, \ldots, G_k)$ grows if $G_1, \ldots, G_k$ grow (large)
- study $R(G_1, \ldots, G_k)$ for fixed $G_1, \ldots, G_k$ (small)

Radziszowski - *Small Ramsey Numbers*
Electronic Journal of Combinatorics - Survey

Flag algebras

Seminal paper:
Razborov, Flag Algebras, *Journal of Symbolic Logic* **72** (2007), 1239–1282.
David P. Robbins Prize by AMS for Razborov in 2013

Flag algebras

Seminal paper:
Razborov, Flag Algebras, *Journal of Symbolic Logic* **72** (2007), 1239–1282.
David P. Robbins Prize by AMS for Razborov in 2013

Example (Goodman, Razborov)

If density of edges is at least $\rho > 0$, what is the minimum density of triangles?

- designed to attack extremal problems.
- works well if constraints as well as desired value can be computed by checking small subgraphs (or average over small subgraphs)
- the results are in limit (very large graphs)

Applications (incomplete list)

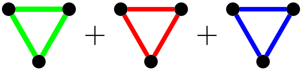
Author	Year	Application/Result
Razborov	2008	edge density vs. triangle density
Hladký, Kráľ, Norin	2009	Bounds for the Caccetta-Haggvist conjectu
Razborov	2010	On 3-hypergraphs with forbidden 4-vertex co
Hatami, Hladký,Kráľ,Norine,Razborov / Grzesik	2011	Erdős Pentagon problem
Hatami, Hladký, Kráľ, Norin, Razborov	2012	Non-Three-Colourable Common Graphs Exist
Balogh, Hu, L., Liu / Baber	2012	4-cycles in hypercubes
Reiher	2012	edge density vs. clique density
Shagnik, Huang, Ma, Naves, Sudakov	2013	minimum number of k-cliques
Baber, Talbot	2013	A Solution to the 2/3 Conjecture
Falgas-Ravry, Vaughan	2013	Turán density of many 3-graphs
Cummings, Kráľ, Pfender, Sperfeld, Treglown, Young	2013	Monochromatic triangles in 3-edge colored
Kramer, Martin, Young	2013	Boolean lattice
Balogh, Hu, L., Pikhurko, Udvari, Volec	2013	Monotone permutations
Norin, Zwols	2013	New bound on Zarankiewicz's conjecture
Huang, Linial, Naves, Peled, Sudakov	2014	3-local profiles of graphs
Balogh, Hu, L., Pfender, Volec, Young	2014	Rainbow triangles in 3-edge colored graphs
Balogh, Hu, L., Pfender	2014	Induced density of C_5
Goaoc, Hubard, de Verclos, Séréni, Volec	2014	Order type and density of convex subsets
Coregliano, Razborov	2015	Tournaments
…	…	…

Applications to graphs, oriented graphs, hypergraphs, hypercubes, permutations, crossing number of graphs, order types, geometry, . . . Razborov: Flag Algebra: an Interim Report

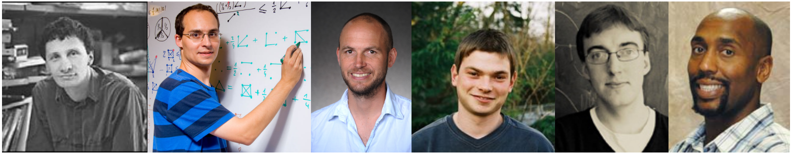
Inspiration

Theorem (Cummings, Kráľ, Pfender, Sperfeld, Treglown, Young)

In every 3-edge-colored complete graph on n vertices, there are at least $\frac{1}{25}\binom{n}{3} + o(n^3)$ monochromatic triangles.



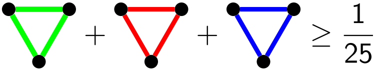
$$\geq \frac{1}{25} + o(1)$$

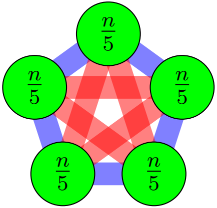


Inspiration

Theorem (Cummings, Kráľ, Pfender, Sperfeld, Treglown, Young)

In every 3-edge-colored complete graph on n vertices, there are at least $\frac{1}{25}\binom{n}{3} + o(n^3)$ *monochromatic triangles.*

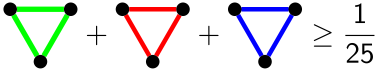


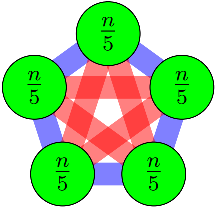


Inspiration

Theorem (Cummings, Kráľ, Pfender, Sperfeld, Treglown, Young)

In every 3-edge-colored complete graph on n vertices, there are at least $\frac{1}{25}\binom{n}{3} + o(n^3)$ monochromatic triangles.

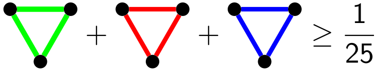


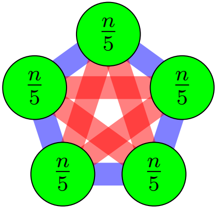


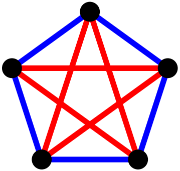
Inspiration

Theorem (Cummings, Kráľ, Pfender, Sperfeld, Treglown, Young)

In every 3-edge-colored complete graph on n vertices, there are at least $\frac{1}{25}\binom{n}{3} + o(n^3)$ *monochromatic triangles.*



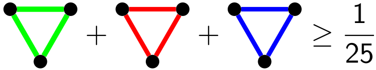


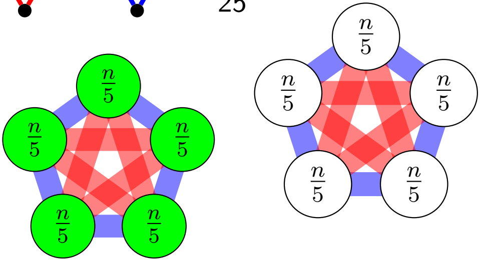


Inspiration

Theorem (Cummings, Kráľ, Pfender, Sperfeld, Treglown, Young)

In every 3-edge-colored complete graph on n vertices, there are at least $\frac{1}{25}\binom{n}{3} + o(n^3)$ *monochromatic triangles.*

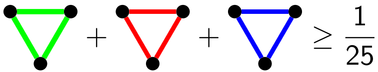


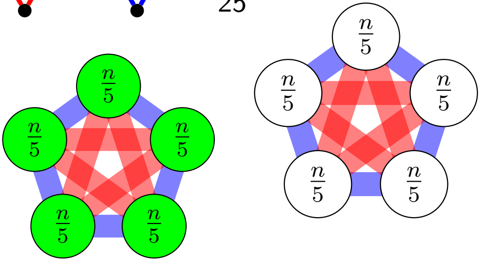


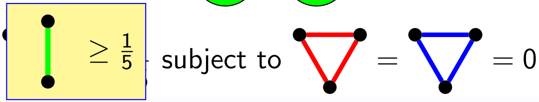
Inspiration

Theorem (Cummings, Kráľ, Pfender, Sperfeld, Treglown, Young)

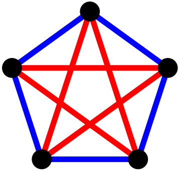
In every 3-edge-colored complete graph on n vertices, there are at least $\frac{1}{25}\binom{n}{3} + o(n^3)$ monochromatic triangles.

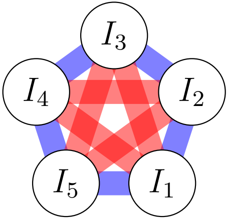






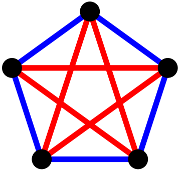
Example

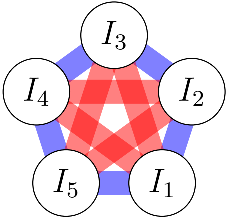




What is number of non-edges in a blow-up?

Example

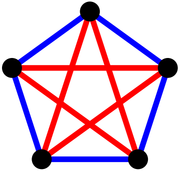


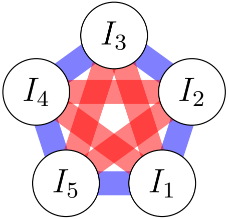


What is number of non-edges in a blow-up?

$$\sum_{i=1}^{5} \binom{|I_i|}{2} \geq \sum_{i=1}^{5} \binom{n/5}{2} \geq 5 \binom{n/5}{2} \approx \frac{1}{5} \binom{n}{2}$$

Example





What is number of non-edges in a blow-up?

$$\sum_{i=1}^{5}\binom{|I_i|}{2} \geq \sum_{i=1}^{5}\binom{n/5}{2} \geq 5\binom{n/5}{2} \approx \frac{1}{5}\binom{n}{2}$$

Observation (Key observation)

If a Ramsey graph G has k vertices, then the density of non-edges in any blow-up of G is at least $\frac{1}{k}+o(1)$.

Outline of idea

Observation (Key observation)

If a Ramsey graph G has k vertices, then the density of non-edges in any blow-up of G is at least $\frac{1}{k}+o(1)$.

Outline of idea

Observation (Key observation)

If a Ramsey graph G has k vertices, then the density of non-edges in any blow-up of G is at least $\frac{1}{k} + o(1)$.

- Let $\mathcal{G}$ be 2-edge-colored complete graphs with no monochromatic triangle.
- Consider all blow-ups $\mathcal{B}$ of graphs in $\mathcal{G}$
- $\forall B \in \mathcal{B}$, density of non-edges in B is at least $\frac{1}{k} = \frac{1}{5}$.

Outline of idea

Observation (Key observation)

If a Ramsey graph G has k vertices, then the density of non-edges in any blow-up of G is at least $\frac{1}{k}+o(1)$.

- Let $\mathcal{G}$ be 2-edge-colored complete graphs with no monochromatic triangle.
- Consider all blow-ups $\mathcal{B}$ of graphs in $\mathcal{G}$
- $\forall B \in \mathcal{B}$, density of non-edges in B is at least $\frac{1}{k} = \frac{1}{5}$.

Observation

If density of non-edges ρ is $> \frac{1}{k+1}$ over all $B \in \mathcal{B}$, then Ramsey graph has as most k vertices.

Outline of idea

Observation (Key observation)

If a Ramsey graph G has k vertices, then the density of non-edges in any blow-up of G is at least $\frac{1}{k}+o(1)$.

- Let $\mathcal{G}$ be 2-edge-colored complete graphs with no monochromatic triangle.
- Consider all blow-ups $\mathcal{B}$ of graphs in $\mathcal{G}$
- $\forall B \in \mathcal{B}$, density of non-edges in B is at least $\frac{1}{k} = \frac{1}{5}$.

Observation

If density of non-edges ρ is $> \frac{1}{k+1}$ over all $B \in \mathcal{B}$, then Ramsey graph has as most k vertices.

If one can prove $\rho > \frac{1}{6}$, then there is no Ramsey graphs on 6 vertices.

Outline of idea

Observation (Key observation)

If a Ramsey graph G has k vertices, then the density of non-edges in any blow-up of G is at least $\frac{1}{k}+o(1)$.

- Let $\mathcal{G}$ be 2-edge-colored complete graphs with no monochromatic triangle.
- Consider all blow-ups $\mathcal{B}$ of graphs in $\mathcal{G}$
- $\forall B \in \mathcal{B}$, density of non-edges in B is at least $\frac{1}{k} = \frac{1}{5}$.

Observation

If density of non-edges ρ is $> \frac{1}{k+1}$ over all $B \in \mathcal{B}$, then Ramsey graph has as most k vertices.

If one can prove $\rho > \frac{1}{6}$, then there is no Ramsey graphs on 6 vertices.

Notice that any lower bound on ρ in $(\frac{1}{k+1}, \frac{1}{k}]$ gives Ramsey graph has at most k vertices.

Outline of idea

Observation (Key observation)

If a Ramsey graph G has k vertices, then the density of non-edges in any blow-up of G is at least $\frac{1}{k}+o(1)$.

$$R(G_1, \ldots, G_n) \leq 1 + 1/\rho$$

- Let $\mathcal{G}$ be 2-edge-colored comp … triangle.
- Consider all blow-ups $\mathcal{B}$ of graphs in $\mathcal{G}$
- $\forall B \in \mathcal{B}$, density of non-edges in B is at least $\frac{1}{k} = \frac{1}{5}$.

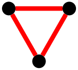
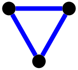
Observation

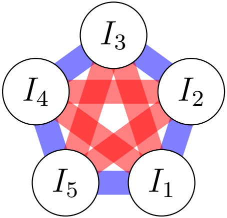
If density of non-edges ρ is $> \frac{1}{k+1}$ over all $B \in \mathcal{B}$, then Ramsey graph has as most k vertices.

If one can prove $\rho > \frac{1}{6}$, then there is no Ramsey graphs on 6 vertices.

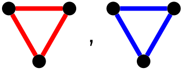
Notice that any lower bound on ρ in $(\frac{1}{k+1}, \frac{1}{k}]$ gives Ramsey graph has at most k vertices.

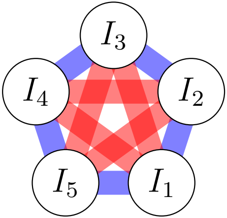
Blow-ups in Flag Algebra

How to characterize blow-ups $\mathcal{B}$ of graphs with no  ,  ?

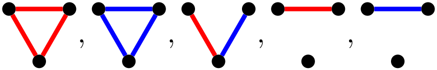


Blow-ups in Flag Algebra

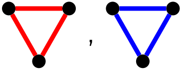
How to characterize blow-ups $\mathcal{B}$ of graphs with no  ?

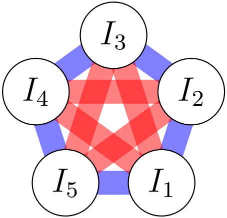


Forbidden subgraphs :

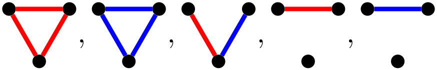


Blow-ups in Flag Algebra

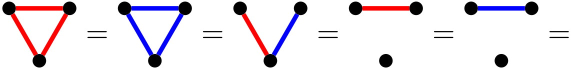
How to characterize blow-ups $\mathcal{B}$ of graphs with no  ?



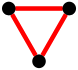
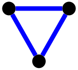
Forbidden subgraphs :

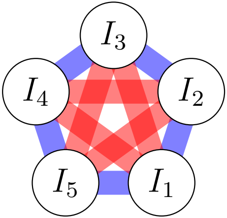


minimize 

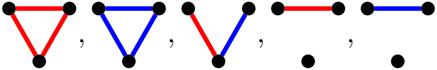
subject to  $= 0$

Blow-ups in Flag Algebra

How to characterize blow-ups $\mathcal{B}$ of graphs with no  ,  ?

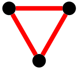
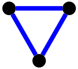
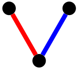
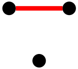
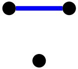


Forbidden subgraphs :



minimize 

Flag Algebra question! Easy to modify.

subject to  =  =  =  =  = 0

New upper bounds (so far)

Problem	Lower	New upper	Old upper
$R(K_4^-, K_4^-, K_4^-)$	28	28	30
$R(K_3, K_4^-, K_4^-)$	21	23	27
$R(K_4, K_4^-, K_4^-)$	33	47	59
$R(K_4, K_4, K_4^-)$	55	104	113
$R(C_3, C_5, C_5)$	17	18	21?
$R(K_4, K_7^-)$	37	52	59
$R(K_{2,2,2}, K_{2,2,2})$	30	32	60?
$R(K_5^-, K_6^-)$	31	38	39
$R(K_5, K_6^-)$	43	62	67

Example of Computation

Lemma

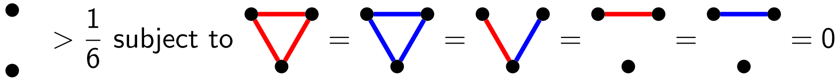
$R(K_3, K_3) \leq 6$

Example of Computation

Lemma

$R(K_3, K_3) \leq 6$

Our goal is to show:

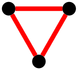
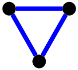
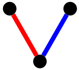
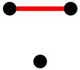
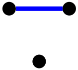


Example of Computation

Lemma

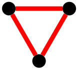
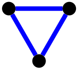
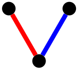
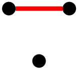
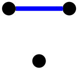
$R(K_3, K_3) \leq 6$

Our goal is to show:

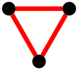
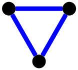
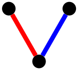
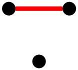
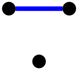
 $> \frac{1}{6}$ subject to  $=$  $=$  $=$  $=$  $= 0$

We show perhaps the most complicated proof of the lemma!

Our goal is to show:

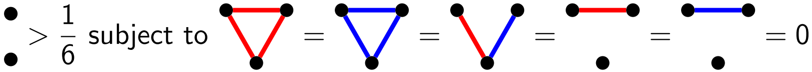
 $> \frac{1}{6}$ subject to  $=$  $=$  $=$  $=$  $= 0$

Our goal is to show:

 $> \frac{1}{6}$ subject to  $=$  $=$  $=$  $=$  $= 0$

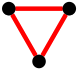
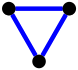
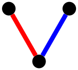
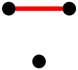
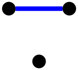
Observe that  and  can be swapped.

Our goal is to show:



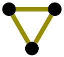
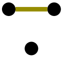
Observe that  and  can be swapped. Change to a colorblind setting.  is a monochromatic triangle (red or blue).

Our goal is to show:

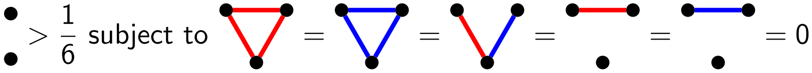
 $> \frac{1}{6}$ subject to  $=$  $=$  $=$  $=$  $= 0$

Observe that  and  can be swapped. Change to a colorblind setting.  is a monochromatic triangle (red or blue).

Our new goal is to show:

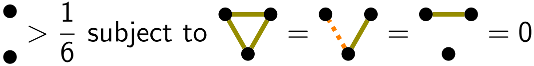
 $> \frac{1}{6}$ subject to  $=$  $=$  $= 0$

Our goal is to show:



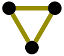
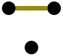
Observe that  and  can be swapped. Change to a colorblind setting.  is a monochromatic triangle (red or blue).

Our new goal is to show:

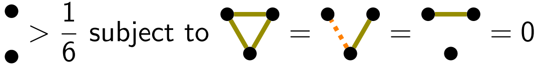


Colorblind setting will allow to fit the computation on these slides. Also important for bigger applications.

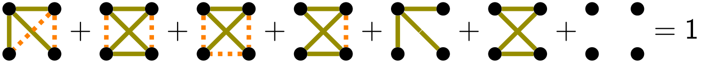
Our goal is to show:

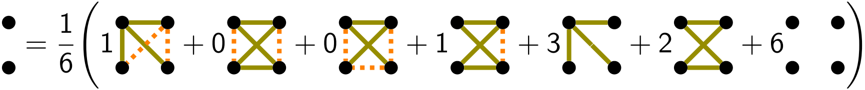
 $> \frac{1}{6}$ subject to  $=$  $=$  $= 0$

Our goal is to show:

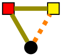
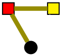


Basic equations:



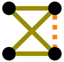


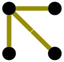
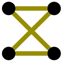
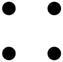
We use flags with type σ_1 of size two

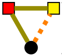
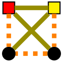
$$F = \Big($$  ,  ,  $$\Big)^T .$$

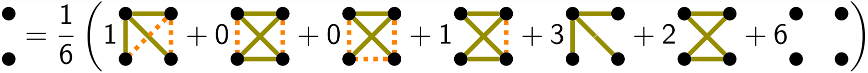
For a positive semidefinite matrix M

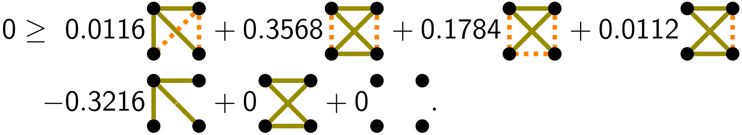
$$0 \leq \left[\!\left[F^T M F \right]\!\right]_{\sigma_1} = \left[\!\left[F^T \begin{pmatrix} 0.0744 & -0.0223 & -0.0520 \\ -0.0223 & 0.0238 & -0.0014 \\ -0.0520 & -0.0014 & 0.0536 \end{pmatrix} F \right]\!\right]_{\sigma_1}$$

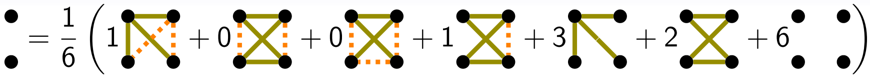
$= -0.0116$  $-\, 0.3568$  $-\, 0.1784$  $-\, 0.0112$ 

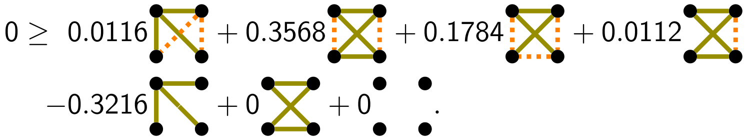
$+\, 0.3216$  $+\, 0$  $+\, 0$  .

$\Big[\!\Big[$  $\times$  $\Big]\!\Big]_{\sigma_1} = \Big[\!\Big[\frac{1}{2}$  $+ \frac{1}{2}$  $\Big]\!\Big]_{\sigma_1} = \frac{1}{2} \Big(\frac{8}{12}$  $+ \frac{4}{12}$  $\Big)$.

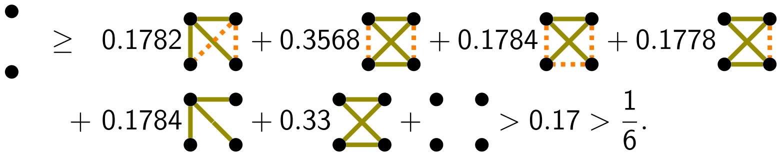

= 1/6 (1 + 0 + 0 + 1 + 3 + 2 + 6)

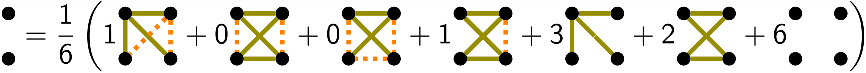

0 ≥ 0.0116 + 0.3568 + 0.1784 + 0.0112
−0.3216 + 0 + 0 .

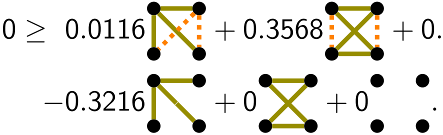




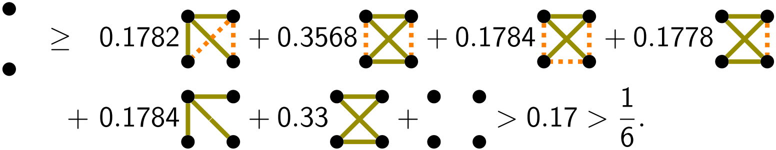
We sum the equations and obtain







We sum the equations and obtain



Note that the matrix M was not unique or tight (easy rounding).
(bound $\geq \frac{1}{5}$ is obtainable)

What have we tried (so far)

Problem	Lower	Upper	Our upper	*Graphs*
$R(K_4^-, K_4^-, K_4^-)$	28	30	28	2589
$R(K_3, K_4^-, K_4^-)$	21	27	23	4877
$R(K_4, K_4^-, K_4^-)$	33	59	47	9476
$R(K_4, K_4, K_4^-)$	55	113	104	11410
$R(C_3, C_5, C_5)$	17	21?	18	5291
$R(K_4, K_7^-)$	37	52	49	11747
$R(K_{2,2,2}, K_{2,2,2})$	30	60?	32	8792
$R(K_5^-, K_6^-)$	31	39	38	14889
$R(K_5, K_6^-)$	43	67	62	18186
$R(K_4, K_6)$	36	41	44	11667
$R(K_4, K_7)$	49	61	67	11765
$R(K_5, K_5)$	43	49	53	8722
$R(K_5, K_5^-)$	30	34	35	14169
$R(K_3, K_3, K_4)$	30	31	33	7878
$R(K_3, K_3, K_5)$	45	57	61	8433
$R(K_3, K_4, K_4)$	55	79	85	15625
$R(K_3, K_3, K_3, K_3)$	51	62	65	18571
$R(K_4, K_6^-)$	30	33	33	11372
$R(K_3, C_4, K_4)$	27	32	32	9928
$R(C_4, C_4, K_4)$	20	22	22	11857

Thank you for your attention!